THE QCD PHASE DIAGRAM FROM THE LATTICE

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OUTLINE OF THE TALK

1. Introduction: Grand Canonical QCD on the lattice
2. The crossover temperature $T_c(T, \mu_B)$
3. Critical endpoint search
4. Summary

QCD IN THE GRAND CANONICAL ENSEMBLE

In pure QCD (unlike full SM) the # of net u,d,s is separately conserved

$$Z = \mathrm{Tr}\left(e^{-(H-\mu_u N_u-\mu_d N_d-\mu_s N_s)/T}\right) = \mathrm{Tr}\left(e^{-(H-\mu_B B-\mu_S S-\mu_Q Q)/T}\right)$$

$$\hat{p} \coloneqq \frac{p}{T^4} = \frac{1}{(LT)^3}\log Z \text{(dimensionless pressure)}$$

We set $\mu_Q = 0$ in what follows and typically $\mu_S(T,\mu_B)$ is tuned such that $\mathrm{n_S}(T,\mu_B,\mu_S(T,\mu_B)) = 0$ to make the phase diagram two dimensional.

$$\chi_{ij}^{BS} = \frac{\partial^{i+j}\hat{p}}{\partial\hat{\mu}_B^i\partial\hat{\mu}_S^j} \qquad \left(\hat{\mu}_{B/S} \coloneqq \frac{\mu_{B/S}}{T}\right) \quad \text{(generalized susceptibilities)}$$

DERIVATIVES ⇔ FLUCTUATIONS AND CORRELATIONS:

$\chi_1^B \propto \langle B\rangle \propto n_B$;

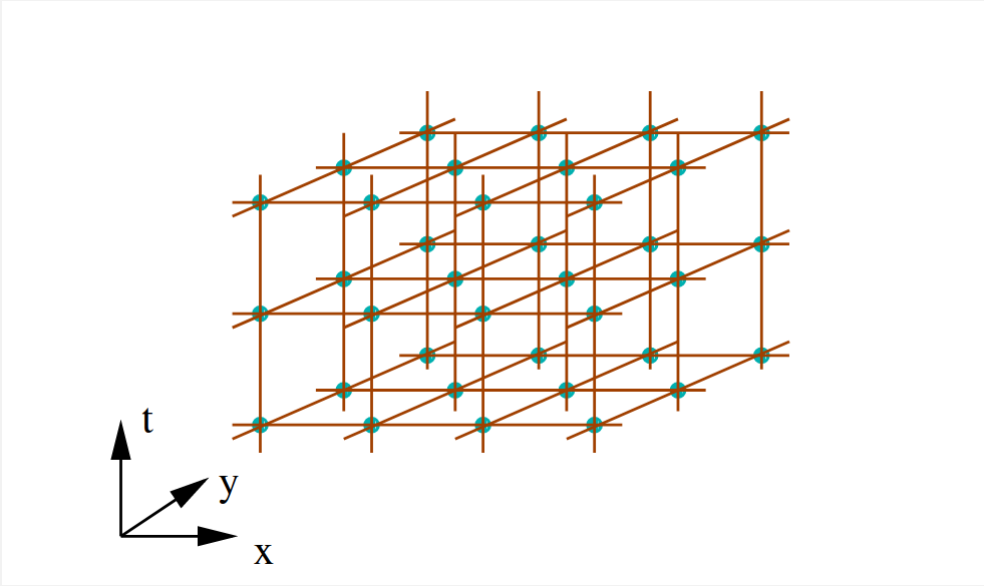
$\chi_2^B \propto \langle B^2\rangle - \langle B\rangle^2$; $\chi_{11}^{BS} \propto \langle BS\rangle - \langle B\rangle\langle S\rangle$;

$\chi_3^B \propto \langle B^3\rangle - 3\langle B^2\rangle\langle B\rangle + 2\langle B\rangle^3$; …

LATTICE FIELD THEORY

Finite space-time lattice: $N_s^3 N_t$

Equilibrium physics: $T = \frac{1}{N_t a}$



1. Continuum limit:

   For fixed temperature $a \to 0 \Leftrightarrow N_t \to \infty$

   (Fixed N_t: Lower $T \Rightarrow$ Larger a (coarser))

2. Thermodynamic limit:

   Size is often measured in units of 1/T

   Aspect ratio: $LT = N_s/N_t$

   Infinite volume limit: $LT \to \infty$

$$Z = \int DA_\mu D\bar{\psi} D\psi e^{-S_{YM} - \bar{\psi} M(A_\mu, m, \mu)\psi} = \int DA_\mu \det M(A_\mu, m, \mu)\, e^{-S_{YM}}$$

Can be simulated with Monte Carlo if $\det M\, e^{-S_{YM}}$ is real and positive: $\mu = 0, \mathrm{Re}(\mu) = 0, \mu_u = -\mu_d$

Otherwise: complex action/sign problem $\Rightarrow$ simulations are exponentially hard in the volume (thermo limit!)

Common workaround: simulate in a situation with no sign problem (imaginary chemical potentials or calculate derivatives at $\mu_B = 0$) and extrapolate/analytically continue to the physically interesting region

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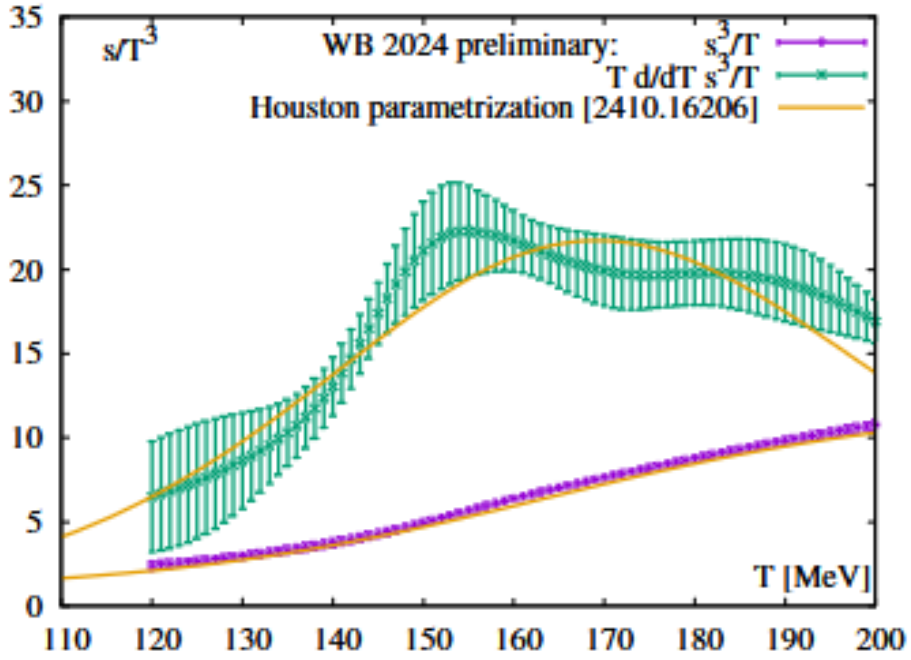
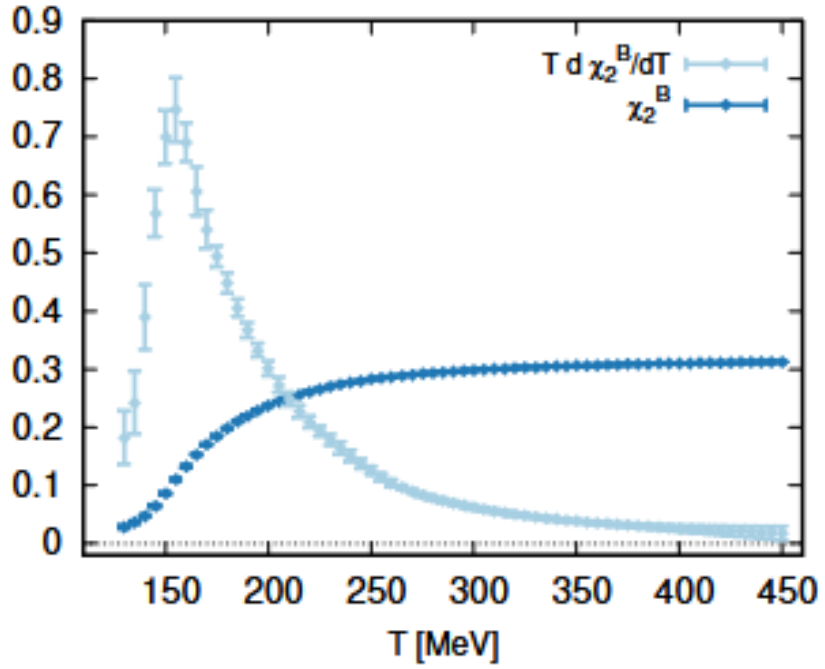
FROM THE EOS? ENTROPY/EFF. DEGREES OF FREEDOM

Rough idea: if quarks are free, there are more effective degrees of freedom





$$\frac{s}{T^3} \xrightarrow[T\to\infty]{} \#N_f + \#(N_c^2 - 1)$$

$$\chi_2^B = \left(\frac{1}{T^3}\frac{\partial^2 p}{\partial \mu_B^2}\right)_{\mu_B=0} \xrightarrow[T\to\infty]{} \#N_f$$

CHIRAL SYMMETRY RESTORATION

- **Chiral symmetry:**
- $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$ for $m_q \rightarrow 0$ (chiral limit)
- spontaneously broken at low T (likely 2nd order for N_f=2, unclear for N_f=3)

  For N_f=2 see [HotQCD, PRL123 (2019)] and [Kotov, Lombardo, Trunin, PLB823 (2021)]

- **Chiral condensate:** order parameter for chiral symmetry breaking in the two-flavour chiral limit

$$\langle\bar{\psi}\psi\rangle = \frac{T}{V}\frac{\partial \log Z}{\partial\, m_{ud}} \qquad \langle\bar{\psi}\psi\rangle_R = -\left[\langle\bar{\psi}\psi\rangle_T - \langle\bar{\psi}\psi\rangle_0\right]\frac{m_{ud}}{f_\pi^4}$$

- **Chiral susceptibility:**

$$\chi_{full} = \frac{T}{V}\frac{\partial^2 \log Z}{\partial\, m_{ud}^2} \qquad \chi_{disc} = \frac{T}{V}\left(\frac{\partial^2 \log Z}{\partial\, m_u \partial m_d}\right)_{m_u=m_d} \qquad \chi_R = [\chi_T - \chi_0]\frac{m_{ud}^2}{f_\pi^4}$$

- Peak postition: crossover temperature
  Peak height and/or inverse width: strength of the crossover

CENTER SYMMETRY BREAKING

- **Center symmetry:**
  - discrete Z_3 symmetry for $m_q \to \infty$ (pure gauge theory)
  - spontaneously broken at high T (weak 1st order)
- **Polyakov-loop:** $P = \frac{1}{V}\frac{1}{N_c}\sum_{\vec{x}} \mathrm{Tr} \prod_{t=0}^{N_t-1} U_4(\vec{x}, t)$
- An order parameter for the deconfinement transition in pure gauge theory

$$|\langle P \rangle| = e^{-F_Q/T} \Rightarrow |\langle P \rangle| = 0 \leftrightarrow F_Q = \infty$$

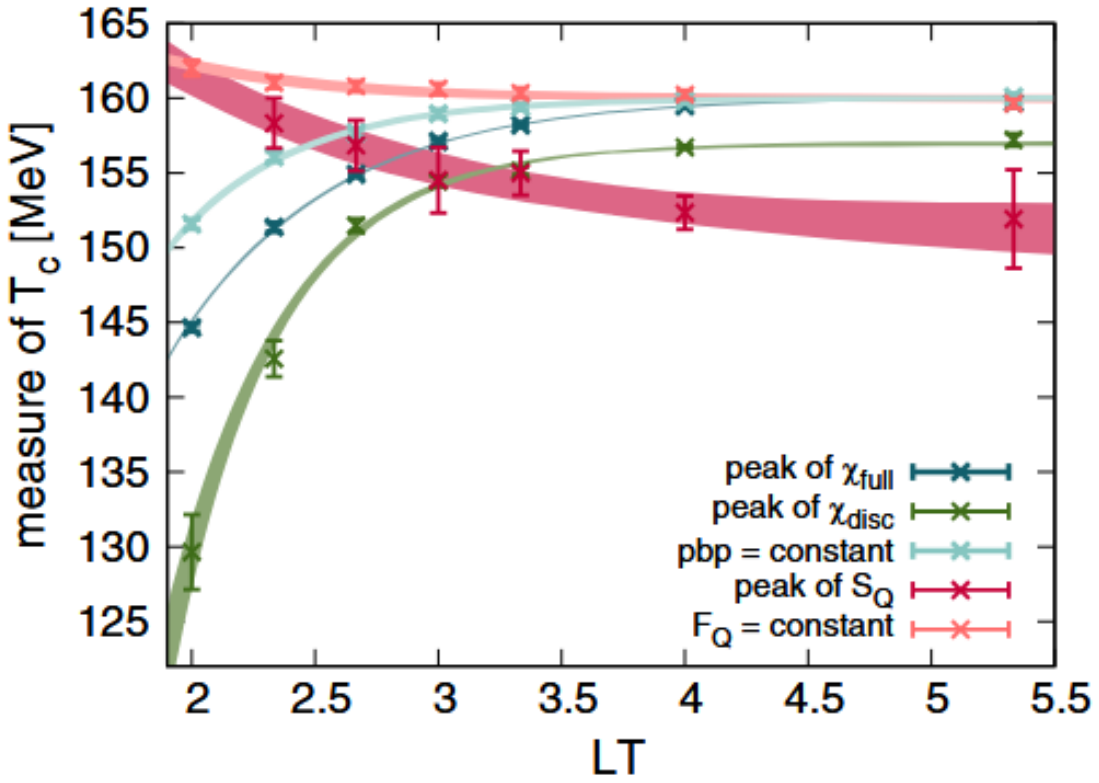
- **Static quark free energy** $F_Q = -T \log |\langle P \rangle|$

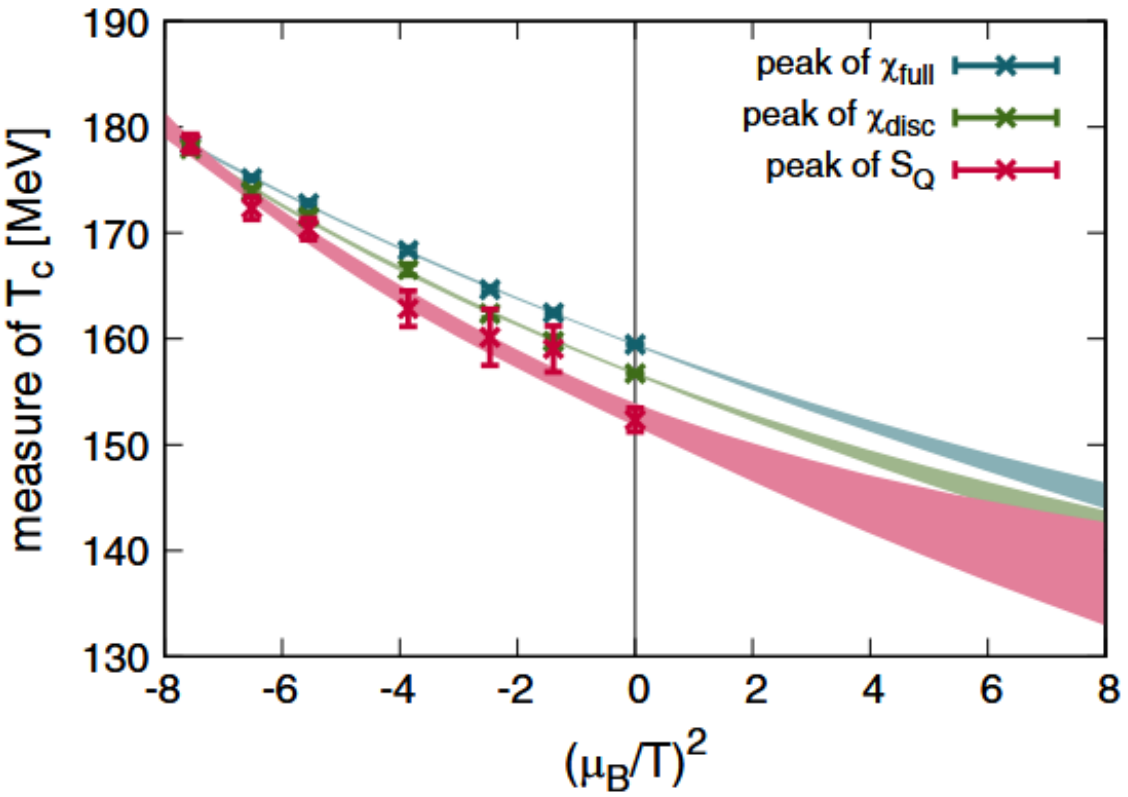
  needs additive renormalization, e.g. $F_Q^R = F_Q(T) - F_Q(T_0)$

- **Static quark entropy** $S_Q = -\frac{\partial F_Q}{\partial T}$ peak position renorm. scheme independent

[Bazavov et al, PRD93 (2016)]

THE CHIRAL AND S_Q DEFINITIONS GIVE A SIMILAR TC



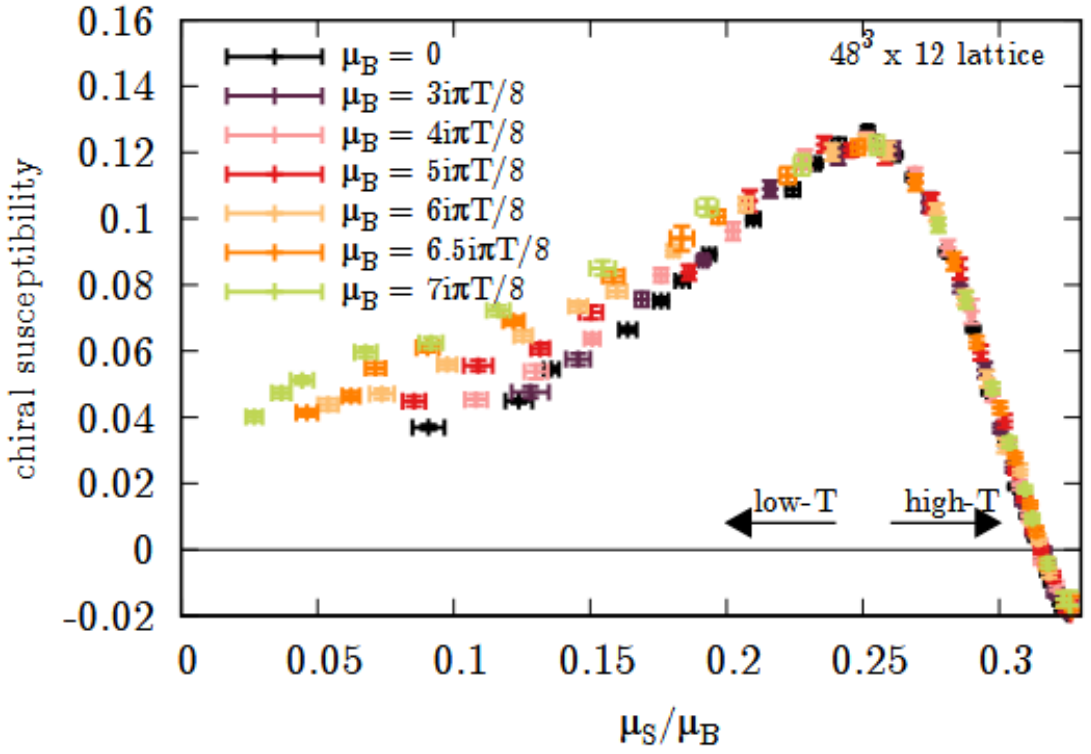


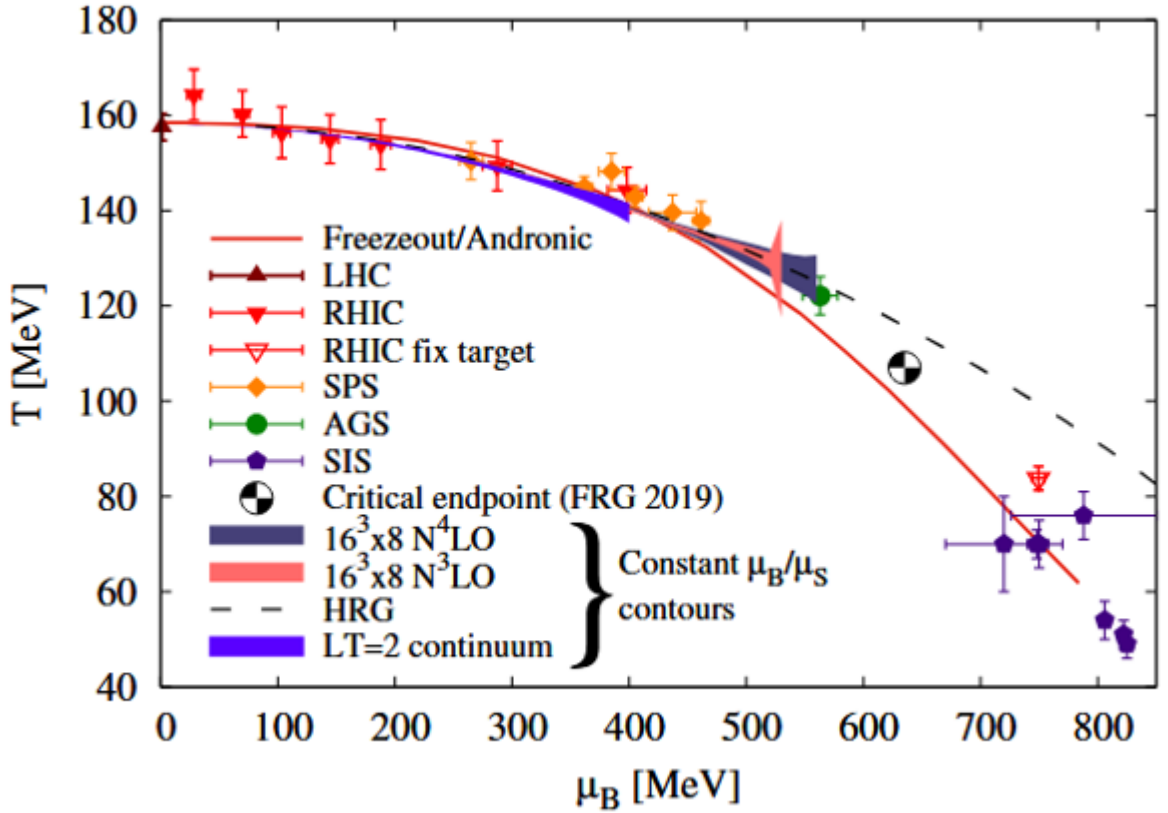
POLYAKOV LOOP BASED OBSERVABLES HAVE SMALLER FINITE VOLUME EFFECTS, BUT ARE NOISIER

COMPATIBLE CURVATURES, NO CLEAR CONVERGENCE TO THE SAME VALUE

[WB, PRD111 (2025)]

A STRANGENESS BASED PROXY FOR THE CROSSOVER





$$\lim_{\mu_B \to 0} \frac{\mu_S}{\mu_B} = -\frac{\chi_{11}^{BS}}{\chi_2^S}$$

[Borsanyi et al, 2510.26455]

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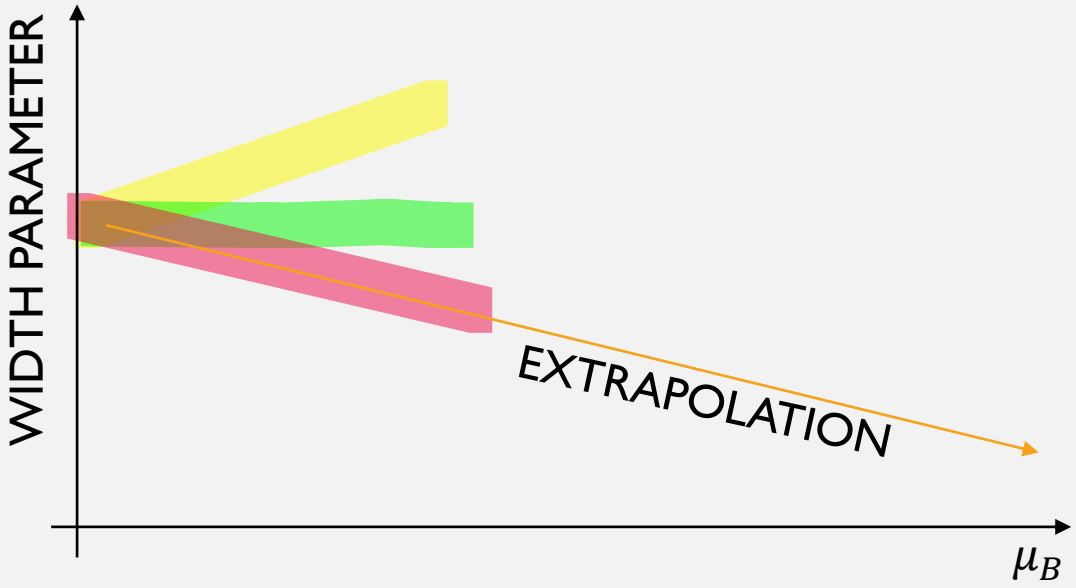
HOW TO GUESSTIMATE THE CEP POSITION?

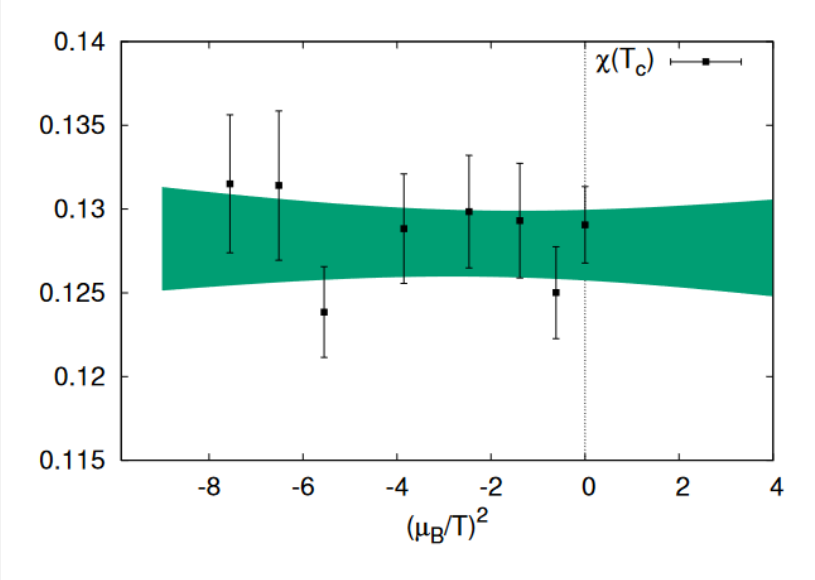
- Define **width** (e.g. curvature of some susceptibility at peak) or **strength** (e.g. value of some susceptibility at peak) for the crossover.



FOR A CROSSOVER, ALL THREE BEHAVIORS COULD BE OBSERVED, DEPENDING ON THE SPECIFIC WIDTH/STRENGTH PARAMETER

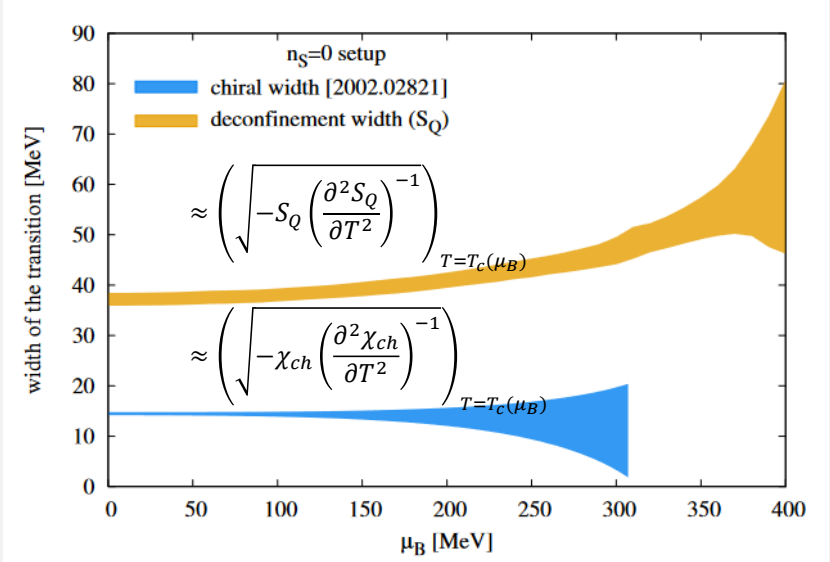
WHAT THIS STRATEGY GIVES YOU

CHIRAL PEAK HEIGHT ~ CONSTANT
CHIRAL WIDTH ~ CONSTANT



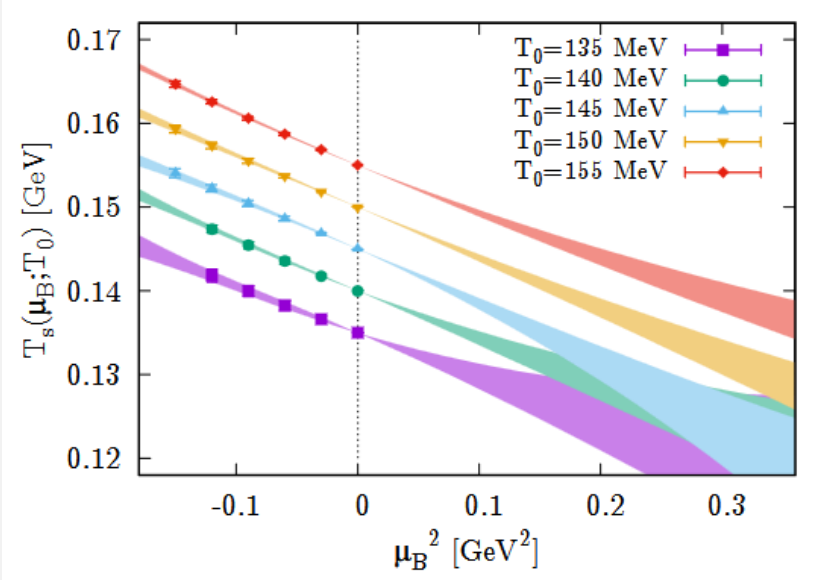
Wuppertal-Budapest PRL125 (2020)

CONFINEMENT WIDTH ~ GROWS



Wuppertal-Budapest PRD110 (2024)

CURVES OF CONSTANT S
~ PARALLEL
~ CONSTANT STRENGTH



Wuppertal-Budapest PRD 112 (2025)

NO ROBUST CEP PREDICTION FROM ORDINARY ANALYTIC CONTINUATION METHODS (FROM IMAGINARY CHEMICAL POTENTIAL AND TAYLOR COEFFICIENTS)

TWO SIGMA BOUND:
$T_{CEP} < 130\text{MeV}$
$\mu_{CEP} > 450\text{MeV}$

CONSTRAINED EXANPSION: THE BASIC IDEA

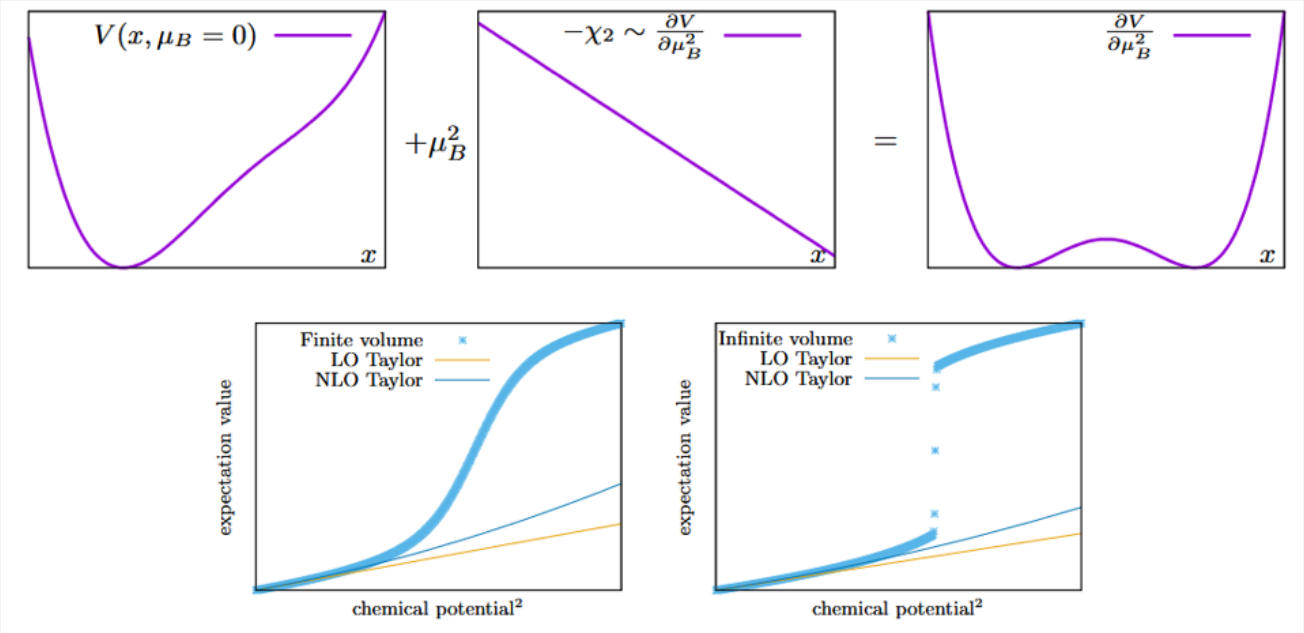
Idea: create configurations with a constraint for the real part of the Polyakov loop, make a Taylor expansion in each separately

$$Z = \int dc \int DU\, e^{-S[U,\mu_B]} \sqrt{\frac{K}{2\pi}}\, e^{-\frac{K}{2}(Re\ P\ -c)^2} = \int dc\ \rho(c,\mu_B)$$

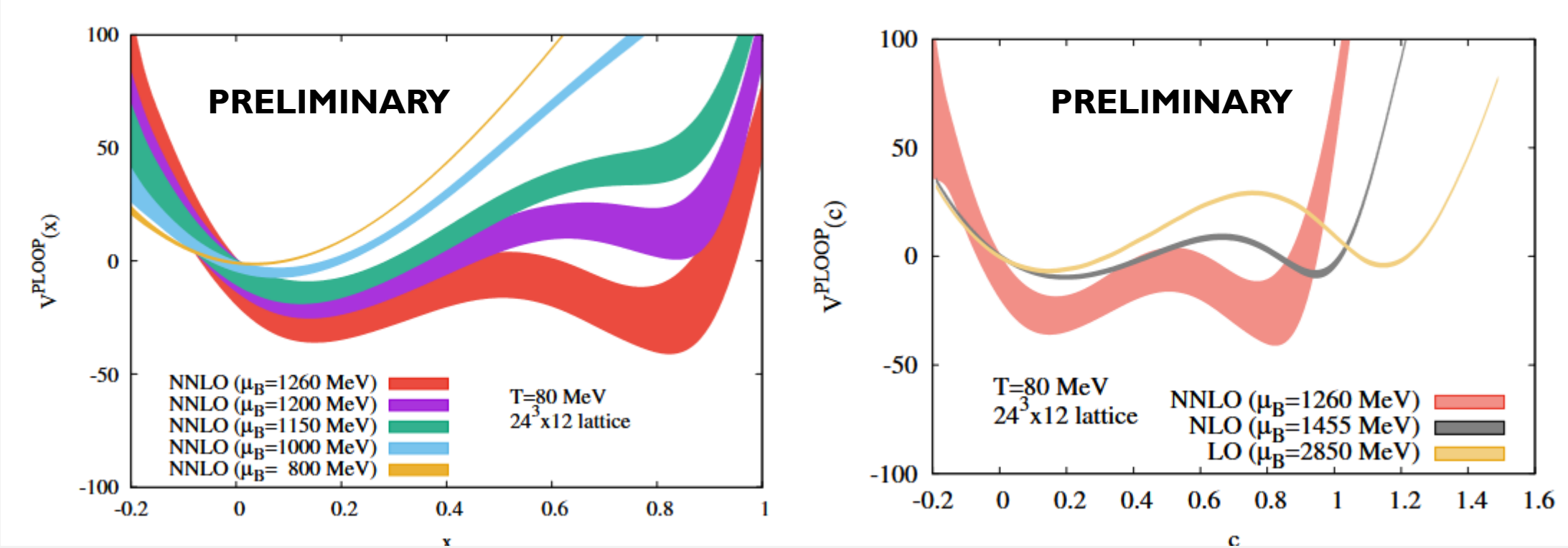
$$\rho(c,\mu_B) = e^{-L^3\, V_{eff}(c,\mu_B)}$$

$$V_{eff}(c,\mu_B) = (\#\mu_B^2 + \#\mu_B^4 + \cdots)$$

Expand the effective potential and NOT the full free energy



HINTS OF A FIRST ORDER TRANSITION (16^3X8)



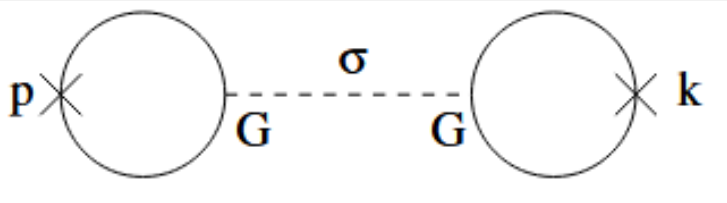
- At T=80MeV, three consecutive terms in the constrained expansion predict a first order transition!
- μ value and the shape of the potential is not stable between the orders
- Caveat: no continuum limit, no infinite volume limit of the potential

SUMMARY AND OUTLOOK

- Up to $\mu_B \approx 400$MeV, the freeze-out line, the chiral crossover line, the Polyakov-loop based crossover line, and the strangness based proxies all agree with each other
- Above 400MeV, the freeze-out line and the strangeness based crossover proxies start to deviate
- Strangeness based proxy available up to around 550MeV (close to HRG proxy, but above the freeze-out line)
- Robust critical endpoint exclusion based on contours of constant entropy: $\mu_{B,CEP} > 450$MeV and $T_{CEP} < 130$MeV
- Less robust (16^3X8) critical endpoint exclusion based on Lee-Yang zeros: $T_{CEP} < 103$MeV (2507.13254, PRD 113)
- Preliminary analysis of the Polyakov loop effective potential hints at a hirst order for $T_{CEP} = 80$MeV (no continuum, no volume scaling yet)

THE CRITICAL PART IS SUPPRESSED IN STRANGESS FLUCTUATIONS

[Hatta and Stephanov, PRL91 (2003)]



$$\langle \Delta N^i \Delta N^j \rangle = V \int_p \int_k \left\langle \delta n_p^i \delta n_k^j \right\rangle$$

$$V \left\langle \delta n_p^i \delta n_k^j \right\rangle = \frac{G^2}{m_\sigma^2 T} \frac{4 m_i m_j}{E_p^i E_k^i} \times \left[n_p^{i,+} \left(1 - n_p^{i,+}\right) - n_p^{i,-} \left(1 - n_p^{i,-}\right) \right] \times \left[n_k^{j,+} \left(1 - n_k^{j,+}\right) - n_k^{j,-} \left(1 - n_k^{j,-}\right) \right]$$

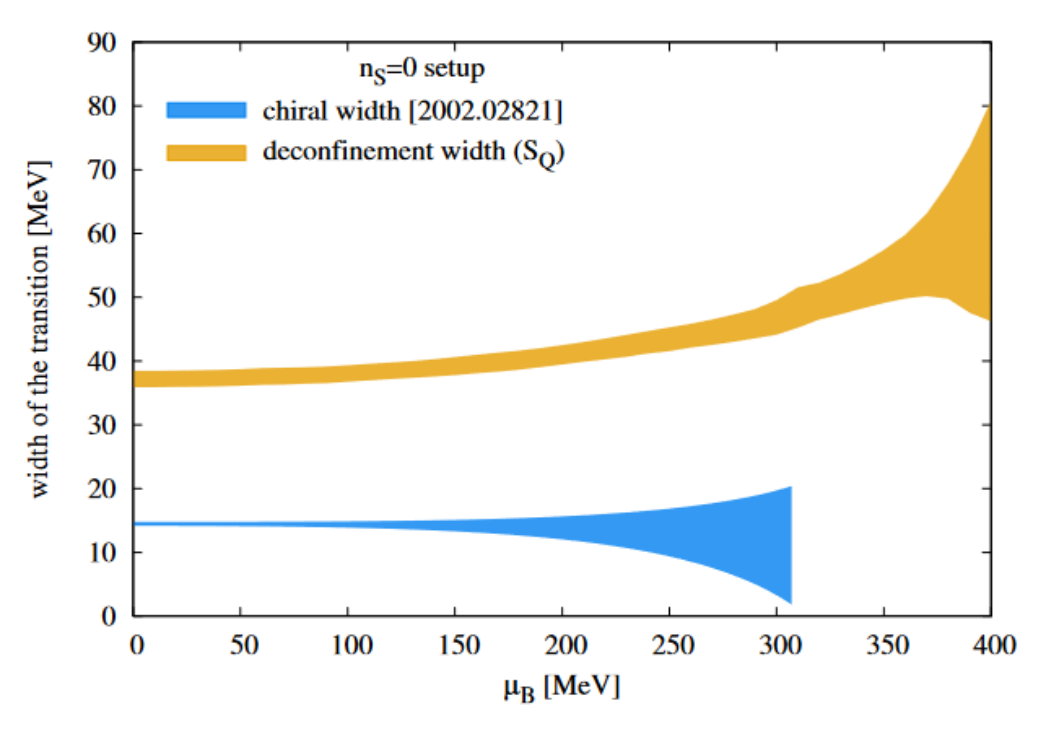
- We saw on the prev. slide that $\frac{\mu_S}{\mu_B} \approx \frac{1}{4}$
- Typical CEP predictions nowaday (DSE, FRG, holographic etc.) give $T_{CEP} \sim 100 - 120$MeV and $\mu_B \sim 600 - 650$MeV
- Consituent quark mass $m_{ud} = 170$MeV and $m_s = 330$MeV we get

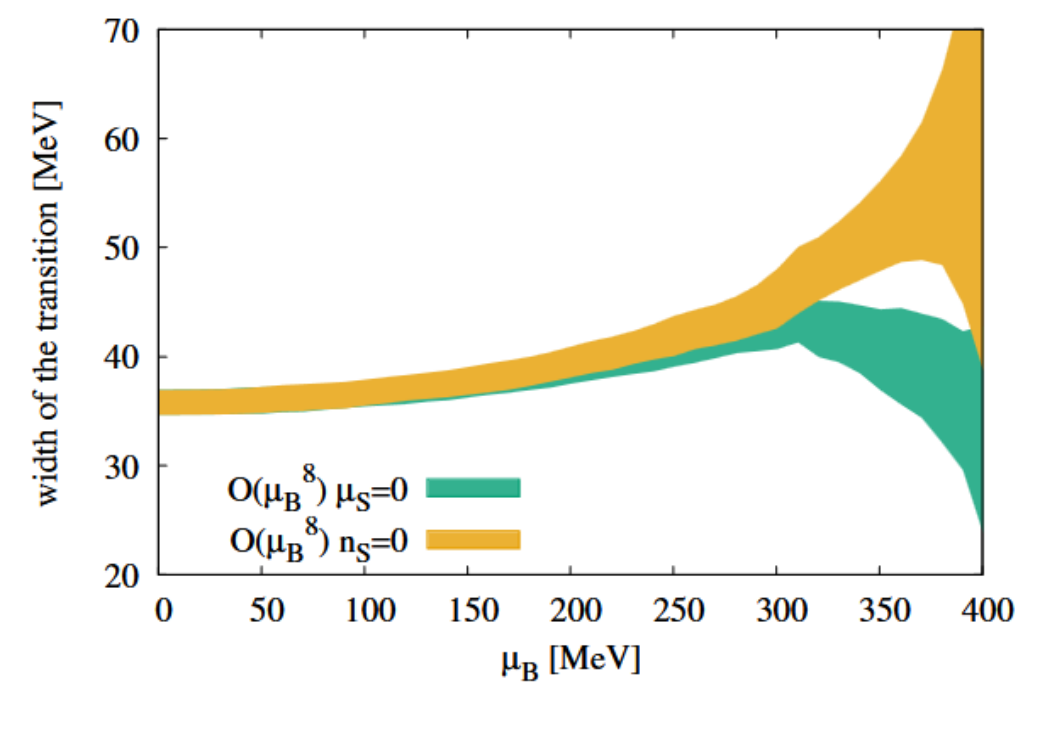
For the strangeness neutral case one estimates:

$$\chi_2^{S(crit)} \approx \frac{1}{100} \chi_2^{B(crit)}$$

[Borsanyi et al, 2510.26455]

DECONFINEMENT WIDTH: WITH OR WITHOUT STRANGENESS NEUTRALITY





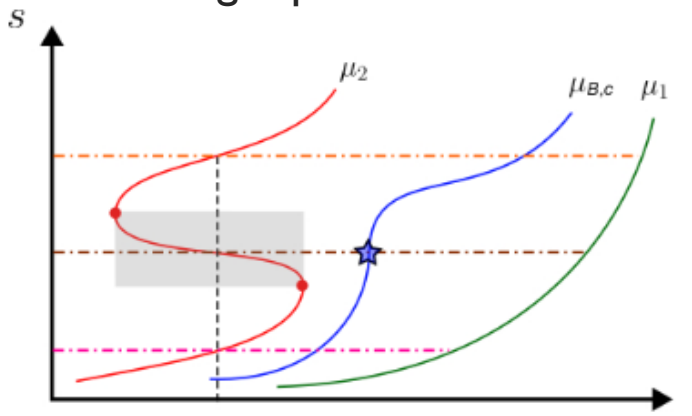
CAVEAT:

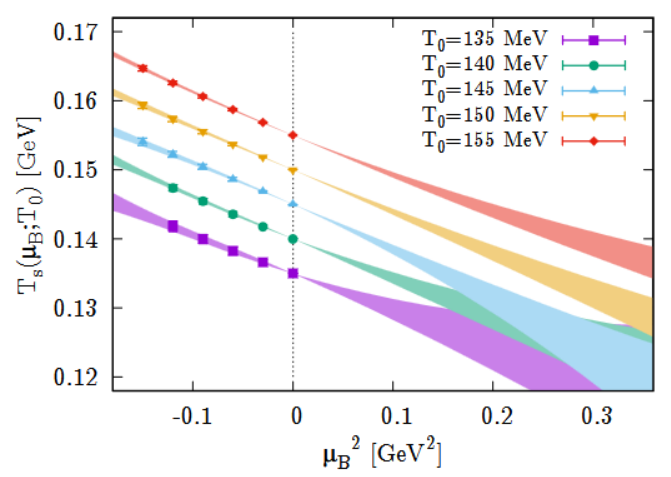
The reversing trend only happens at the highest order of the Taylor expansion we calculated.

CONTOURS OF CONSTANT ENTROPY

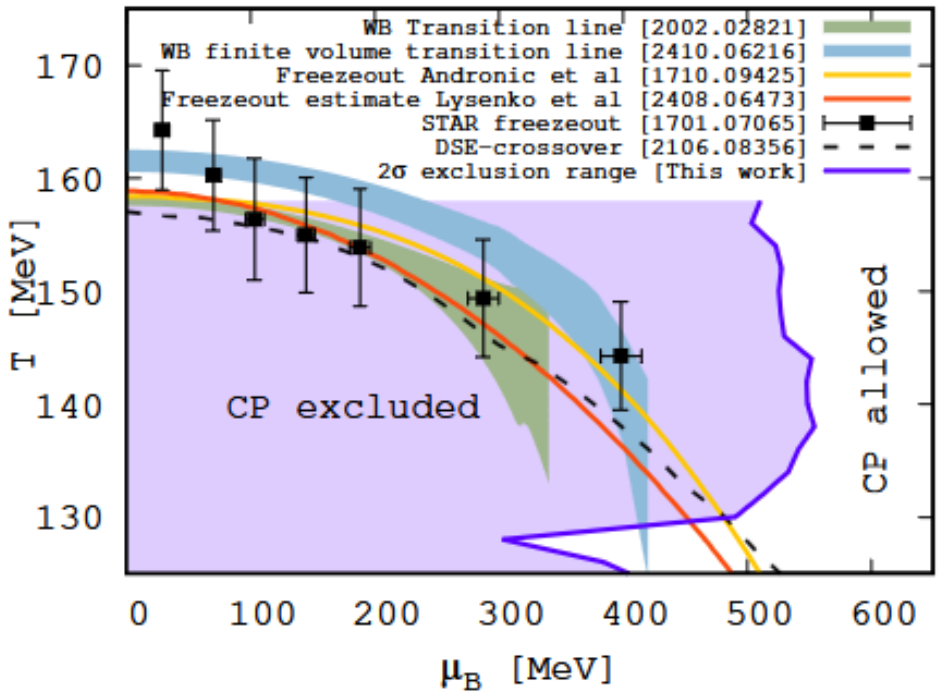
idea: [2410.16206] (leading order extrapolation in μ_B, μ_S=0)
improved implementation: [2502.10267] (new lattice data, beyond leading order, n_S=0)

- Inverse strength parameter = dT/ds or dT/dT_0





Uses continuum extrapolated data in a large volume (LT=4)



Contours of constant entropy are parallel (within errors)