Complete 2^{nd} order fluid dynamics from the Boltzmann equation

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Notation

- For simplicity we choose the flat space-time, the metric is $g^{\mu\nu} \equiv g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- The normalized 4-flow of matter is denoted by $u^{\mu}(t, \mathbf{x})$, where $u^{\mu} u_{\mu} = 1, (c^2 = 1)$.
- The local rest frame (LRF) is defined as, $u^{\mu}_{LRF} = (1, 0, 0, 0)$.
- The projection tensor is defined perpendicular to the 4-flow of matter, $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu} u^{\nu}$, where $\Delta^{\mu\nu} u_{\mu} = 0$, and $\Delta^{\mu\nu} \Delta_{\mu\nu} = 3$.
- The comoving time-derivative or proper-time derivative in LRF of an arbitrary 4-vector A is denoted by, $\dot{A} = u^{\mu} \partial_{\mu} A$ while, the comoving spatial-derivative or gradient is denoted by, $\nabla^{\mu} A = \Delta^{\mu\nu} \partial_{\nu} A$.
- The symmetric, traceless and orthogonal part of any tensor is denoted by, $A^{\langle\mu\nu\rangle} = \left[\frac{1}{2}\left(\Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta} + \Delta^{\nu}_{\alpha}\Delta^{\mu}_{\beta}\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] A^{\alpha\beta}$.

Basics

Conservation laws for a simple (single component) fluid

$$\partial_\mu N^\mu = 0 \quad \textbf{charge conservation}$$
$$\partial_\mu T^{\mu\nu} = 0 \quad \textbf{energy-momentum conservation}$$

Thermodynamic constraint, $S^\mu = S^\mu\left(N^\mu, T^{\mu\nu}\right)$, such that

$$\partial_\mu S^\mu \geq 0$$

Perfect Fluids I.

Conservation laws for a simple (single component) perfect fluid (no dissipation)

$$\partial_\mu N_0^\mu = 0 \qquad \textbf{charge conservation} \qquad \Rightarrow \textbf{1 eq.}$$

$$\partial_\mu T_0^{\mu\nu} = 0 \qquad \textbf{energy-momentum conservation} \qquad \Rightarrow \textbf{4 eqs.}$$

Perfect fluids - decomposition with respect to u^μ

$$N_0^\mu = n_0 u^\mu$$

$$T_0^{\mu\nu} = e_0 u^\mu u^\nu - p_0 \Delta^{\mu\nu}$$

$$n_0 = N_0^\mu u_\mu \qquad \textbf{(net)charge density}$$

$$e_0 = T_0^{\mu\nu} u_\mu u_\nu \qquad \textbf{energy density}$$

$$p_0 = -\frac{1}{3}\Delta_{\mu\nu} T_0^{\mu\nu} \qquad \textbf{isotropic (equilibrium) pressure}$$

- 5 equations with 6 unknowns: $n_0(1), e_0(1), p_0(1)$ and $u^\mu(3)$ uniquely defined! These equations are *postulated*!

Perfect Fluids II.

- The assumption of local thermal equilibrium! Provides closure:

Equation of State (EoS)

$$p_0 = p_0(e_0, n_0) \qquad \text{EoS} \Rightarrow \mathbf{1\ eq.}$$

and $s_0 = s_0(e_0, n_0)$, $p_0(T_0, \mu_0)$.

- Auxiliary, $S_0^\mu = s_0 u^\mu$, where $s_0 = S_0^\mu u_\mu$, hence

$$\partial_\mu S_0^\mu \geq 0$$

entropy is maximal in local thermal equilibrium.

- Thermodynamic relations are derived from, $T_0 S_0^\mu = -\mu_0 N_0^\mu + T_0^{\mu\nu} u_\nu + p_0 u^\mu$, that is, $T\partial_\mu(su^\mu) = \partial_\mu(eu^\mu) + p(\partial_\mu u^\mu) - \mu\partial_\mu(nu^\mu)$, hence

$$\begin{aligned} T_0 s_0 &= e_0 + p_0 - \mu n_0 \\ T_0 \dot{s}_0 &= \dot{e}_0 - \mu_0 \dot{n}_0 \\ \dot{p}_0 &= s_0 \dot{T}_0 + n_0 \dot{\mu}_0 \end{aligned}$$

Dissipative Fluids I.

Conservation laws for a simple (single component) dissipative fluid

$$\partial_\mu N^\mu = 0 \quad \textbf{charge conservation} \quad \Rightarrow \textbf{1 eq.}$$
$$\partial_\mu T^{\mu\nu} = 0 \quad \textbf{energy-momentum conservation} \quad \Rightarrow \textbf{4 eqs.}$$

General decomposition with respect to u^μ

$$N^\mu = n u^\mu + V^\mu$$
$$T^{\mu\nu} = e u^\mu u^\nu - p\Delta^{\mu\nu} + W^\mu u^\nu + W^\nu u^\mu + \pi^{\mu\nu}$$
$$n = N^\mu u_\mu \quad \textbf{charge density}$$
$$e = T^{\mu\nu} u_\mu u_\nu \quad \textbf{energy density}$$
$$p = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu} \quad \textbf{isotropic pressure}$$
$$V^\mu = \Delta^{\mu\alpha} N_\beta \quad \textbf{charge flow} \quad V^\mu u_\mu = 0$$
$$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta \quad \textbf{energy-momentum flow} \quad W^\mu u_\mu = 0$$
$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle} \quad \textbf{stress tensor} \quad \pi^{\mu\nu} u_\mu = \pi^{\mu\nu} g_{\mu\nu} = 0$$

- 5 equations with $14 + 3$ unknowns, $n(1), e(1), p(1), V^\mu(3), W^\mu(3), \pi^{\mu\nu}(5)$ and $u^\mu(3)$

Dissipative Fluids II.

Generally, $N^\mu = N_0^\mu + \delta N^\mu$ and $T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}$

Simplifications (I): Matching to equilibrium and the EOS

$$\begin{aligned} n &= n_0 + \delta n \\ e &= e_0 + \delta e \\ p(e,n) &= p_0(e_0, n_0) + \delta p \end{aligned}$$

- Convenient choice, $\delta n = 0$, $\delta e = 0$, $\delta p = \Pi$ hence, $n = n_0, e = e_0$, $p(e,n) = p_0(e_0, n_0) + \Pi$, while $T = T_0$ and $\mu = \mu_0$, but $s = s_0 + \delta s$!
- Still with $14 + 3$ unknowns! $n(1), e(1), \Pi(1)$, $V^\mu(3), W^\mu(3), \pi^{\mu\nu}(5)$ and $u^\mu(3)$.
- Other choices also possible ! e.g., $e = e_0 + 3\Pi$, etc.

Simplifications (II): Fixing the Local Rest Frame

$$u_E^\mu = N^\mu/n \quad \Leftrightarrow V^\mu = 0 \quad \Rightarrow q^\mu = W^\mu \qquad \textbf{Eckart}$$

$$u_L^\mu = T^{\mu\nu} u_{L\nu}/e \quad \Leftrightarrow W^\mu = 0 \quad \Rightarrow q^\mu = -\frac{e+p}{n} V^\mu \qquad \textbf{Landau \& Lifsitz}$$

- We eliminated (3) unknowns, such that $u_L^\mu = u_E^\mu + \delta u^\mu$.
- We are left with 14 unknowns! $n(1), e(1), u^\mu(3)$ and $p(1), q^\mu(3), \pi^{\mu\nu}(5)$.

Dissipative Fluids III. - The relativistic Navier-Stokes theory

- The entropy current is also modified $S^\mu \equiv S_0^\mu + \delta S^\mu = (s_0 + \delta s)u^\mu + \Phi^\mu$, where $s \equiv S^\mu u_\mu = (s_0 + \delta s)$ and $\Phi^\mu = \Delta^{\mu\nu} S_\nu$.

2nd law of thermodynamics (Eckart's frame)

$$\partial_\mu S^\mu = \partial_\mu \left[\Phi^\mu - \frac{q^\mu}{T}\right] - \frac{q^\mu}{T}\left(\frac{1}{T}\partial_\mu T - \dot{u}_\mu\right) - \frac{\Pi}{T}\partial_\mu u^\mu + \frac{\pi^{\mu\nu}}{T}\partial_\mu u_\nu \geq 0$$

- Assuming that $s = s_0$, i.e., $\delta s = 0$, while $\Phi^\mu = q^\mu / T$
- For small gradients, linear relations between thermodynamical forces and fluxes!

The relativistic Navier-Stokes relations

$$\Pi_{NS} = -\zeta \nabla_\mu u^\mu$$
$$\pi^{\mu\nu}_{NS} = 2\eta \nabla^{\langle\mu} u^{\nu\rangle}$$
$$q^\mu_{NS} = -\kappa T \frac{T n}{e + p} \nabla^\mu \left(\frac{\mu}{T}\right)$$

- $\zeta \geq 0$, $\eta \geq 0$ bulk and shear viscosity coefficients, $\kappa \geq 0$ coefficient of thermal conductivity.
- Now, the equations of fluid dynamics are closed, but the relativistic Navier-Stokes theory leads to accausal signal propagation and stability issues.

Dissipative Fluids IV. - 2nd order theories

Generalization by Müller (1967), Israel (1976) and Stewart (1971, 1977)

$$S^{\mu} \equiv s_0 u^{\mu} + \frac{q^{\mu}}{T} - \left(\beta_0 \Pi^2 - \beta_1 q^{\mu} q_{\mu} + \beta_2 \pi^{\mu\nu} \pi_{\mu\nu}\right) \frac{u^{\mu}}{2T} - \frac{\alpha_0 \Pi q^{\mu}}{T} + \frac{\alpha_1 \pi^{\mu\nu} q_{\nu}}{T} + O_3$$

- the newly introduced coefficients $\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1$ are related to the relaxation time/length of bulk viscosity, heat conductivity and shear viscosity

$$\begin{aligned}
\tau_{\Pi} &= \zeta \beta_0 \\
\tau_q &= \kappa T \beta_1 \\
\tau_{\pi} &= 2\eta \beta_2 \\
l_{\Pi q} &= \zeta \alpha_0 \\
l_{q \Pi} &= \kappa T \alpha_0 \\
l_{q \pi} &= \kappa T \alpha_1 \\
l_{\pi q} &= 2\eta \alpha_1
\end{aligned}$$

- The $\beta_0, \beta_1, \beta_2, \alpha_0, \alpha_1$ coefficients are frame dependent and remain undetermined in phenomenological theories!

Dissipative Fluids V. - Equations of motion

Relaxation equations in Eckart's frame (Israel 1976)

$$\begin{aligned} \tau_\Pi \dot{\Pi} + \Pi &= \Pi_{NS} + l_{\Pi q} \nabla_\mu q^\mu \\ \tau_q \Delta^\mu_\alpha \dot{q}^\alpha + q^\mu &= q^\mu_{NS} + l_{q\Pi} \nabla^\mu \Pi - l_{q\pi} \Delta^\mu_\alpha \partial_\nu \pi^{\alpha\nu} \\ \tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= \pi^{\mu\nu}_{NS} + l_{\pi q} \nabla^{\langle\mu} q^{\nu\rangle} \end{aligned}$$

- The equations determine the time evolution of Π, q^μ, and $\pi^{\mu\nu}$
- The Navier-Stokes theory appears if the relaxation times and length scales $\tau_i \to 0$, $l_i \to 0$ with ζ, η and κ_q fixed, (asymptotic solution)
- Later O_2 corrections ($\sim \nabla_\mu \alpha_i, \sim \nabla_\mu \beta_i, \ldots$, etc.) were added by Israel and Stewart (1977-1979), Hiscock and Lindblom (1983), Relativistic Extended Thermodynamics of Liu, Müller and Ruggeri (1983) etc.

We need kinetic theory to motivate and determine the above introduced phenomenological equations self-consistently!

The relativistic Boltzmann equation and conservation equations

In a dilute gas, the space-time evolution of the single-particle distribution function $f_{\mathbf{k}} = f(t, \mathbf{x}, k^0, \mathbf{k})$ due to particle motion and binary collisions is given by the

The relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = \frac{1}{2} \int dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} \left(f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'} \right)$$

$k^\mu = (k^0, \mathbf{k})$ is the four-momenta of particles with mass m and energy $k^0 = \sqrt{\mathbf{k}^2 + m^2}$. $\tilde{f}_{\mathbf{k}} = 1 - a f_{\mathbf{k}}$, with $a = 0$ for Boltzmann, $a = 1$ for Fermi, $a = -1$ for Bose statistics. The inv. phase-space element is, $dK = g d^3\mathbf{k} / \left[(2\pi)^3 k^0 \right]$. The transition rate W is invariant, to final state momenta $W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} = W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}'\mathbf{p}}$, and satisfies detailed balance, $W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} = W_{\mathbf{p}\mathbf{p}' \to \mathbf{k}\mathbf{k}'}$.

Conservation laws from the Boltzmann equation

$$\partial_\mu N^\mu \equiv \int dK\, k^\mu \partial_\mu f_{\mathbf{k}} = \int dK\, C[f_{\mathbf{k}}] = 0 \qquad \textbf{charge cons.}$$

$$\partial_\mu T^{\mu\nu} \equiv \int dK\, k^\nu k^\mu \partial_\mu f_{\mathbf{k}} = \int dK\, k^\nu C[f_{\mathbf{k}}] = 0 \qquad \textbf{energy-momentum cons.}$$

Conservation laws are obtained, 5-collisional invariants, but we still need the solution of the Boltzmann equation $f_{\mathbf{k}}$!

Local thermal equilibrium

In local thermal equilibrium $\partial S_0^\mu = 0$, hence $f_{\mathbf{k}} \to f_{0\mathbf{k}}$, where $f_{0\mathbf{k}} = f(\alpha_0, \beta_0, E_{\mathbf{k}})$

$$f_{0\mathbf{k}} \equiv \left[\exp\left(-\alpha_0 + \beta_0 E_{\mathbf{k}}\right) + a\right]^{-1}$$ **Jüttner distribution**

where $\alpha_0 = \mu_0/T_0$ is the chemical potential, $\beta_0 = 1/T_0$ is the inverse temperature, $E_{\mathbf{k}} = k^\mu u_\mu$, and $a = 0$ for Boltzmann, $a = 1$ for Fermi, $a = -1$ for Bose statistics.

Moments of the equilibrium distribution function

$$N_0^\mu(t, \mathbf{x}) \equiv \int dK\, k^\mu f_{0\mathbf{k}} = \langle k^\mu \rangle_0$$ **charge current**

$$T_0^{\mu\nu}(t, \mathbf{x}) \equiv \int dK\, k^\mu k^\nu f_{0\mathbf{k}} = \langle k^\mu k^\nu \rangle_0$$ **energy-momentum tensor**

$$S_0^\mu(t, \mathbf{x}) \equiv \int dK\, k^\mu f_{0\mathbf{k}} \left(\ln f_{0\mathbf{k}} - 1\right) = \langle k^\mu \Psi \rangle_0$$ **entropy current**

Conservation eqs. for and ideal gas

$$\partial_\mu N_0^\mu = 0, \qquad \partial_\mu T_0^{\mu\nu} = 0, \qquad \partial_\mu S_0^\mu \geq 0$$

$(a = 0)$ hence $p_0 = n_0 T_0$ ideal gas!

General kinetic decompositions and definitions

Decompose the momenta, $k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}$, where $E_{\mathbf{k}} = k^\mu u_\mu$ and $k^{\langle\mu\rangle} = \Delta^\mu_\nu k^\nu$

$$
\begin{aligned}
N^\mu &= \langle E_{\mathbf{k}} \rangle u^\mu + \langle k^{\langle\mu\rangle} \rangle \\
T^{\mu\nu} &= \langle E_{\mathbf{k}}^2 \rangle u^\mu u^\nu + \frac{1}{3}\Delta^{\mu\nu}\langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle + \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle u^\nu + \langle E_{\mathbf{k}} k^{\langle\nu\rangle} \rangle u^\mu + \langle k^{\langle\mu} k^{\nu\rangle} \rangle
\end{aligned}
$$

where $k^{\langle\mu} k^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta$ and $\langle k^{\mu_1} k^{\mu_2} \ldots k^{\mu_n} \rangle \Rightarrow \int dK\, k^{\mu_1} \ldots k^{\mu_n} f_{\mathbf{k}}$ for any $f_{\mathbf{k}}$

$$
\begin{aligned}
n &\equiv \langle E_{\mathbf{k}} \rangle \qquad e \equiv \langle E_{\mathbf{k}}^2 \rangle \qquad p \equiv -\frac{1}{3}\langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle \\
V^\mu &\equiv \langle k^{\langle\mu\rangle} \rangle \qquad W^\mu \equiv \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle \qquad \pi^{\mu\nu} \equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle
\end{aligned}
$$

In equilibrium $\langle k^{\mu_1} k^{\mu_2} \ldots k^{\mu_n} \rangle_0 \Rightarrow \int dK\, k^{\mu_1} \ldots k^{\mu_n} f_{0\mathbf{k}}$, hence

$$
\begin{aligned}
n_0 &\equiv \langle E_{\mathbf{k}} \rangle_0 \qquad e_0 \equiv \langle E_{\mathbf{k}}^2 \rangle_0 \qquad p_0 \equiv -\frac{1}{3}\langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle_0 \\
V^\mu &\equiv \langle k^{\langle\mu\rangle} \rangle_0 = 0 \qquad W^\mu \equiv \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle_0 = 0 \qquad \pi^{\mu\nu} \equiv \langle k^{\langle\mu} k^{\nu\rangle} \rangle_0 = 0
\end{aligned}
$$

Out of equilibrium

- For a system out of equilibrium

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}$$

- Let us define the following **irreducible moment**

$$\rho_{(r)}^{\mu_1 \cdots \mu_\ell} \equiv \left\langle (E_{\mathbf{k}})^r \, k^{\langle \mu_1} \ldots k^{\mu_\ell \rangle} \right\rangle_\delta$$

where $\langle \ldots \rangle_\delta \equiv \langle \ldots \rangle - \langle \ldots \rangle_0 = \int dK \ldots f_{\mathbf{k}} - \int dK \ldots f_{0\mathbf{k}} = \int dK \ldots \delta f_{\mathbf{k}}$

$$\begin{aligned} \delta n &\equiv \rho_{(1)} \qquad \delta e \equiv \rho_{(2)} \qquad \Pi \equiv -\frac{m^2}{3}\rho_{(0)} \\ V^\mu &\equiv \rho_{(0)}^\mu \qquad W^\mu \equiv \rho_{(1)}^\mu \qquad \pi^{\mu\nu} \equiv \rho_{(0)}^{\mu\nu} \end{aligned}$$

Matching: Non-equilibrium to equilibrium

$$\begin{aligned} \delta n \equiv \langle E_{\mathbf{k}} \rangle_\delta = 0 \quad &\Rightarrow \quad n = n_0 \\ \delta e \equiv \langle E_{\mathbf{k}}^2 \rangle_\delta = 0 \quad &\Rightarrow \quad e = e_0 \\ \Pi \equiv -\frac{1}{3}\langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle_\delta \quad &\Rightarrow \quad p(e,n) = p_0(e_0, n_0) + \Pi \end{aligned}$$

General equations of motion

Equations of motion for the irreducible moments

$$\dot{\rho}_{(r)}^{\langle\mu_1\cdots\mu_\ell\rangle} = \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} \frac{d}{d\tau} \int dK \, (E_{\mathbf{k}})^r \, k^{\langle\nu_1} \ldots k^{\nu_\ell\rangle} \delta f_{\mathbf{k}}$$

where $\dot{A} \equiv u^\mu \partial_\mu A \equiv dA/d\tau$ and $\dot{\rho}_{(r)}^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} \; \dot{\rho}_{(r)}^{\nu_1\cdots\nu_\ell}$

Now, using the Boltzmann equation $k^\mu \partial_\mu f_{\mathbf{k}} = C\,[f]$ in the following form

$$\delta \dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu \delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1} C\,[f]$$

where $\nabla_\mu = \Delta^\nu_\mu \partial_\nu$, we obtain the *exact* equations for the comoving derivatives of the irreducible moments, $\rho_{(r)}^{\mu_1\cdots\mu_l}$

General equations of motion

$$\dot{\rho}_{(r)} - C_{(r-1)} = -\beta_{\zeta}^{(r)}\theta + r\rho_{(r-1)}^{\mu}\dot{u}_{\mu} + \frac{G_{2r}}{D_{20}}W^{\mu}\dot{u}_{\mu} - \nabla_{\mu}\rho_{(r-1)}^{\mu} + \frac{G_{3r}}{D_{20}}\partial_{\mu}V^{\mu} - \frac{G_{2r}}{D_{20}}\partial_{\mu}W^{\mu}$$
$$+\frac{1}{3}\left[(r-1)\,m^{2}\rho_{(r-2)} - (r+2)\,\rho_{(r)} - 3\frac{G_{2r}}{D_{20}}\Pi\right]\theta + \left[(r-1)\,\rho_{(r-2)}^{\mu\nu} + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\right]\sigma_{\mu\nu}, \qquad (1)$$

$$\dot{\rho}_{(r)}^{\langle\mu\rangle} - C_{(r-1)}^{\langle\mu\rangle} = \beta_{\kappa}^{(r)}\nabla^{\mu}\alpha_{0} - \alpha_{h}^{(r)}\dot{W}^{\mu} + r\rho_{(r-1)}^{\mu\nu}\dot{u}_{\nu} + \frac{1}{3}\left[m^{2}r\rho_{(r-1)} - (r+3)\,\rho_{(r+1)} - 3\alpha_{h}^{(r)}\Pi\right]\dot{u}^{\mu}$$
$$-\frac{1}{3}\nabla^{\mu}\left[m^{2}\rho_{(r-1)} - \rho_{(r+1)}\right] + \alpha_{h}^{(r)}\nabla^{\mu}\Pi - \Delta_{\nu}^{\mu}\left(\nabla_{\lambda}\rho_{(r-1)}^{\nu\lambda} + \alpha_{h}^{(r)}\partial_{\lambda}\pi^{\nu\lambda}\right)$$
$$+\frac{1}{3}\left[(r-1)\,m^{2}\rho_{(r-2)}^{\mu} - (r+3)\,\rho_{(r)}^{\mu} - 4\alpha_{h}^{(r)}W^{\mu}\right]\theta + \left(\rho_{(r)\nu} + \alpha_{h}^{(r)}W_{\nu}\right)\omega^{\mu\nu}$$
$$+\frac{1}{5}\left[(2r-2)\,m^{2}\rho_{(r-2)}^{\nu} - (2r+3)\,\rho_{(r)}^{\nu} - 5\alpha_{h}^{(r)}W^{\nu}\right]\sigma_{\nu}^{\mu} + (r-1)\,\rho_{(r-2)}^{\mu\nu\lambda}\sigma_{\nu\lambda}, \qquad (2)$$

$$\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} - C_{(r-1)}^{\langle\mu\nu\rangle} = 2\beta_{\eta}^{(r)}\sigma^{\mu\nu} + r\rho_{(r-1)}^{\mu\nu\lambda}\dot{u}_{\lambda} + \frac{2}{5}\left[m^{2}r\rho_{(r-1)}^{\langle\mu} - (r+5)\,\rho_{(r+1)}^{\langle\mu}\right]\dot{u}^{\nu\rangle} - \frac{2}{5}\nabla^{\langle\mu}\left(m^{2}\rho_{(r-1)}^{\nu\rangle} - \rho_{(r+1)}^{\nu\rangle}\right)$$
$$-\Delta_{\alpha\beta}^{\mu\nu}\nabla_{\lambda}\rho_{(r-1)}^{\alpha\beta\lambda} + \frac{1}{3}\left[(r-1)\,m^{2}\rho_{(r-2)}^{\mu\nu} - (r+4)\,\rho_{(r)}^{\mu\nu}\right]\theta + 2\rho_{(r)\lambda}^{\langle\mu}\omega^{\nu\rangle\lambda}$$
$$+\frac{2}{15}\left[(r-1)\,m^{4}\rho_{(r-2)} - (2r+3)\,m^{2}\rho_{(r)} + (r+4)\rho_{(r+2)}\right]\sigma^{\mu\nu}$$
$$+\frac{2}{7}\left[(2r-2)\,m^{2}\rho_{(r-2)}^{\lambda\langle\mu} - (2r+5)\,\rho_{(r)}^{\lambda\langle\mu}\right]\sigma_{\lambda}^{\nu\rangle} + (r-1)\rho_{(r-2)}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa}, \qquad (3)$$

Generalized collision terms

$$C_{(r)}^{\langle\mu_{1}\cdots\mu_{\ell}\rangle} = \int dK\,(E_{\mathbf{k}})^{r}\,k^{\langle\mu_{1}}\ldots k^{\mu_{\ell}\rangle}\,C[f].$$

Collision integral in a Boltzmann gas

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

where $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}} \ll 1$ close to equilibrium

Collision integral for Boltzmann particles

$$C_{(r)}^{\mu_1 \cdots \mu_\ell} = \frac{1}{2} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \to \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \left(E_{\mathbf{k}}\right)^r k^{\mu_1} \ldots k^{\mu_\ell}$$
$$\times \left[\left(\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'}\right) + \left(\phi_{\mathbf{p}}\phi_{\mathbf{p}'} - \phi_{\mathbf{k}}\phi_{\mathbf{k}'}\right)\right]$$

Similar to

Entropy

$$S^\mu = S_0^\mu - \left\langle k^\mu \ln\left(f_{0\mathbf{k}}\right)\right\rangle_\delta - \frac{1}{2}\left\langle k^\mu \phi_{\mathbf{k}}^2 \right\rangle_0$$

$$S^\mu = s_0 u^\mu - \alpha_0 V^\mu + \beta_0 W^\mu$$
$$- \frac{1}{2}\beta_0 \left(\beta_0^\Pi \Pi^2 - \beta_1^V V^\alpha V_\alpha - \beta_1^W W^\alpha W_\alpha - \beta_1^{VW} V^\alpha W_\alpha + \pi^{\alpha\beta}\pi_{\alpha\beta}\right) u^\mu$$
$$- \beta_0 \left(\alpha_0^V V^\mu + \alpha_0^W W^\mu\right) \Pi - \beta_0 \left(\alpha_1^V V_\alpha + \alpha_1^W W_\alpha\right) \pi^{\alpha\mu}$$

The relaxation equations with a linearised collision integral

The original Israel-Stewart derivation neglected several terms; corrections added later.
More terms from the non-linearized collision integral!

$$\tau_\Pi \dot{\Pi} + \Pi = \Pi_{NS} + \tau_{\Pi q}\, q\cdot\dot{u} - \ell_{\Pi q}\,\partial\cdot q - \zeta\,\hat{\delta}_0\,\Pi\,\theta + \lambda_{\Pi q}\, q\cdot\nabla\alpha + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu}$$

$$\tau_q\,\Delta^{\mu\nu}\dot{q}_\nu + q^\mu = q^\mu_{NS} - \tau_{q\Pi}\,\Pi\,\dot{u}^\mu - \tau_{q\pi}\,\pi^{\mu\nu}\,\dot{u}_\nu + \ell_{q\Pi}\,\nabla^\mu\Pi - \ell_{q\pi}\,\Delta^{\mu\nu}\,\partial^\lambda\pi_{\nu\lambda} + \tau_q\,\omega^{\mu\nu}\,q_\nu - \frac{\kappa}{\beta}\,\hat{\delta}_1\,q^\mu\,\theta - \lambda_{qq}\,\sigma^{\mu\nu}\,q_\nu + \lambda_{q\Pi}\,\Pi\,\nabla^\mu\alpha + \lambda_{q\pi}\,\pi^{\mu\nu}\,\nabla_\nu\alpha$$

$$\tau_\pi\,\dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} = \pi^{\mu\nu}_{NS} + 2\,\tau_{\pi q}\,q^{<\mu}\dot{u}^{\nu>} + 2\,\ell_{\pi q}\,\nabla^{<\mu}q^{\nu>} + 2\,\tau_\pi\,\pi_\lambda{}^{<\mu}\omega^{\nu>\lambda} - 2\,\eta\,\hat{\delta}_2\,\pi^{\mu\nu}\,\theta - 2\,\tau_\pi\,\pi_\lambda{}^{<\mu}\sigma^{\nu>\lambda} - 2\,\lambda_{\pi q}\,q^{<\mu}\nabla^{\nu>}\alpha + 2\,\lambda_{\pi\Pi}\,\Pi\,\sigma^{\mu\nu}$$

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The complete 2^{nd} order relaxation equations

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} + \mathcal{A}^{(r)}_{\Pi\Pi}\Pi^2 + \mathcal{A}^{(r)}_{VV} V^\mu V_\mu + \mathcal{A}^{(r)}_{VW} V^\mu W_\mu + \mathcal{A}^{(r)}_{WW} W^\mu W_\mu + \mathcal{A}^{(r)}_{\pi\pi} \pi^{\mu\nu}\pi_{\mu\nu}$$

$$= -\frac{\zeta^{(r)}}{\tau^{(r)}_\Pi}\theta + \tau^{(r)}_{\Pi W} W_\mu \dot{u}^\mu - l^{(r)}_{\Pi W}\partial_\mu W^\mu + \lambda^{(r)}_{\Pi W} W^\mu \nabla_\mu \alpha_0$$

$$- \delta^{(r)}_\Pi \Pi\theta + \tau^{(r)}_{\Pi V} V_\mu \dot{u}^\mu - l^{(r)}_{\Pi V}\partial_\mu V^\mu + \lambda^{(r)}_{\Pi V} V^\mu \nabla_\mu \alpha_0 + \lambda^{(r)}_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} + \mathcal{R}$$

$$\dot{V}^\mu + \psi^{(r)}_W \dot{W}^\mu + \frac{V^\mu}{\tau^{(r)}_V} + \psi^{(r)}_W \frac{W^\mu}{\tau^{(r)}_W} + \mathcal{B}^{(r)}_{V\Pi}\Pi V^\mu + \psi^{(r)}_W \mathcal{B}^{(r)}_{W\Pi}\Pi W^\mu + 2\mathcal{B}^{(r)}_{V\pi}\Delta^{\mu(\alpha} V^{\beta)}\pi_{\alpha\beta} + 2\psi^{(r)}_W \mathcal{B}^{(r)}_{W\pi}\Delta^{\mu(\alpha} W^{\beta)}\pi_{\alpha\beta}$$

$$= \frac{\kappa^{(r)}_q}{\tau^{(r)}_V \beta_0^2 h_0^2}\nabla^\mu \alpha_0 + \psi^{(r)}_W W_\nu \omega^{\mu\nu} + V_\nu \omega^{\mu\nu} - \psi^{(r)}_W \delta^{(r)}_W W^\mu \theta + \delta^{(r)}_V V^\mu \theta$$

$$- \tau^{(r)}_{q\Pi}\Pi \dot{u}^\mu - \tau^{(r)}_{q\pi}\pi^{\mu\nu}\dot{u}_\nu + l^{(r)}_{q\Pi}\nabla^\mu \Pi - l^{(r)}_{q\pi}\Delta^\mu_\alpha \partial_\nu \pi^{\alpha\nu}$$

$$+ \lambda^{(r)}_{q\Pi}\Pi\nabla^\mu \alpha_0 + \lambda^{(r)}_{q\pi}\pi^{\mu\nu}\nabla_\nu \alpha_0 - \psi^{(r)}_W \lambda^{(r)}_{WW} W_\nu \sigma^{\mu\nu} + \lambda^{(r)}_{VV} V_\nu \sigma^{\mu\nu} + \mathcal{R}^\mu$$

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau^{(r)}_\pi} + \mathcal{C}^{(r)}_{VV} V^{\langle\mu} V^{\nu\rangle} + \mathcal{C}^{(r)}_{VW} V^{\langle\mu} W^{\nu\rangle} + \mathcal{C}^{(r)}_{WW} W^{\langle\mu} W^{\nu\rangle} + \mathcal{C}^{(r)}_{\pi\Pi}\Pi\pi^{\mu\nu} + \mathcal{C}^{(r)}_{\pi\pi}\pi^{\langle\mu}_\alpha \pi^{\nu\rangle\alpha}$$

$$= \frac{2\eta^{(r)}}{\tau^{(r)}_\pi}\sigma^{\mu\nu} + 2\lambda^{(r)}_{\pi\Pi}\Pi\sigma^{\mu\nu} + 2\tau^{(r)}_{\pi W} W^{\langle\mu}\dot{u}^{\nu\rangle} + 2\tau^{(r)}_{\pi V} V^{\langle\mu}\dot{u}^{\nu\rangle} + 2l^{(r)}_{\pi W}\nabla^{\langle\mu} W^{\nu\rangle} + 2l^{(r)}_{\pi V}\nabla^{\langle\mu} V^{\nu\rangle}$$

$$- 2\lambda^{(r)}_{\pi W} W^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 - 2\lambda^{(r)}_{\pi V} V^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 - 2\delta^{(r)}_\pi \pi^{\mu\nu}\theta - 2\lambda^{(r)}_{\pi\pi}\pi^{\langle\mu}_\alpha \sigma^{\nu\rangle\alpha} + 2\pi^{\langle\mu}_\alpha \omega^{\nu\rangle\alpha} + \mathcal{R}^{\mu\nu},$$

So Many Jokes, So Little Time ...

Seriously...

- The equations and coefficients are in better agreement with the non-equilibrium solution of the Boltzmann equation than the others out there!
- Hard to see the fine details, though !
- Different choices (of methods, moments, coefficients etc.) lead to the same asymptotic (Navier-Stokes) solutions, but not the other way around!

Backup slides

Backup

Power Counting

We power count with

- Knudsen number

$$\mathrm{Kn} = \frac{\ell_{\mathrm{micr}}}{L_{\mathrm{macr}}}$$

- e.g. $\frac{2\eta}{P_0}\sigma^{\mu\nu} = O(\mathrm{Kn})$
- inverse Reynolds number

$$\mathrm{R}_{\Pi}^{-1} \sim \frac{|\Pi|}{P_0}, \ \mathrm{R}_{n}^{-1} \sim \frac{|n^{\mu}|}{n_0}, \ \mathrm{R}_{\pi}^{-1} \sim \frac{|\pi^{\mu\nu}|}{P_0}$$

- Note that in general Knudsen number and inverse Reynolds number are not equivalent (only in the NS/Burnett-limit)
- Aim is to derive fluid dynamics with some definite order in Kn and R^{-1}

Expansion of $f_{\mathbf{k}}$

The previous definitions from kinetic theory are only useful if we can specify $f_{\mathbf{k}}$ or $\delta f_{\mathbf{k}}$

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

where $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}} \ll 1$ close to equilibrium

- Expand $\phi_{\mathbf{k}}$ using an orthogonal basis i.e., $1, k_{\langle\mu_1\rangle}, k_{\langle\mu_1} k_{\mu_2\rangle}, \ldots$

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \qquad \textbf{deviation from equilibrium}$$

$$\lambda_{\mathbf{k}}^{\langle\mu_1\cdots\mu_\ell\rangle} = \sum_{n=0}^{N_\ell} c_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} P_{\mathbf{k}}^{(n\ell)} \equiv \sum_{n=0}^{\infty} \chi_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} \left(E_{\mathbf{k}}\right)^n \qquad \textbf{coefficients}$$

where

$$P_{\mathbf{k}}^{(n\ell)} = \sum_{r=0}^{n} a_{nr}^{(\ell)} \left(E_{\mathbf{k}}\right)^r \qquad \chi_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \sum_{m=n}^{\infty} c_{(m)}^{\langle\mu_1\cdots\mu_\ell\rangle} a_{mn}^{(\ell)}$$

$$\int dK \omega_\ell P_{\mathbf{k}}^{(n\ell)} P_{\mathbf{k}}^{(m\ell)} = \delta^{mn}, \qquad \omega_\ell \equiv \frac{\mathcal{N}_\ell}{(2\ell+1)!!} \left(\Delta^{\alpha\beta} k_\alpha k_\beta\right)^\ell f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}$$

Expansion of $f_{\mathbf{k}}$ II

Orthogonality conditions imply that

$$c_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{\mathcal{N}_\ell}{\ell!}\left\langle P_{\mathbf{k}}^{(n\ell)} k^{\langle\mu_1} \ldots k^{\mu_\ell\rangle}\right\rangle_\delta \equiv \frac{\mathcal{N}_\ell}{\ell!}\sum_{m=0}^{n} \rho_{(m)}^{\mu_1\cdots\mu_\ell} a_{nm}^{(\ell)}$$

Full expansion of $f_{\mathbf{k}}$ reads:

$$f_{\mathbf{k}} = f_{0\mathbf{k}}\left[1 + \sum_{\ell,n=0}^{\infty} \rho_{(n)}^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1}\ldots k_{\mu_\ell\rangle}\left(\frac{\mathcal{N}_\ell}{\ell!}\sum_{m=n}^{\infty} a_{mn}^{(\ell)} P_{\mathbf{k}}^{(m\ell)}\right)\right]$$

$$\chi_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} = \frac{\mathcal{N}_\ell}{\ell!}\sum_{m=n}^{\infty}\sum_{k=0}^{m} \rho_{(n)}^{\mu_1\cdots\mu_\ell} a_{mn}^{(\ell)} a_{mk}^{(\ell)}$$

$$\rho_{(r)}^{\mu_1\cdots\mu_\ell} = \ell!\sum_{n=0}^{\infty} \chi_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} I_{r+n+2\ell,\ell}$$

where the thermodynamical integrals are

$$I_{n+r,q} = \frac{1}{(2q+1)!!}\left\langle (E_{\mathbf{k}})^{n+r-2q}\left(\Delta^{\alpha\beta}k_\alpha k_\beta\right)^q\right\rangle_0$$

Approximate solutions: Grad's method of moments

The previous definitions from kinetic theory are only useful if we can specify $f_{\mathbf{k}}$ or $\delta f_{\mathbf{k}}$

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} \simeq f_{0\mathbf{k}} \left(1 + \phi_{\mathbf{k}}\right)$$

where $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}} \ll 1$ close to equilibrium, $f_{0\mathbf{k}} \equiv \left[\exp\left(-\alpha_0 + \beta_0 E_{\mathbf{k}}\right) + a\right]^{-1}$, where $\alpha_0 = \mu_0 / T_0$ is the chemical potential, $\beta_0 = 1/T_0$ is the inverse temperature, $E_{\mathbf{k}} = k^{\mu} u_{\mu}$, and $a = 0$ for Boltzmann, $a = 1$ for Fermi, $a = -1$ for Bose statistics.

Relativistic Grad's approach: Stewart (1972), Israel and Stewart (1977-1979)

$$\phi_{\mathbf{k}} \equiv \sum_{l=0}^{\infty} \epsilon_{(l)}^{\mu_1 \dots \mu_l} k_{\mu_1} \dots k_{\mu_l} = \epsilon_{(0)} + \epsilon_{(1)}^{\mu} k_{\mu} + \epsilon_{(2)}^{\mu\nu} k_{\mu} k_{\nu} + \dots$$

Assuming that in $\phi_{\mathbf{k}}$ only $l = 0, 1, 2$-rank tensors appear $\lambda(1)$, $\lambda^{\mu}(4)$ and $\lambda^{\mu\nu}(9)$ where $\lambda^{\mu}_{\mu} = 0$, we provide a closure for the hierarchy (truncation) and be able to determine the 14 moments of dissipative fluid dynamics!

Applying the 14-method

Non-equilibrium 14-moment ansatz

$$f_{\mathbf{k}} \equiv f_{0\mathbf{k}}\Big[1+\epsilon_{(0)}+E_{\mathbf{k}}\epsilon^{\mu}_{(1)}u_{\mu}+\epsilon^{\langle\mu\rangle}_{(1)}k_{\langle\mu\rangle}+E^2_{\mathbf{k}}\epsilon^{\mu\nu}_{(2)}u_{\mu}u_{\nu}+\frac{1}{3}\left(\Delta^{\alpha\beta}k_{\alpha}k_{\beta}\right)\Delta_{\mu\nu}\epsilon^{\mu\nu}_{(2)}$$
$$+\ E_{\mathbf{k}}\epsilon^{\langle\mu\rangle\nu}_{(2)}k_{\langle\mu\rangle}u_{\nu}+E_{\mathbf{k}}\epsilon^{\langle\mu\rangle\nu}_{(2)}k_{\langle\nu\rangle}u_{\mu}+\epsilon^{\langle\mu\nu\rangle}_{(2)}k_{\langle\mu}k_{\nu\rangle}\Big]$$

which is basically an expansion in $E^n_{\mathbf{k}}$ and $k_{\langle\mu_1}\dots k_{\mu_n\rangle}$

Decomposition and matching

$$\epsilon_{(0)}-\alpha_0 = A_0\Pi$$
$$\epsilon^{\mu}_{(1)}u_{\mu}-\beta_0 = A_1\Pi$$
$$\epsilon^{\mu\nu}_{(2)}u_{\mu}u_{\nu} = A_2\Pi$$
$$\epsilon^{\langle\mu\rangle}_{(1)} = B_0q^{\mu}+B_1V^{\mu}$$
$$\epsilon^{\langle\mu\rangle\beta}_{(2)}u_{\beta} = C_0q^{\mu}+C_1V^{\mu}$$
$$\epsilon^{\mu\nu}_{(2)} = A_3(3u^{\mu}u^{\nu}-\Delta^{\mu\nu})\Pi+2C_0u^{(\mu}q^{\nu)}+2C_1u^{(\mu}V^{\nu)}+D_0\pi^{\mu\nu}$$
$$\epsilon^{\langle\mu\nu\rangle}_{(2)} = D_0\pi^{\mu\nu}$$

The coefficients $\epsilon_{(0)}$, $\epsilon^{\mu}_{(1)}$, $\epsilon^{\mu\nu}_{(2)}$ expressed in terms of dissipative fields, Π, $q^{\mu}(V^{\mu})$, $\pi^{\mu\nu}$

The relaxation equations

Using the 14-moment ansatz, the 3rd moment, $\langle k^\mu k^\nu k^\lambda \rangle = F(e, n, p, u^\mu, \Pi, q^\mu, \pi^{\mu\nu})$, therefore, using the Boltzmann transport eq., $\partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle = \langle C^{\mu\nu} \rangle$, we get:

Israel-Stewart (1979)

$$
\begin{aligned}
u_\mu u_\nu \partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle &= u_\mu u_\nu \langle C^{\mu\nu} \rangle && \text{Bulk eq.} \\
\Delta^\alpha_\mu u_\nu \partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle &= \Delta^\alpha_\mu u_\nu \langle C^{\mu\nu} \rangle && \text{Heat-flow eq.} \\
\Delta^{\alpha\beta}_{\mu\nu} \partial_\lambda \langle k^\mu k^\nu k^\lambda \rangle &= \Delta^{\alpha\beta}_{\mu\nu} \langle C^{\mu\nu} \rangle && \text{Shear viscosity eq.}
\end{aligned}
$$

The linearised collision integral

$$
\begin{aligned}
u_\mu u_\nu \langle C^{\mu\nu} \rangle &= C_\Pi \Pi \\
\Delta^\alpha_\mu u_\nu \langle C^{\mu\nu} \rangle &= C_q q^\alpha \\
\Delta^{\alpha\beta}_{\mu\nu} \langle C^{\mu\nu} \rangle &= C_\pi \pi^{\alpha\beta}
\end{aligned}
$$

where $C_\Pi, C_q, C_\pi \sim \langle \sigma \rangle$