Schwinger's non-commutative coordinates for photons and gravitons

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Publications

1. *Schwinger's non commutative coordinates*
   J.Math.Phys. 66 (2025) 072303 G.S.

2. *Extension of the Poincaré Group and Non-Abelian Tensor Gauge Fields*
   Int.J.Mod.Phys.A 25 (2010) 5765-5785 G.S

3. *Extensions of the Poincare group.*
   J.Math.Phys. 52 (2011) 072303 I.Antoniadis, L.Brink, G.S.

The Poincaré Group

Extension of the Poincaré Group

Schwinger non commutative coordinates

Heisenberg - Schwinger uncertainty relation

Extension of the Poincaré Algebra

$$[P^\mu,\ P^\nu] = 0,$$
$$[M^{\mu\nu},\ P^\lambda] = i(\eta^{\lambda\nu}\ P^\mu - \eta^{\lambda\mu}\ P^\nu),$$
$$[M^{\mu\nu},\ M^{\lambda\rho}] = i(\eta^{\mu\rho}\ M^{\nu\lambda} - \eta^{\mu\lambda}\ M^{\nu\rho} + \eta^{\nu\lambda}\ M^{\mu\rho} - \eta^{\nu\rho}\ M^{\mu\lambda}),$$

$$[P^\mu,\ L_a^{\lambda_1\ldots\lambda_s}] = 0,$$
$$[M^{\mu\nu},\ L_a^{\lambda_1\ldots\lambda_s}] = i(\eta^{\lambda_1\nu} L_a^{\mu\lambda_2\ldots\lambda_s} - \eta^{\lambda_1\mu} L_a^{\nu\lambda_2\ldots\lambda_s} + \ldots + \eta^{\lambda_s\nu} L_a^{\lambda_1\ldots\lambda_{s-1}\mu} - \eta^{\lambda_s\mu} L_a^{\lambda_1\ldots\lambda_{s-1}\nu}),$$

$$[L_a^{\lambda_1\ldots\lambda_i}, L_b^{\lambda_{i+1}\ldots\lambda_s}] = i f_{abc} L_c^{\lambda_1\ldots\lambda_s} \quad (\mu,\nu,\rho,\lambda = 0,1,2,3; \qquad s = 0,1,2,\ldots),$$

Three dimensional operators

$$cP^0 = H + Mc^2, \qquad P_i, \qquad \frac{1}{c}M^{0i} = N_i, \qquad M_{ij} = \varepsilon_{ijk}J_k.$$

The generators P_i, and J_k, are the linear and angular momentum operators, while H is the energy, or the Hamiltonian operator, and N_i is the boost operator

$$[P_n,\ P_m] = 0, \qquad [J_n, J_m] = i\hbar\varepsilon_{nmk}J_k,$$

$$[P_n,\ J_m] = i\hbar\varepsilon_{nmk}P_k, \qquad [N_n,\ N_m] = -i\hbar\frac{1}{c^2}J_{nm},$$

$$[N_n,\ J_m] = i\hbar\varepsilon_{nmk}N_k, \qquad [P_n,\ N_m] = i\hbar\delta_{nm}\frac{P_0}{c},$$

$$[H,\ P_n] = 0,$$
$$[H,\ J_n] = 0,$$
$$[H,\ N_n] = i\hbar P_n.$$

Casimir operators of Poincare algebra

The fully invariant vacuum state $|0\rangle$ is defined as

$$P^{\mu}|0\rangle = 0, \qquad M^{\mu\nu}|0\rangle = 0.$$

$$-P^{\mu}P_{\mu} = (P^{0})^{2} - \vec{P}^{2} = M^{2}c^{2},$$

$$M^{2} > 0 \qquad\qquad P^{0} = +(\vec{P}^{2} + M^{2}c^{2})^{1/2} > 0$$

$$M^{2} = 0 \text{ and then } P^{0} = +|\vec{P}| > 0.$$

Casimir operators of Poincare algebra

Pauli-Lubanski pseudo-vector

$$W^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho} M_{\nu\lambda} P_{\rho}.$$

$$\left[P^{\mu},\ W^{\nu}\right] = 0, \quad \left[W^{\mu},\ W^{\nu}\right] = i\hbar\varepsilon^{\mu\nu\lambda\rho}\ W_{\lambda}P_{\rho}, \qquad P_{\mu}W^{\mu} = 0.$$

it is a space-like pseudo-vector:

$$W^{\mu}W_{\mu} = \rho^{2} \geq 0.$$

Massive Representation with Spin

$$[S_i,\ S_l] = i\hbar\varepsilon_{ilk}S_k$$ Particle angular momentum

$$\vec{J} = \vec{R} \times \vec{P} + \vec{S}.$$

$$[R_i, R_j] = 0, \quad [S_n, R_m] = [S_n, P_m] = 0, \quad [R_i, P_k] = i\hbar\delta_{ik}.$$

One should find the boost operator

$$\vec{N} = \vec{P}x^0 - \frac{1}{c}\{P^0\vec{R}\} + \frac{1}{c^2}\frac{1}{P^0 + Mc}\vec{S} \times \vec{P}.$$

Poincaré algebra for particles with spin in terms of the coordinate, momentum and spin operators.

Massive Representation with Spin

It is possible to invert these formulas and express the coordinate and spin operators in terms of the momentum, angular momentum, and boost operators

$$M\vec{R} = -\vec{N} + \frac{1}{P^0(P^0 + Mc)}\vec{P}(\vec{P}\cdot\vec{N}) + \frac{1}{P^0 + Mc}\vec{J}\times\vec{P},$$

$$M\vec{S} = \frac{1}{c}P^0\vec{J} - \frac{1}{c}\frac{1}{P^0 + Mc}\vec{P}(\vec{P}\cdot\vec{J}) + \vec{N}\times\vec{P}.$$

The components of the Pauli-Lubanski vector

$$W^0 = \vec{P}\cdot\vec{J} = \vec{P}\cdot\vec{S},$$

$$\vec{W} = P^0\vec{J} - c\vec{P}\times\vec{N} = cM\vec{S} + \frac{(\vec{P}\cdot\vec{S})}{P^0 + Mc}\vec{P}.$$

Casimir operators of Poincare algebra

$$W^2 = c^2 M^2 \vec{S}^2,$$

$$-P^2 = M^2 c^2$$

the representation is characterized by two invariants, the mass of the particles and their spin

Massless Representation and Non-commutative Coordinates

consider the limit $M^2 \to 0$ when $\vec{S}^2$ is kept fixed. $M^2 = -P^\mu P_\mu = 0$,

$$W^0 = \vec{P} \cdot \vec{S}, \quad \vec{W} = \frac{(\vec{P} \cdot \vec{S})}{P} \vec{P}, \quad W^2 = 0.$$

can be written in the following form:

$$W^\mu = \lambda P^\mu,$$

$$\lambda = \frac{\vec{P} \cdot \vec{S}}{P}.$$

as far as it is Lorentz invariant, the system exhibits only two values of helicity $\pm s$.

not all of the $2s + 1$ spin magnetic quantum number states exist in the massless limit.

Massless Representation and Non-commutative Coordinates

In order to eliminate the operator $\vec{S}$ from a massless representation Schwinger suggested that the new coordinates can be defined

$$\hat{\vec{R}} = \vec{R} - \frac{\vec{S} \times \vec{P}}{P^2}.$$

commutation relation of the coordinate $\hat{\vec{R}}$ and the momentum operator $\vec{P}$ remains intact:

$$[P_n, P_m] = 0, \qquad [\hat{R}_n, P_m] = i\hbar\delta_{nm},$$

but

$$[\hat{R}_n, \hat{R}_m] = -i\hbar\lambda\varepsilon_{nmk}\frac{P_k}{P^3},$$

$$[[\hat{R}_1, \hat{R}_2], \hat{R}_3] + [[\hat{R}_3, \hat{R}_1], \hat{R}_2] + [[\hat{R}_2, \hat{R}_3], \hat{R}_1] = \lambda\hbar^2 \triangle_P\left(\frac{1}{P}\right) = -4\pi\lambda\hbar^2\delta^{(3)}(\vec{P}).$$

Massless Representation and Non-commutative Coordinates

the angular momentum operator in terms of new coordinates and helicity operators:

$$\vec{J} = \hat{\vec{R}} \times \vec{P} + \lambda \frac{\vec{P}}{P}.$$

the boost operator N will take the following form:

$$\vec{N} = \vec{P} x^0 - \frac{1}{c} P^0 \hat{\vec{R}}.$$

without any reference to spin operators. It is also true that $[\lambda, \hat{R}] = 0$.

completely define the massless representation of the Poincaré algebra in terms of new non-commuting

coordinates $\hat{\vec{R}}$, momentum $\vec{P}$, and helicity operator λ.

Heisenberg-Schwinger Uncertainty Relation

$$[\hat{R}_n, \hat{R}_m] = -i\hbar\lambda\varepsilon_{nmk}\frac{P_k}{P^3},$$

$$\triangle \hat{R}_n \;\; \triangle \hat{R}_m \geq \hbar\frac{|\lambda|}{2}\left|\left\langle \frac{P_k}{P^3} \right\rangle\right|, \qquad n \neq m \neq k,$$

where $\triangle \hat{R}_n{}^2 = \langle (\hat{R}_n - \langle \hat{R}_n \rangle)^2 \rangle$ is the mean square of the deviation of $\hat{R}_n$ from its mean value $\langle \hat{R}_n \rangle$

For a momentum state with some degree of directionality along a given axis

$$(\triangle \hat{\mathbf{R}})^2 \geq \hbar|\lambda|\left\langle \frac{1}{P^2} \right\rangle,$$

where $\frac{2\pi\hbar}{P} = \Lambda$ is the wavelength of the massless particle indicating that the wavelength sets the scale of coordinate uncertainty.

Violation of Associativity Relation by 3-Cocycle

$$[P_n, P_m] = 0, \qquad [\hat{R}_n, P_m] = i\hbar\delta_{nm},$$

$$[\hat{R}_n, \hat{R}_m] = -i\hbar\lambda\varepsilon_{nmk}\frac{P_k}{P^3},$$

$$[[\hat{R}_1, \hat{R}_2], \hat{R}_3] + [[\hat{R}_3, \hat{R}_1], \hat{R}_2] + [[\hat{R}_2, \hat{R}_3], \hat{R}_1] = -4\pi\lambda\hbar^2\delta^{(3)}(\vec{P}).$$

We will consider now the *momentum translation operator*

$$U(\vec{b}) = e^{-\frac{i}{\hbar}\vec{b}\,\hat{\vec{R}}}, \qquad U(\vec{b})\Psi(\vec{P}) = \Psi(\vec{P} + \vec{b})$$

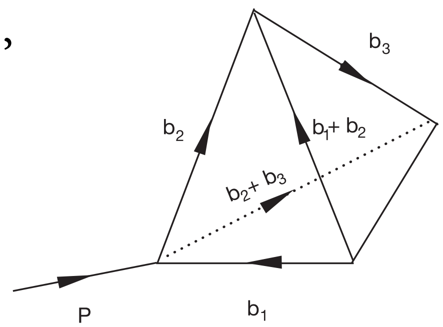
Violation of Associativity Relation by 3-Cocycle

would like to analyze the associativity relation

$$U(\vec{b}_1)\ (U(\vec{b}_2)U(\vec{b}_3)) = e^{i\Phi/\hbar}\ (U(\vec{b}_1)U(\vec{b}_2))\ U(\vec{b}_3).$$

The 3-cocycle ω_3 appears when the associativity is violated by a phase factor

$$U(g_1)\ (U(g_2)U(g_3)) = e^{2\pi i\ \omega_3(x;g_1,g_2,g_3)}\ (U(g_1)U(g_2))\ U(g_3),$$

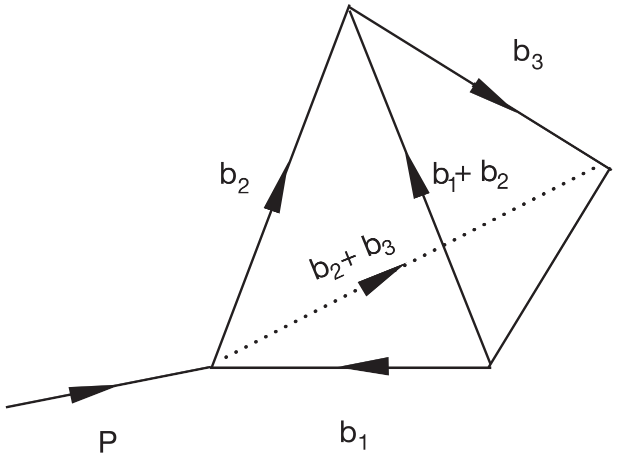


calculate the phase in the associativity relation

$$e^{\frac{i}{\hbar}\Phi} = U(\vec{b}_1)\ (U(\vec{b}_2)\ U(\vec{b}_3))\ ((U(\vec{b}_1)\ U(\vec{b}_2))\ U(\vec{a}_3))^{-1}$$

$$= \exp\left[\frac{i\lambda}{2\hbar}\{(\vec{b}_2 \times \vec{b}_3) + (\vec{b}_1 \times (\vec{b}_2 + \vec{b}_3)) - (\vec{b}_1 \times \vec{b}_2) - ((\vec{b}_1 + \vec{b}_2) \times \vec{b}_3)\} \cdot \vec{P}/P^3\right]$$

Violation of Associativity Relation by 3-Cocycle



the total flux of the momentum $\vec{P}/P^3$ through the surface

$$\Phi = \lambda \oint \frac{\vec{P}}{P^3} d\vec{S} = \lambda \int \left(\vec{\nabla} \cdot \frac{\vec{P}}{P^3} \right) d^3\vec{P} = \lambda \int 4\pi \delta^{(3)}(\vec{P})\ d^3\vec{P} = 4\pi\lambda.$$

$$\frac{\Phi}{\hbar} = \frac{4\pi\lambda}{\hbar} = 2\pi n,\ \lambda = \frac{\hbar}{2} n, \quad n = \pm 1, \pm 2, \ldots.$$

The quantum mechanical consistency requires the quantization of the particles helicity.

Correspondence with Dirac Quantisation Condition

Consider a particle of mass m and charge e moving in a magnetic field of a monopole of charge g_m.

$$H = \frac{1}{2m}(\vec{p} - e\vec{A})^2,$$

the Heisenberg commutation relations are

$$[x_n, x_m] = 0, \quad [x_n, p_m] = i\hbar\delta_{nm}, \quad [p_n, p_m] = i\hbar\frac{e}{c}\varepsilon_{nmk}H_k,$$

Jacobi identity of triple momentum commutator has the following form:

$$[[p_1, p_2], p_3] + [[p_2, p_3], p_1] + [[p_3, p_1], p_2] = \hbar^2 \frac{eg_m}{c} \triangle_r\left(\frac{1}{r}\right) = -4\pi\hbar^2 \frac{eg_m}{c} \delta^3(\vec{r}).$$

coordinate translation operator $\quad U(\vec{a}) = e^{\frac{i}{\hbar}\vec{a}\vec{p}}, \quad U(\vec{a})\psi(\vec{r}) = \psi(\vec{r} + \vec{a})$

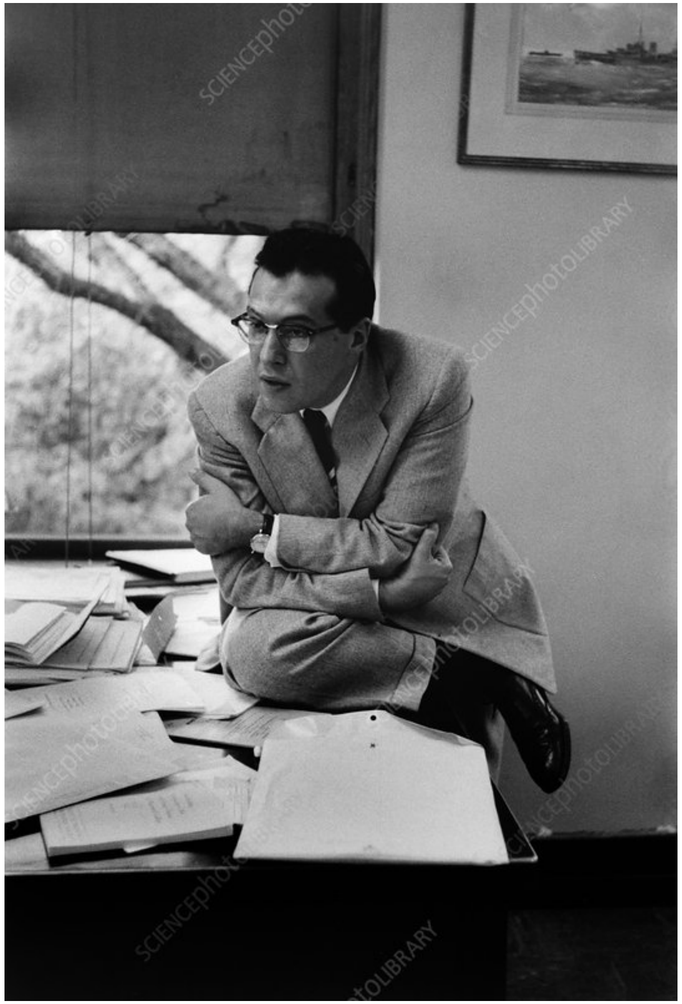
$$\Phi_m = \frac{eg_m}{c} \oint \frac{\vec{r}}{r^3} d\vec{S} = \frac{eg_m}{c} \int \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^3}\right) d^3\vec{r} = \frac{eg_m}{c} \int 4\pi\delta^{(3)}(\vec{r})\, d^3\vec{r} = 4\pi\frac{eg_m}{c}.$$

$$\frac{\Phi_m}{\hbar} = 4\pi\frac{eg_m}{\hbar c} = 2\pi n, \qquad \frac{eg_m}{c} = \frac{\hbar}{2}n, \quad n = \pm 1, \pm 2, \ldots,$$

Correspondence with Dirac Quantisation Condition

$$\hat{\vec{R}} \longleftrightarrow \vec{p}, \qquad \vec{P} \longleftrightarrow -\vec{r}, \qquad\qquad \lambda \longleftrightarrow \frac{eg_m}{c}.$$

In summary, we have demonstrated that there is a well defined correspondence (duality) (75) between helicity quantization condition of massless relativistic particles (65) and the Dirac quantization condition of the magnetic charge(74), which is based on the analyses of 3-cocycles in the corresponding associativity relations (60) and (71).



$$\triangle \hat{R}_n \, \triangle \hat{R}_m \geq \hbar \frac{|\lambda|}{2} \left| \langle \frac{P_k}{P^3} \rangle \right|, \qquad n \neq m \neq k$$

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the product $P^0\vec{R}$ is symmetrized because these operators are not commuting:

$$\frac{1}{i\hbar}\left[\vec{R}, P^0\right] = \frac{\partial P^0}{\partial \vec{P}} = \frac{\vec{P}}{P^0}.$$