Self-Similar Solution of the three dimensional Navier-Stokes Equation

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Outline

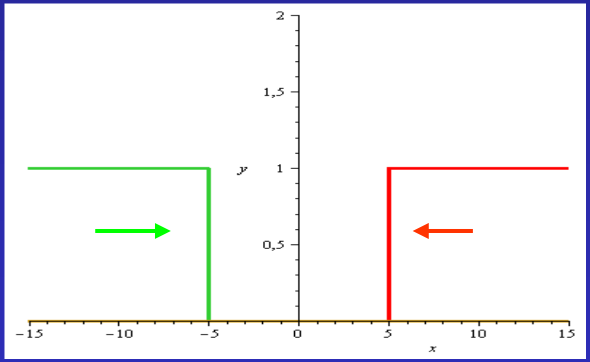
- **Solutions of PDEs** *self-similar, various heat conduction examples*
- ***Navier-Stokes equation***
- ***My 3D Ansatz & geometry*** *my solution + other solutions*
- ***Summary & Outlook*** *additional new systems to study*

Physically important solutions of PDEs

- Travelling waves:

  arbitrary wave fronts

  u(x,t) ~ g(x-ct), g(x+ct)



- Self-similar

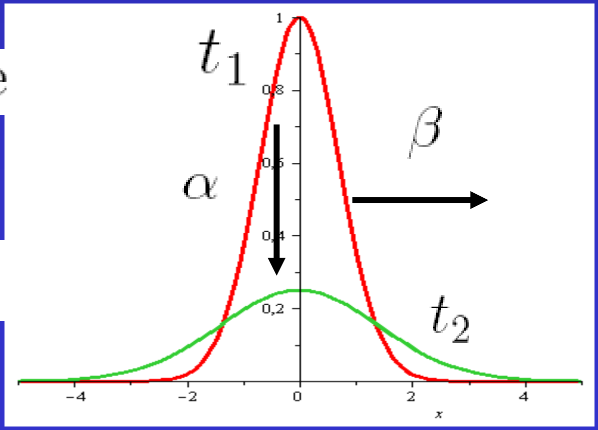
$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

Sedov, Barenblatt, Zeldovich

α and β are of primary physical importance

α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)



$t_1 < t_2$

in Fourier heat-conduction

Ordinary diffusion/heat conduction equation

$$\mathbf{q} = -k\nabla U(x,t), \quad \nabla \mathbf{q} = -\gamma \frac{\partial U(x,t)}{\partial t}$$

J(x,t) temperature distribution
Fourier law + conservation law

- $$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$

  abolic PDA, no time-reversal sym.

- *strong maximum principle ~ solution is smeared out in time*
- *the fundamental solution:*
- *general solution is:*

$$\Phi(x,t) = \int \frac{1}{\sqrt{4\pi k t}} exp\left(-\frac{x^2}{4kt}\right)$$

$$u(x,t) = \int \Phi(x-y,t) g(y) dy$$

$$u(x,0) = g(x) \;\; for -\infty < x < \infty \;\; and \;\; 0 < t < \infty$$

- *kernel is non compact = inf. prop. Speed paradox of heat cond.*
- *Problem from a long time ☹*
- *But have self-similar solution ☺*

$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

General derivation for heat conduction law

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k \nabla T(x,t)$$

Cattaneo heat conduction law, there is a general way to derive

$$q = -\int_{-\infty}^{t} Q(t-t') \frac{\partial T(x,t')}{\partial x} \mathrm{d}t'$$

T(x,t) temperature distribution
q heat flux

Joseph D D and Preziosi L 1989 *Rev. Mod. Phys.* **61** 41
Joseph D D and Preziosi L 1990 *Rev. Mod. Phys.* **62** 375

$$Q(t-t') = \frac{k\tau^{l}}{(t-t'+\omega)^{l}}$$

the kernel can have microscopic interpretation

$$\epsilon \frac{\partial^2 T(x,t)}{\partial t^2} + \frac{a}{t}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2}$$

telegraph-type time dependent eq. with self sim. solution

$$T(x,t) = t^{-\alpha} f(\eta) \;\; with \;\; \eta = \frac{x}{t^{\beta}}$$

Solutions

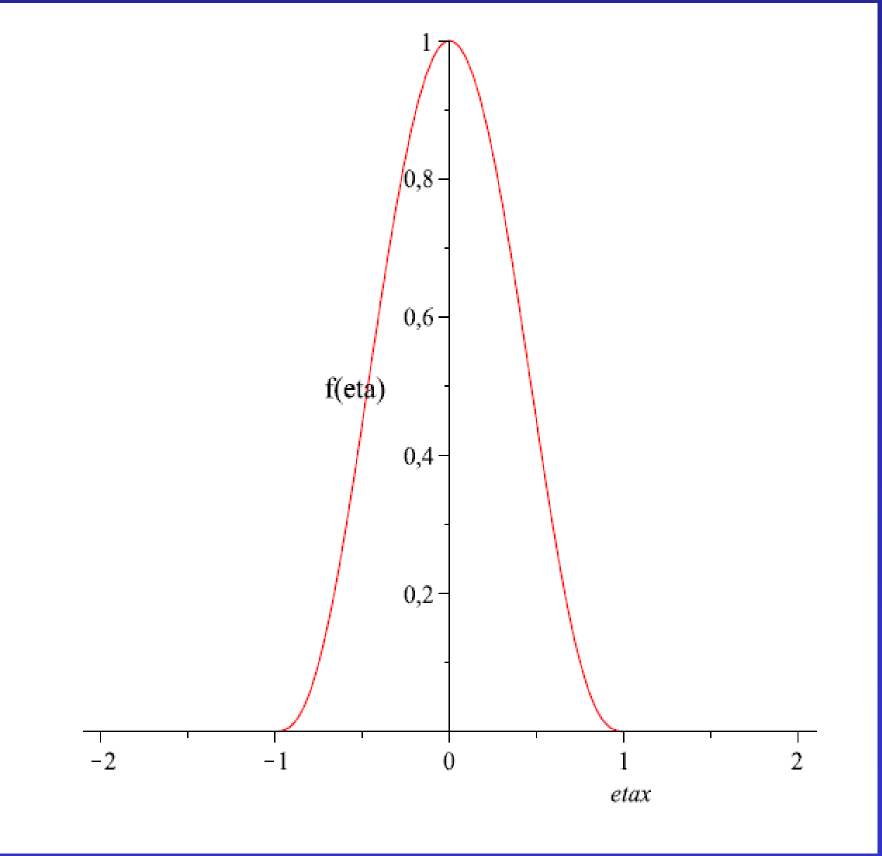
J. Phys. A: Math. Theor. 43 (2010) 375210

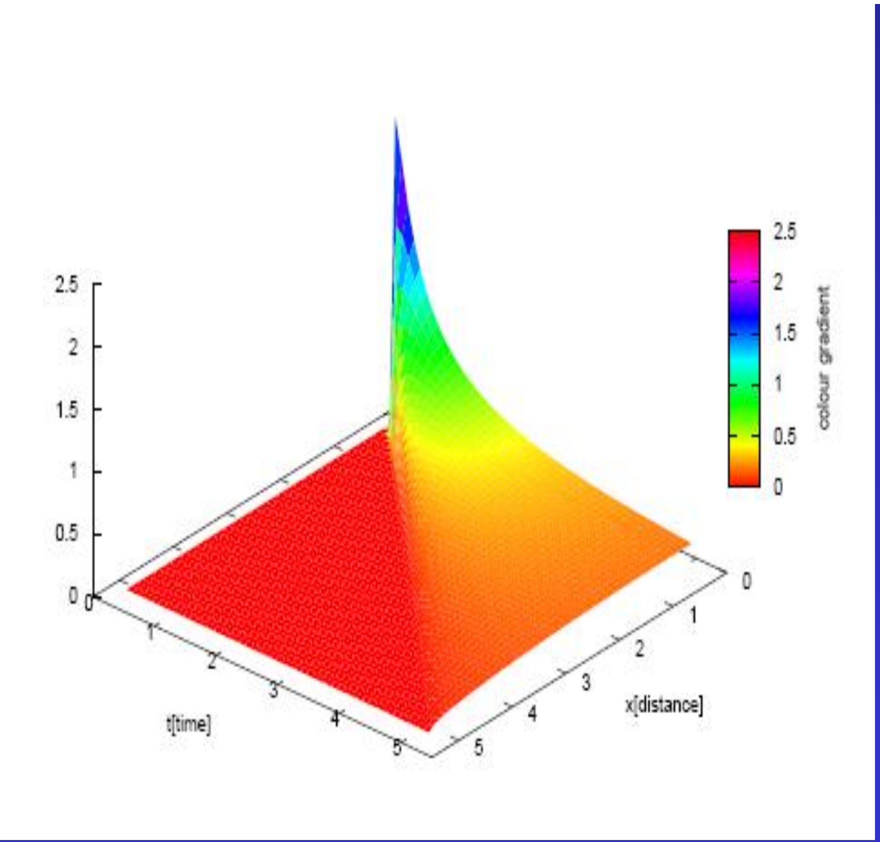
$c_1 = 0$

$a = 4.1, \epsilon = 1$

$$f(\eta) = (1 - \epsilon\eta^2)^{\frac{a}{2\epsilon}-1}$$

$$T(x,t) = \frac{1}{t}\left(1 - \epsilon\frac{x^2}{t^2}\right)^{\frac{a}{2\epsilon}-1}$$





2 Important new feature: the solution is a product of 2 travelling wavefronts

$if\ a > 4\epsilon,\ f'(\eta) = 0$ *no flux conservation problem*

$$T(x,t) \sim U(x - ct)U(x + ct)$$

The Navier-Stokes equation

$$\nabla \mathbf{v} = 0,$$
$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = \nu \triangle \mathbf{v} - \frac{\nabla p}{\rho} + a$$

3 dimensional cartesian coordinates,
Euler description
v velocity field, p pressure, a external field
ν **kinematic viscosity,** ρ **constant density**

$\mathbf{v}(x,y,z,t) =$ $u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ $p(x,y,z,t)$

Consider the most general case

$$u_x + v_y + w_z = 0$$
$$u_t + uu_x + vu_y + wu_z = \nu(u_{xx} + u_{yy} + u_{zz}) - \frac{p_x}{\rho}$$
$$v_t + uv_x + vv_y + wv_z = \nu(v_{xx} + v_{yy} + v_{zz}) - \frac{p_y}{\rho}$$
$$w_t + uw_x + vw_y + ww_z = \nu(w_{xx} + w_{yy} + w_{zz}) - \frac{p_z}{\rho} + a.$$

just to write out all the coordinates

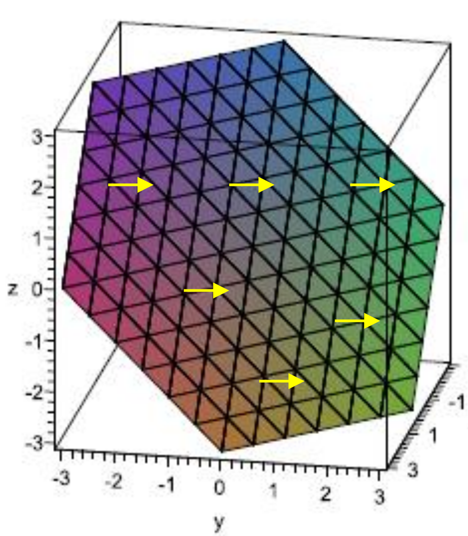
My 3 dimensional Ansatz

$u(x,t) = t^{-\alpha} f(x/t^{\beta})$ → $u(x,y,z,t) = t^{-\alpha} f\left(\frac{F(x,y,z)}{t^{\beta}}\right) := t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right) := t^{-\alpha} f(\omega)$

$F(x,y,z) = x + y + z = 0$

A more general function does not work for N-S

~~$u(x,y,z,t) = t^{-\alpha} f\left(\frac{\sqrt{x^2+y^2+z^2}-a}{t^{\beta}}\right)$~~



The graph of the $x+y+z=0$ plane.

The final applied forms

$$u(x,y,z,t) = t^{-\alpha} f\left(\frac{x+y+z}{t^{\beta}}\right), \quad v(x,y,z,t) = t^{-\gamma} g\left(\frac{x+y+z}{t^{\delta}}\right)$$

$$w(x,y,z,t) = t^{-\epsilon} h\left(\frac{x+y+z}{t^{\zeta}}\right), \quad p(x,y,z,t) = t^{-\eta} l\left(\frac{x+y+z}{t^{\theta}}\right)$$

The obtained ODE system

$$
\begin{aligned}
f'(\omega) + g'(\omega) + h'(\omega) &= 0 \\
-\frac{1}{2}f(\omega) - \frac{1}{2}\omega f'(\omega) + [f(\omega) + g(\omega) + h(\omega)]f'(\omega) &= 3\nu f''(\omega) - \frac{l'(\omega)}{\rho} \\
-\frac{1}{2}g(\omega) - \frac{1}{2}\omega g'(\omega) + [f(\omega) + g(\omega) + h(\omega)]g'(\omega) &= 3\nu g''(\omega) - \frac{l'(\omega)}{\rho} \\
-\frac{1}{2}h(\omega) - \frac{1}{2}\omega h'(\omega) + [f(\omega) + g(\omega) + h(\omega)]h'(\omega) &= 3\nu h''(\omega) - \frac{l'(\omega)}{\rho} + a.
\end{aligned}
$$

as constraints we got for the exponents:

$$\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \theta = 1/2, \qquad \eta = 1$$

$$u(x,y,z,t) = t^{-1/2} f\left(\frac{x+y+z}{t^{1/2}}\right) = t^{-1/2} f(\omega), \quad v(x,y,z,t) = t^{-1/2} g(\omega),$$
$$w(x,y,z,t) = t^{-1/2} h(\omega), \quad p(x,y,z,t) = t^{-1} l(\omega),$$

Continuity eq. helps us to get additional constraints:

$$f(\omega) + g(\omega) + h(\omega) = c, \quad \textit{and} \quad f''(\omega) + g''(\omega) + h''(\omega) = 0$$

Solutions of the ODE

a single Eq. remains

$$9\nu f''(\omega) - 3(\omega + c)f'(\omega) + \frac{3}{2}f(\omega) - \frac{c}{2} + a = 0.$$

$$f(\omega) = c_1 \cdot KummerU\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + c_2 \cdot KummerM\left(-\frac{1}{4}, \frac{1}{2}, \frac{(\omega+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}$$

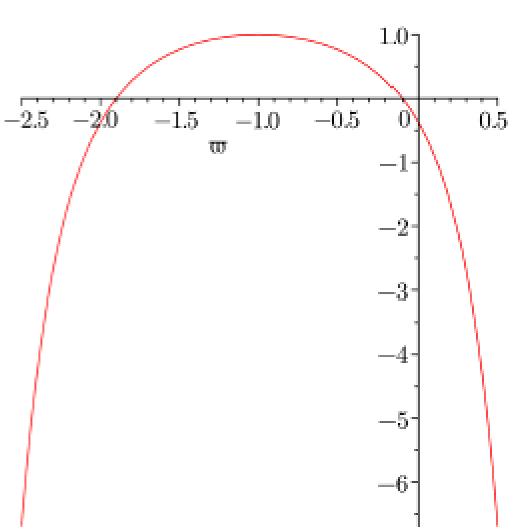


Fig. 3 The KummerM$(-1/4, 1/2, (\omega+c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

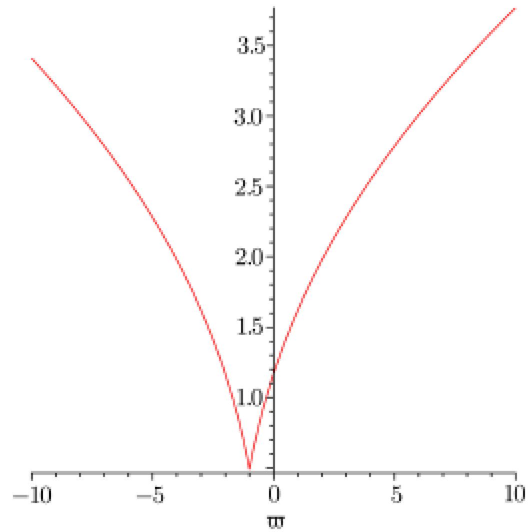


Fig. 4 The KummerU$(-1/4, 1/2, (\omega+c)^2/6\nu)$ function for $c = 1$ and $\nu = 0.1$.

$(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), (a)_0 = 1$$

$$M(a,b,z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \cdots + \frac{(a)_n z^n}{(b)_n n!},$$

$$U(a,b,z) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)}\right]$$

Solutions of N-S

$$u(x,y,z,t) = t^{-1/2} f(\omega) = t^{-1/2}\left[c_1 \cdot \mathrm{KummerU}\left(\frac{-1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right)\right] + t^{-1/2}\left[c_2 \cdot \mathrm{KummerM}\left(-\frac{1}{4}, \frac{1}{2}, \frac{((x+y+z)/t^{1/2}+c)^2}{6\nu}\right) + \frac{c}{3} - \frac{2a}{3}\right]$$

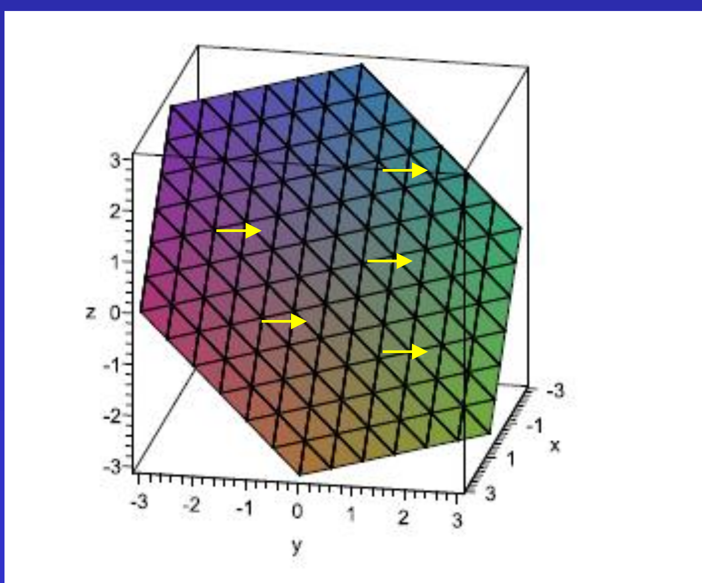
only for one velocity component ☹

Geometrical explanation:

all v components with coordinate constrain x+y+z=0 lie in a plane = equivalent

Naver-Stokes makes a dynamics of this plane

getting a multi-valued surface



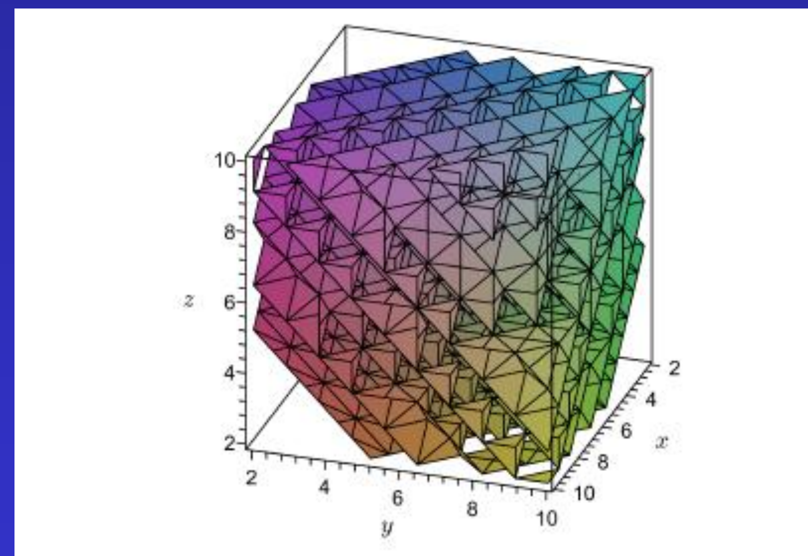


Fig. 5 The implicit plot of the self-similar solution Eq. (17). Only the KummerU function is presented for $t=1$, $c_1=1$, $c_2=0$, $a=0$, $c=1$, and $\nu=0.1$.

I.F. Barna http://arxiv.org/abs/1102.5504
Commun. Theor. Phys. 56 (2011) 745-750

Other analytic solutions I

Without completeness, usually from Lie algebra studies

W. I. Fushchich, W. M. Shtelen and S. L. Slavutsky J. Phys. A: Math. Gen. 24 (1990) 971.

$$\omega = z/\sqrt{t}$$

Presented 19 various solutions one of them is:

$$u(z,t) = \frac{f(\omega)}{\sqrt{t}}, \quad v(y,z) = \frac{g(\omega)}{\sqrt{t}} + \frac{y}{t}, \quad w(z,t) = \frac{h(\omega)}{\sqrt{t}}, \quad p(t,z) = \frac{l(\omega)}{\sqrt{t}}$$

the obtained ODE system

$$\begin{aligned} h'(\omega) + 1 &= 0 \\ -\frac{1}{2}(f(\omega) + \omega f'(\omega)) + h(\omega) f'(\omega) &= f''(\omega), \\ \frac{1}{2}(g(\omega) + \omega g'(\omega)) + h(\omega) g'(\omega) &= g''(\omega), \\ -\frac{1}{2}(h(\omega) + \omega h'(\omega)) + h(\omega) h'(\omega) + l'(\omega) &= f''(\omega). \end{aligned}$$

the solution:

$$\begin{aligned} f(\omega) &= (\frac{3}{2}\omega - c)^{-1/2} exp\left[-\frac{1}{6}(\frac{3}{2}\omega - c)^2\right] w\left[-\frac{1}{12}, \frac{1}{4}, \frac{1}{3}(\frac{3}{2}\omega - c)^2\right] \\ g(\omega) &= (\frac{3}{2}\omega - c)^{-1/2} exp\left[-\frac{1}{6}(\frac{3}{2}\omega - c)^2\right] w\left[-\frac{5}{12}, \frac{1}{4}, \frac{1}{3}(\frac{3}{2}\omega - c)^2\right] \\ h(\omega) &= -\omega + c \\ l(\omega) &= \frac{3}{2}c\omega - \omega^2 + c_1 \end{aligned}$$

with:

$$w(\kappa, \mu, z) = e^{-1/2z} z^{1/2+\mu} KummerM(1/2 + \mu - \kappa, 1 + 2\mu, z).$$

Other analytic solutions II

V. Grassi, R.A. Leo, G. Soliani and P. Tempesta, Physica **286** (2000) 79

The initial Navier-Stokes

velocity components $U_i(y,z,t)$ and π is the pressure,

$$
\begin{aligned}
U_{1t} + cU_1 + U_2U_{1y} + U_3U_{1z} - \nu(U_{1yy} + U_{1zz}) &= 0,\\
U_{2t} + U_2U_{2y} + U_3U_{2z} + \pi_y - \nu(U_{2yy} + U_{2zz}) &= 0,\\
U_{3t} + U_2U_{3y} + U_3U_{3z} + \pi_z - \nu(U_{3yy} + U_{3zz}) &= 0,\\
U_{2y} + U_{3z} + c &= 0
\end{aligned}
$$

After some transformation got a PDE:

$$U_{1t} + k_1yU_{1y} + (\sigma - k_1z)U_{1z} - \nu(U_{1yy} + U_{1zz}) = 0$$

applied Ansatz:

$$U_1 = Y(y)T(z)\Phi(t).$$

Solutions:
where M is a Kummer function

$$
\begin{aligned}
\Phi &= c_1exp(c_2)t\\
Y &= c_3M\left(-c_4, \frac{1}{2}, \frac{y^2}{2\nu}\right) + yc_5M\left(\frac{1}{2} - c_4, \frac{3}{2}, \frac{y^2}{2\nu}\right)\\
T &\approx M\left(c_6, \frac{1}{2}, \frac{z^2}{2\nu}\right) + zM\left(\frac{1}{2} - c_6, \frac{3}{2}, \frac{z^2}{2\nu}\right)
\end{aligned}
$$

Summary and Outlook

we presented the self-similar Ansatz as a tool for non-linear PDEs with some examples for heat conduction

as a new feature we presented a 3d self-similar Ansatz for the fully 3d N-S system with explanation and results

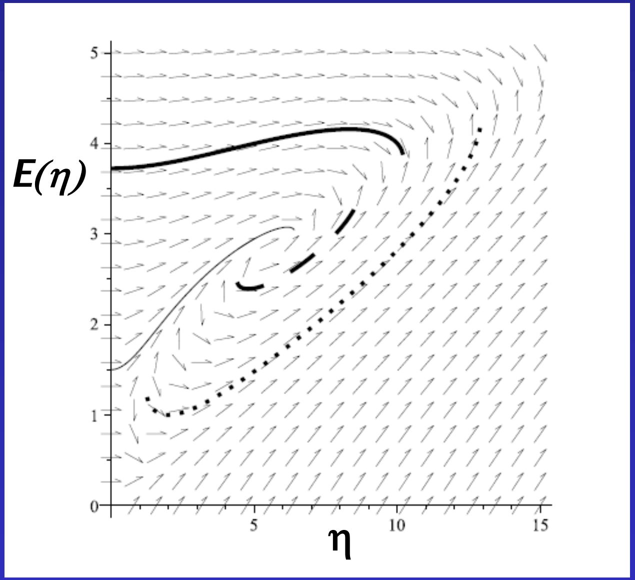
*further work is in progress to clear out all **dark points** and give a bit more generality ($ax - by + z = 0$)*

Outlook

Self-similar model is used for additional systems:

non-linear Maxwell equ. to find shock-wave/compact solutions Idea: power-law field dependent materials $\mu(\mathrm{H}) = a\mathrm{H}^q \qquad \epsilon(\mathrm{E}) = b\mathrm{E}^r$

first results: if q < -1 →



1 dim flow + heat conduction system is under investigation too

$$\rho(x,t)_t + [\rho(x,t)v(x,t)]_x = 0,$$
$$v(x,t)_x + v(x,t)v(x,t)_x = -\frac{1}{\rho(x,t)}p(x,t)_x,$$
$$T(x,t)_t + v(x,t)T(x,t)_x = \lambda T(x,t)_{xx},$$

Thank you for your attention!

Questions, Remarks, Comments?…