LONGITUDINAL DYNAMICS

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with lots of material
from the course by
Joël Le Duff
Many Thanks!

Introductory Level Accelerator Physics Course
Granada, 28 October - 9 November 2012

Overview

- Methods of Acceleration
- Accelerating Structures
- Synchronism Condition and Phase Stability (Linac)
- Bunching and bunch compression
- Circular accelerators: Cyclotron / Synchrotron
- Dispersion Effects in Synchrotron
- Synchrotron Oscillations
- Energy-Phase Equations
- Longitudinal Phase Space Motion
- Stationary Bucket
- Injection Matching

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And CERN Accelerator Schools (CAS) Proceedings

Main Characteristics of an Accelerator

Newton-Lorentz Force on a charged particle:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

2nd term always perpendicular to motion => no acceleration

ACCELERATION is the main job of an accelerator.

- It provides **kinetic energy** to charged particles, hence increasing their **momentum**.
- In order to do so, it is necessary to have an electric field $\vec{E}$, preferably along the direction of the initial momentum.

$$\boxed{\frac{dp}{dt} = eE_z}$$

BENDING is generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\boxed{\frac{p}{e} = B\rho}$$

in practical units: $B\ \rho\ [\mathrm{Tm}] \approx \frac{p\ [\mathrm{GeV/c}]}{0.3}$

FOCUSING is a second way of using a magnetic field, in which the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Energy Gain

In relativistic dynamics, total energy E and momentum p are linked by

$$E^2 = E_0^2 + p^2c^2 \qquad (E = E_0 + W) \qquad W \textit{ kinetic energy}$$

Hence: $$dE = v\,dp$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z\,dz \quad \rightarrow \quad W = e\int E_z\,dz = eV$$

where V is just a potential.

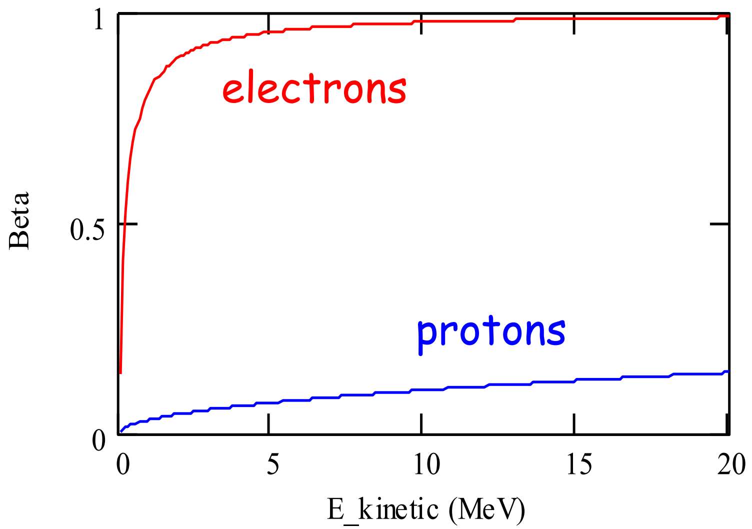
Some relativistic relations:

$$p = mv = \frac{E}{c^2}\beta c = \beta\frac{E}{c} = \beta\gamma m_0 c \qquad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}}$$

Velocity and Energy

normalized velocity

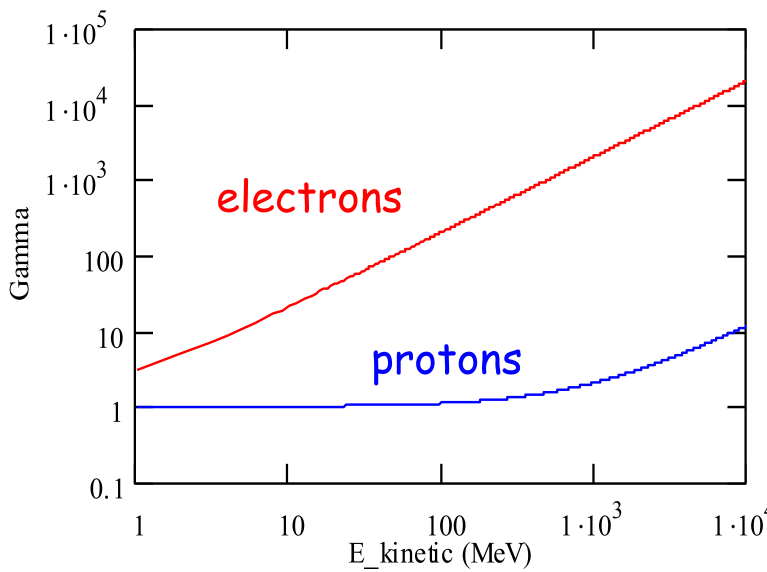
$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$



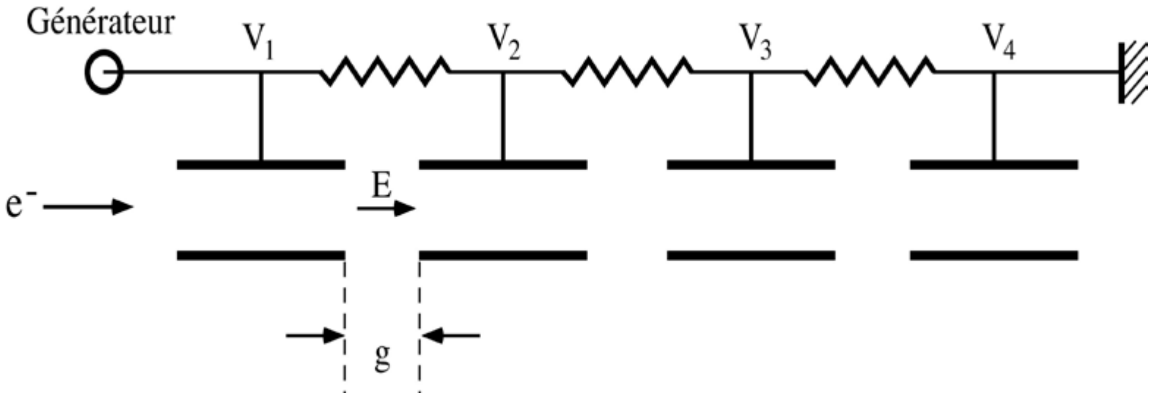
=> electrons almost reach the speed of light very quickly

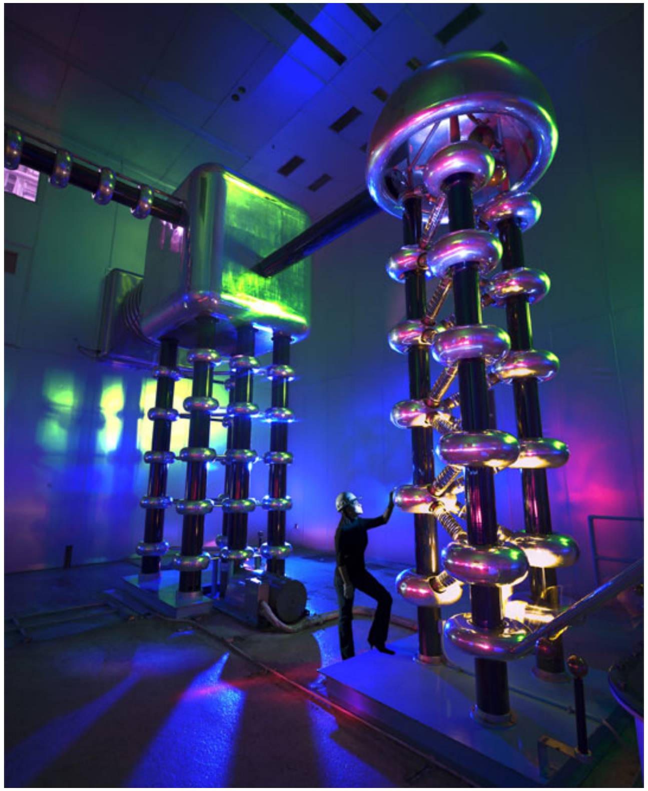
$$\frac{\text{total energy}}{\text{rest energy}}$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$$



Methods of Acceleration: Electrostatic





750 kV Cockroft-Walton generator at Fermilab (Proton source)

Electrostatic Field:

Energy gain: $W = n\ e(V_2 - V_1)$

limitation : $V_{generator} = \Sigma V_i$

$\Rightarrow$ insulation problems
maximum high voltage (~ 10 MV)

used for first stage of acceleration:
particle sources, electron guns
x-ray tubes

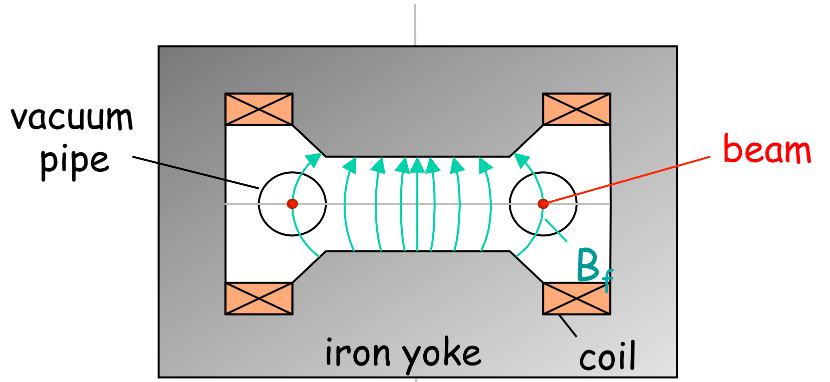
Methods of Acceleration: Induction

From Maxwell's Equations:

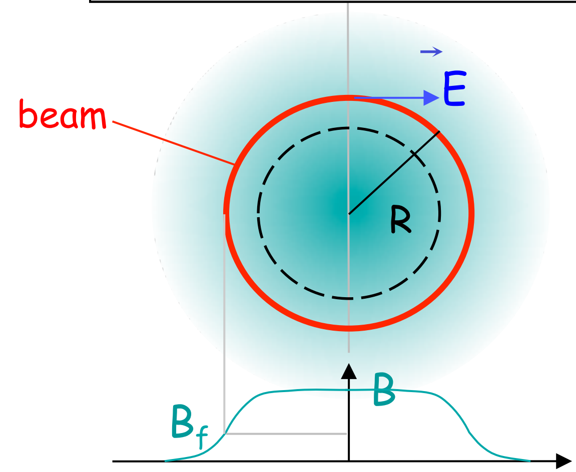
The electric field is derived from a scalar potential φ and a vector potential A
The time variation of the magnetic field H generates an electric field E

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

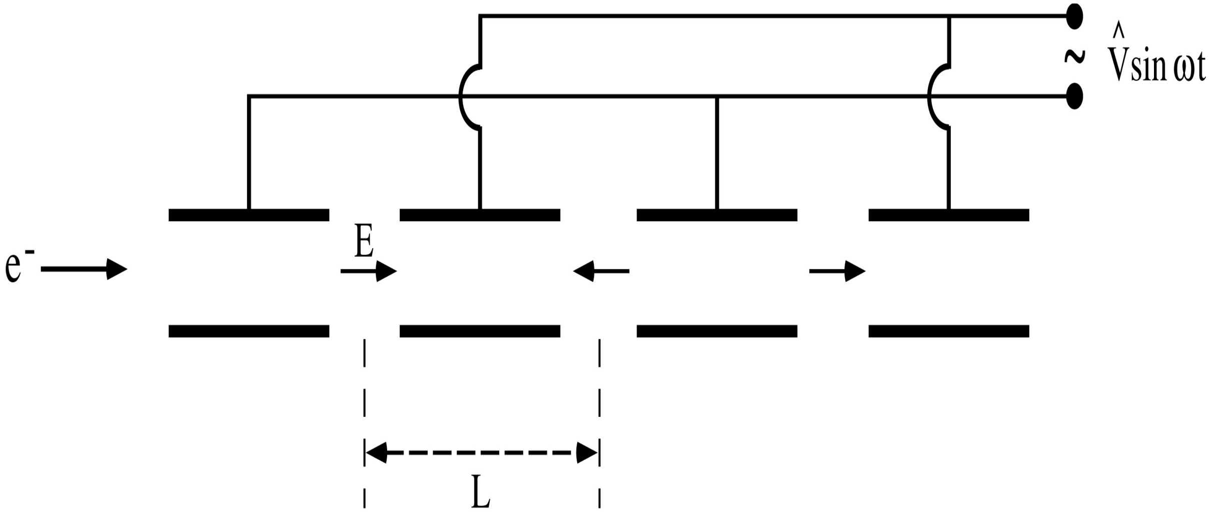
$$\vec{B} = \mu\vec{H} = \vec{\nabla} \times \vec{A}$$



Example: **Betatron**
The varying magnetic field is used to guide particles on a circular trajectory as well as for acceleration.
Limited by saturation in iron



Methods of Acceleration: Radio-Frequency (RF)

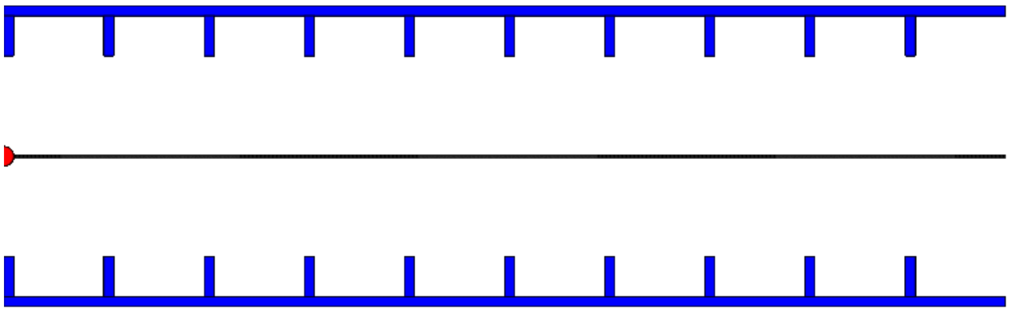


Wideröe-type structure

Cylindrical electrodes (drift tubes) separated by gaps and fed by a RF generator, as shown above, lead to an alternating electric field polarity

Synchronism condition $\longrightarrow$ $L = v\,T/2$

v = particle velocity
T = RF period



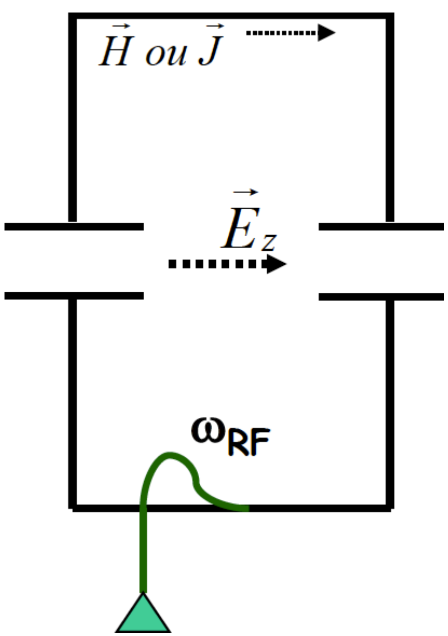
Similar for standing wave cavity as shown (with v≈c)

D.Schulte

The advantages of resonant cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency.
  => The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency.
  => The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

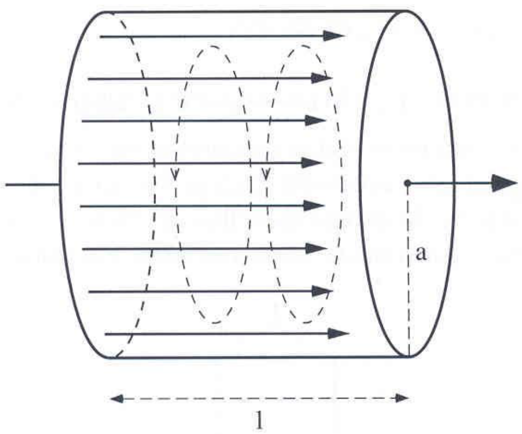


- The electromagnetic power is now constrained in the resonant volume

- Each such cavity can be independently powered from the RF generator

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

The Pill Box Cavity



$\longrightarrow E_z \quad \dashrightarrow H_\theta$

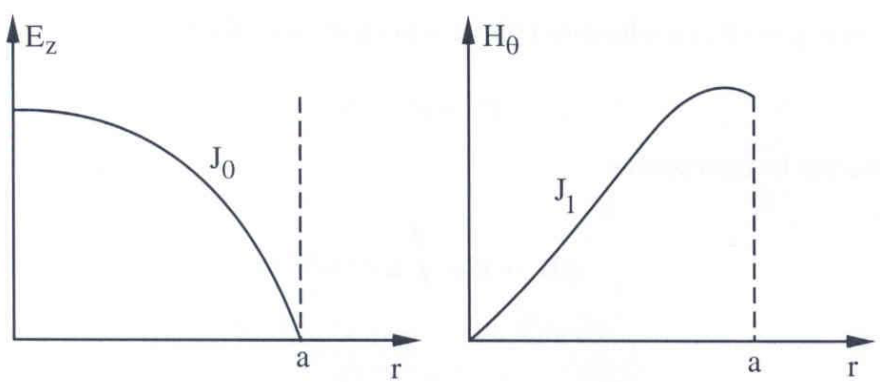
From Maxwell's equations one can derive the wave equations:

$$\nabla^2 A - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \qquad (A = E \text{ or } H)$$

Solutions for E and H are oscillating modes, at discrete frequencies, of types TM_{xyz} (transverse magnetic) or TE_{xyz} (transverse electric).

Indices linked to the number of field knots in polar co-ordinates φ, r and z.

For l<2a the most simple mode, TM_{010}, has the lowest frequency, and has only two field components:

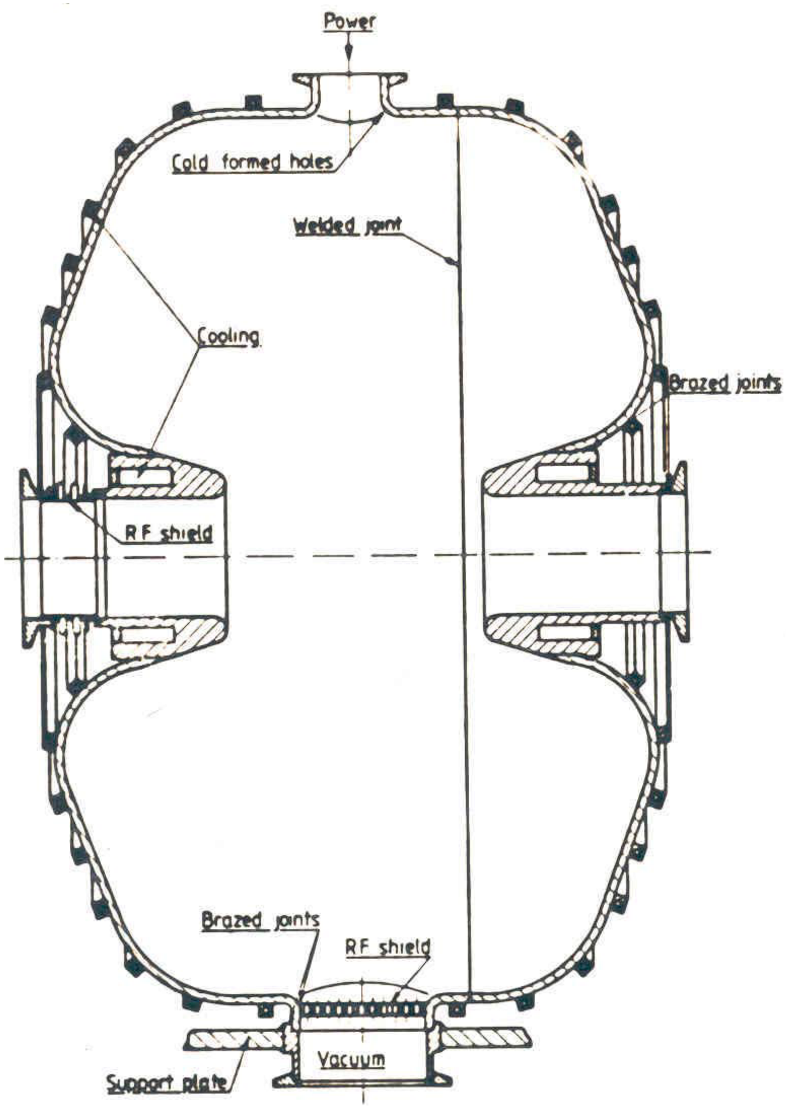


$$E_z = J_0(kr)\, e^{i\omega t}$$

$$H_\theta = -\frac{i}{Z_0} J_1(kr)\, e^{i\omega t}$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \qquad \lambda = 2.62a \qquad Z_0 = 377\Omega$$

The Pill Box Cavity (2)



The design of a pill-box cavity can be sophisticated in order to improve its performances:

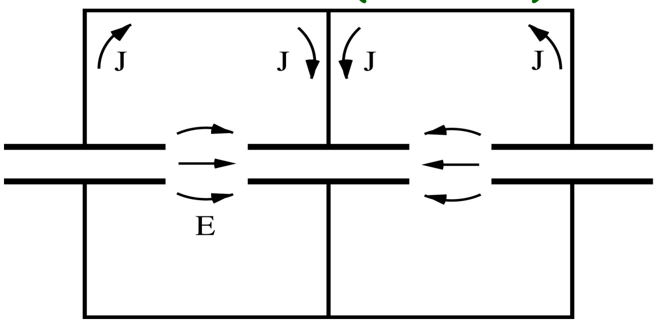
- A nose cone can be introduced in order to concentrate the electric field around the axis

- Round shaping of the corners allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses.
It also prevents from multipactoring effects.

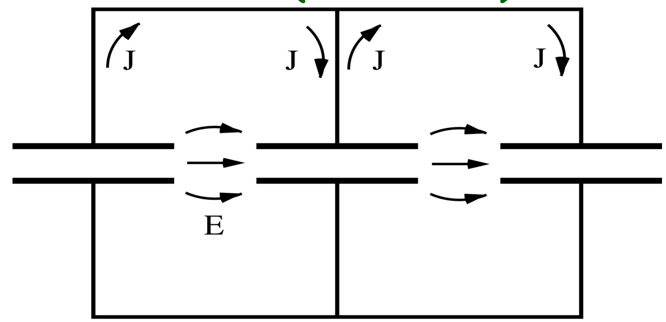
A good cavity is a cavity which efficiently transforms the RF power into accelerating voltage.

Multi-gap Accelerating Structures

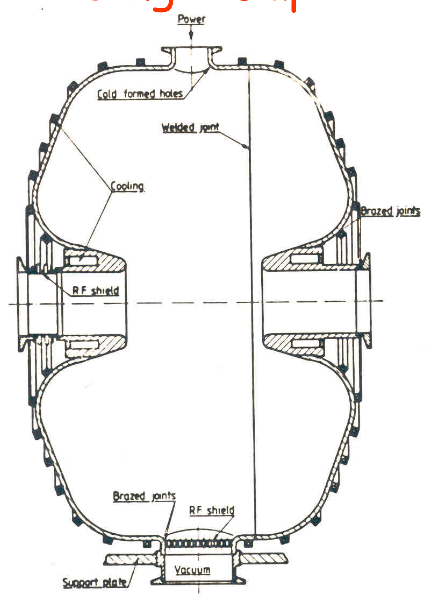
L = vT/2 (π mode)



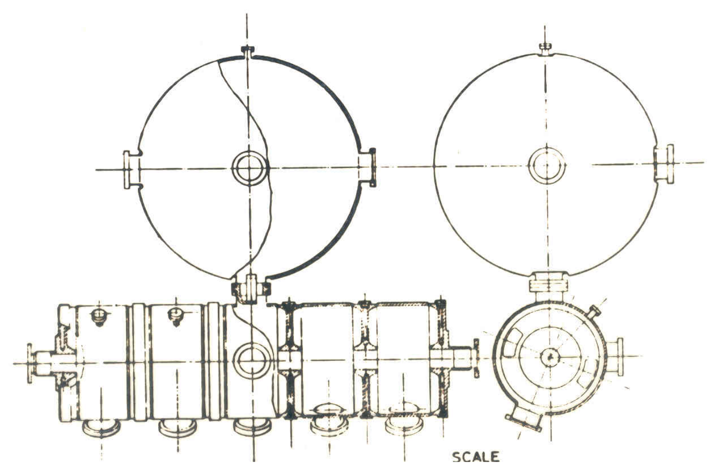
L = vT (2π mode)



Single Gap

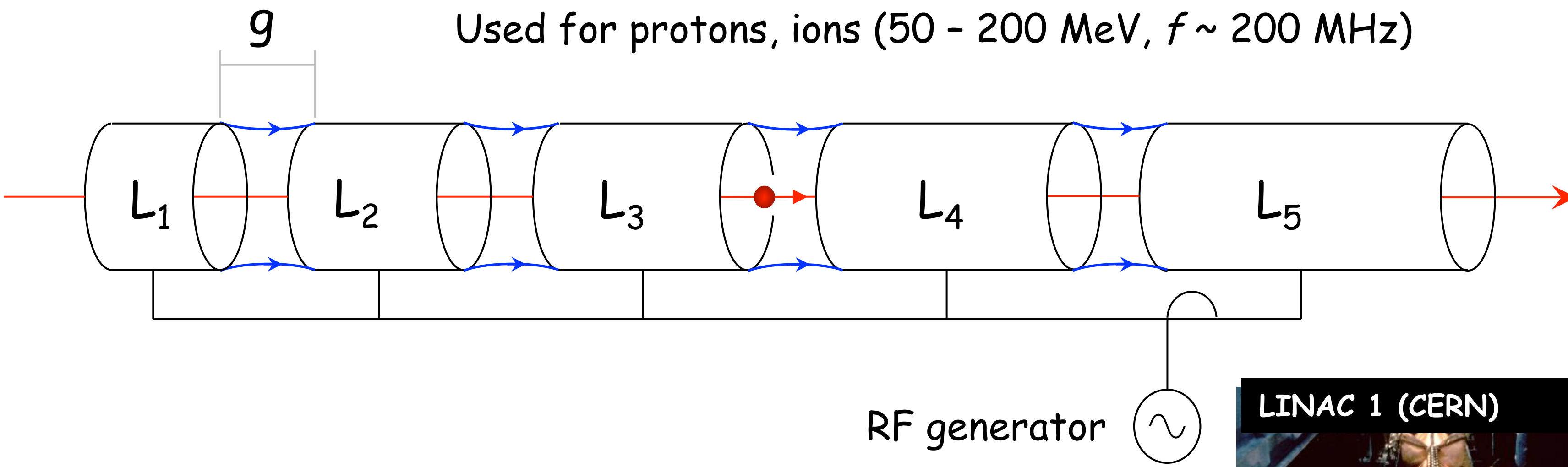


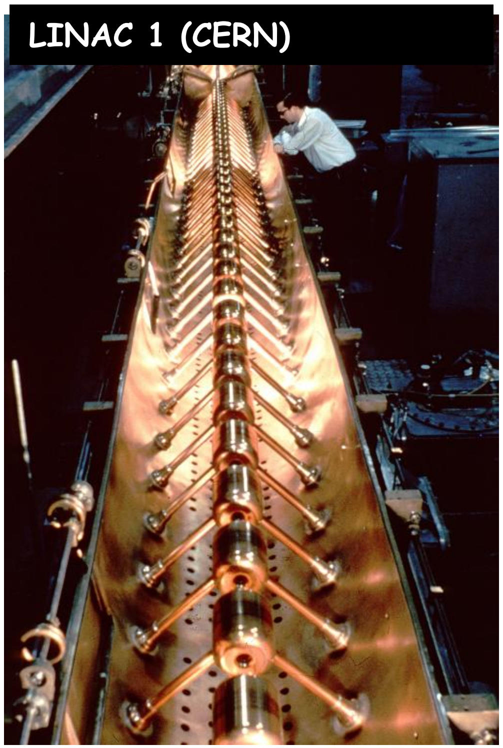
Multi-Gap



RF acceleration: Alvarez Structure

Used for protons, ions (50 – 200 MeV, f ~ 200 MHz)





Synchronism condition $(g << L)$

$$\Rightarrow \quad \boxed{L = v_s T_{RF} = \beta_s \lambda_{RF}}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

Transit time factor

The accelerating field varies during the passage of the particle
=> particle does not see maximum field all the time => effective acceleration smaller

Defined as:

$$T_a = \frac{\text{energy gain of particle with } v = \beta c}{\text{maximum energy gain (particle with } v \to \infty)}$$

In the general case, the transit time factor is:

for $E(s,r,t) = E_1(s,r) \cdot E_2(t)$

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r) \cos\left(\omega_{RF} \frac{s}{v}\right) \mathrm{d}s}{\int_{-\infty}^{+\infty} E_1(s,r) \, \mathrm{d}s}$$

Simple model uniform field:

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

follows:

$$T_a = \left| \sin \frac{\omega_{RF} g}{2v} \middle/ \frac{\omega_{RF} g}{2v} \right|$$

- $0 < T_a < 1$
- $T_a \to 1$ for $g \to 0$, smaller ω_{RF}

Important for low velocities (ions)

Important Parameters of Accelerating Cavities

Shunt Impedance R

$$P_d = \frac{V^2}{R}$$

Relationship between gap voltage V and wall losses P_d

Quality Factor Q

$$Q = \frac{\omega W_s}{P_d}$$

Relationship between stored energy W_s in the volume and dissipated power on the walls

$$\frac{R}{Q} = \frac{V^2}{\omega W_s}$$

Filling Time τ

$$P_d = -\frac{dW_s}{dt} = \frac{\omega}{Q} W_s$$

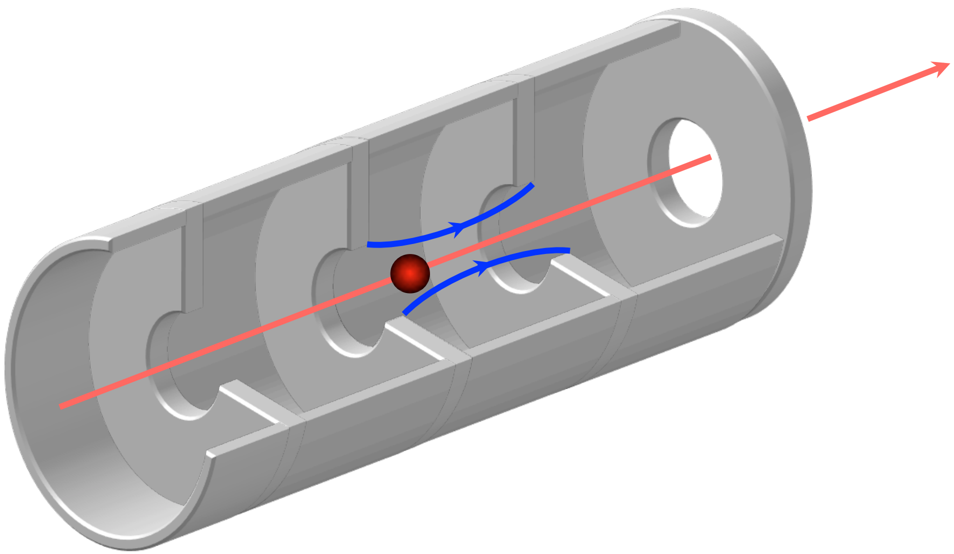
Exponential decay of the stored energy W_s due to losses

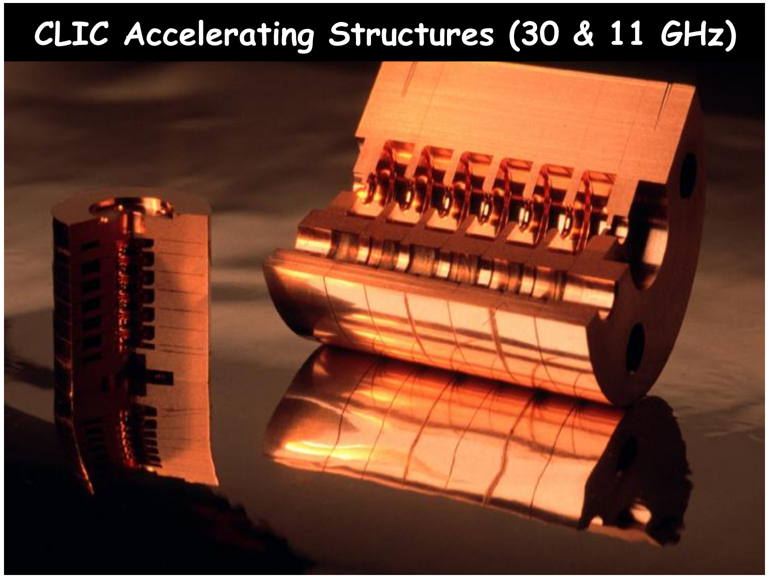
$$\tau = \frac{Q}{\omega}$$

Disc loaded traveling wave structures

-When particles gets ultra-relativistic (v~c) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).

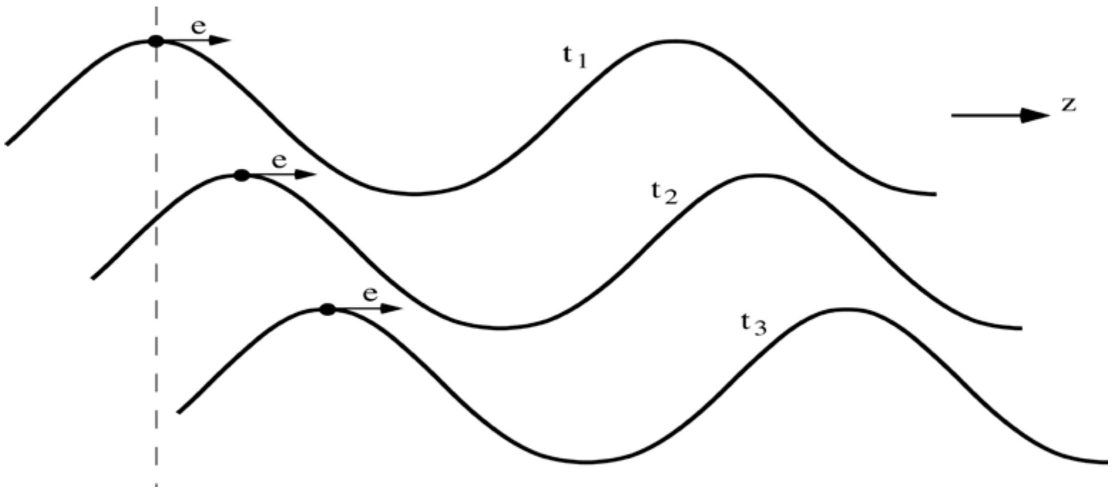
-Next came the idea of suppressing the drift tubes using traveling waves. However to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity.





solution: slow wave guide with irises ==> iris loaded structure

The Traveling Wave Case



$$E_z = E_0 \cos(\omega_{RF} t - kz)$$

$$k = \frac{\omega_{RF}}{v_\varphi}$$ wave number

$$z = v(t - t_0)$$

v_φ = phase velocity
v = particle velocity

The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

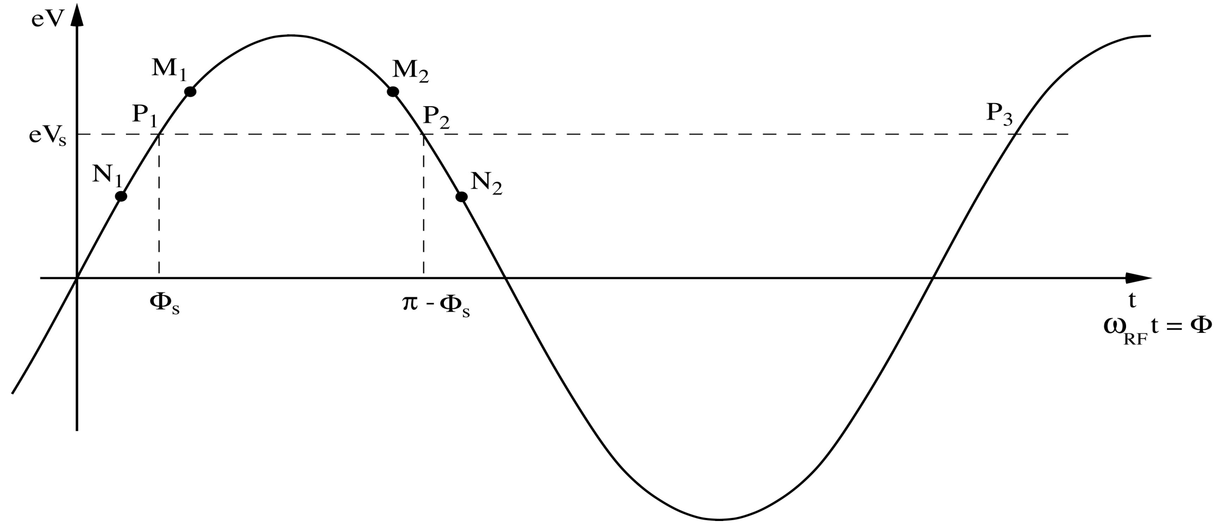
If synchronism satisfied: $v = v_\varphi$ and $E_z = E_0 \cos\phi_0$

where Φ_0 is the RF phase seen by the particle.

Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



$$eV_s = e\hat{V}\sin\Phi_s$$ **is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.**

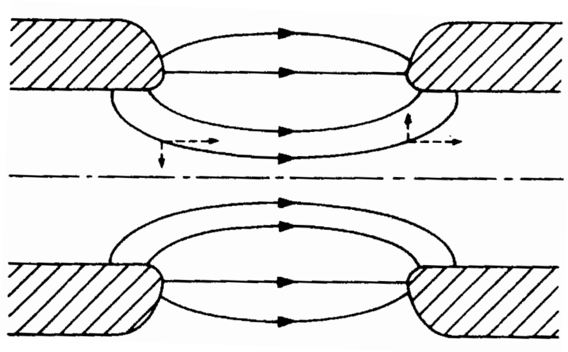
If an energy increase is transferred into a velocity increase =>

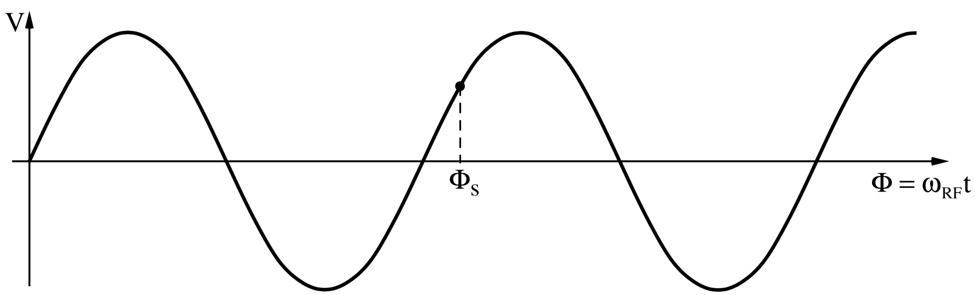
M_1 & N_1 will move towards P_1 => stable

M_2 & N_2 will go away from P_2 => unstable

(Highly relativistic particles have no significant velocity change)

A Consequence of Phase Stability





Transverse focusing fields at the entrance and defocusing at the exit of the cavity.
Electrostatic case: Energy gain inside the cavity leads to focusing
RF case: Field increases during passage => transverse defocusing!

Longitudinal phase stability means : $\dfrac{\partial V}{\partial t} > 0 \Rightarrow \dfrac{\partial E_z}{\partial z} < 0$

defocusing
RF force

The divergence of the field is zero according to Maxwell : $\nabla.\vec{E} = 0 \quad \Rightarrow \quad \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \dfrac{\partial E_x}{\partial x} > 0$

External focusing (solenoid, quadrupole) is then necessary

Energy-Phase Equations

Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin\phi_s$$

Rate of energy gain for a non-synchronous particle, expressed in reduced variables $w = W - W_s = E - E_s$ and $\varphi = \phi - \phi_s$

$$\frac{dw}{dz} = eE_0\left[\sin(\phi_s + \varphi) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad (small\ \varphi)$$

Rate of change of the phase with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}(v - v_s)$$

Since: $$v - v_s = c(\beta - \beta_s) \cong \frac{c}{2\beta_s}\left(\beta^2 - \beta_s^2\right) \cong \frac{w}{m_0 v_s \gamma_s^3}$$

Energy Phase Oscillations

one gets:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Combining the two first order equations into a second order one:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Stable harmonic oscillations imply:

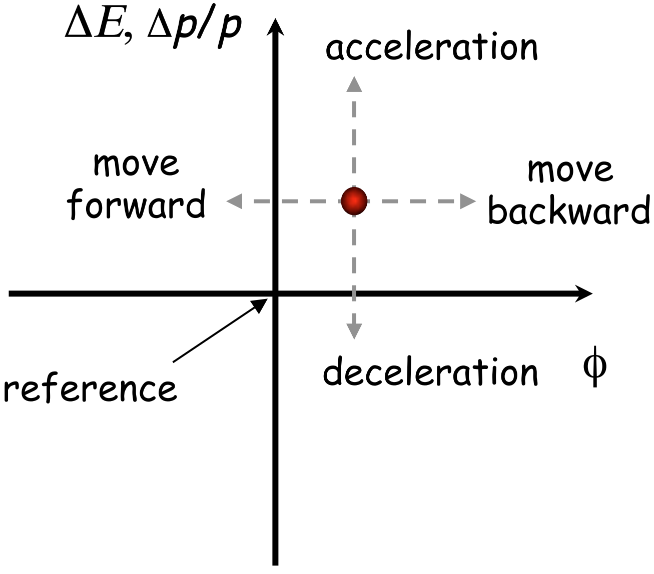
$$\Omega_s^2 > 0 \quad and\ real$$

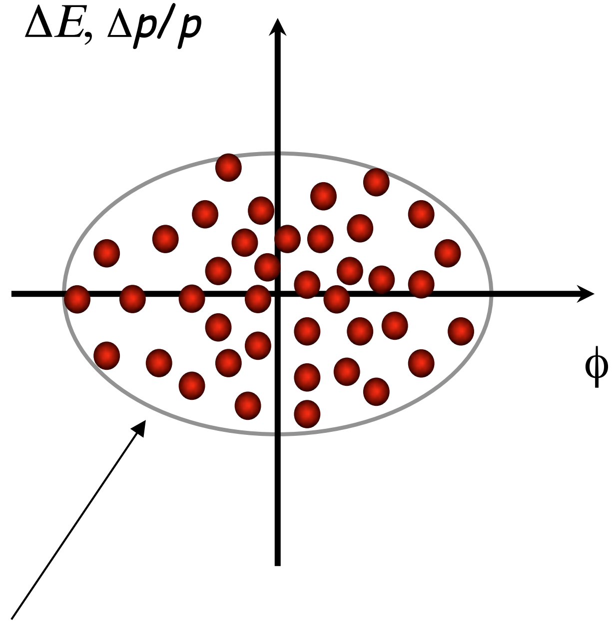
hence: $\cos\phi_s > 0$

And since acceleration also means: $\sin\phi_s > 0$

One finally gets the results: $0 < \phi_s < \frac{\pi}{2}$

Longitudinal phase space





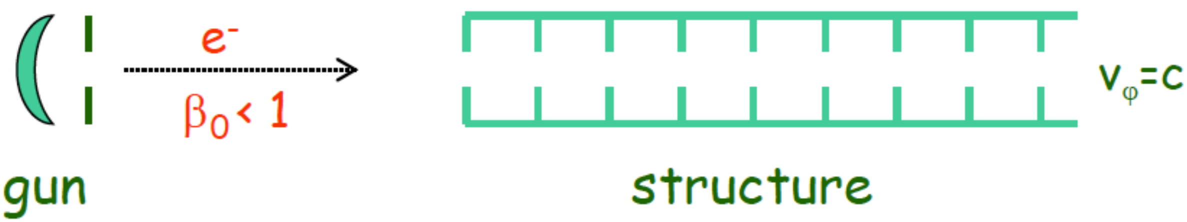
The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.

Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

The Capture Problem

- Previous results show that at ultra-relativistic energies ($\gamma \gg 1$) the longitudinal motion is frozen. Since this is rapidly the case for electrons, all traveling wave structures can be made identical (phase velocity=c).
- Hence the question is: can we capture low kinetic electrons energies ($\gamma < 1$), as they come out from a gun, using an iris loaded structure matched to c ?



$$E_z = E_0 \sin\phi(t)$$

The electron entering the structure, with velocity v < c, is not synchronous with the wave. The path difference, after a time dt, between the wave and the particle is:

$$dz = (c - v)dt$$

Since $\phi = \omega_{RF} t - kz$ with propagation factor $k = \dfrac{\omega_{RF}}{v_\varphi} = \dfrac{\omega_{RF}}{c}$

one gets $dz = \dfrac{c}{\omega_{RF}} d\phi = \dfrac{\lambda_g}{2\pi} d\phi$ and $\dfrac{d\phi}{dt} = \dfrac{2\pi}{\lambda_g} c(1 - \beta)$

The Capture Problem (2)

From Newton-Lorentz:

$$\frac{d}{dt}(mv) = m_0 c \frac{d}{dt}(\beta\gamma) = m_0 c \frac{d}{dt}\left(\frac{\beta}{\left(1-\beta^2\right)^{\frac{1}{2}}}\right) = eE_0 \sin\phi$$

Introducing a suitable variable: $\beta = \cos\alpha$

the equation becomes: $$\frac{d\alpha}{dt} = -\frac{eE_0}{m_0 c}\sin\phi \sin^2\alpha$$

Using $\frac{d\phi}{dt} = \frac{d\phi}{d\alpha}\frac{d\alpha}{dt}$ $\longrightarrow$ $$-\sin\phi\, d\phi = \frac{2\pi m_0 c^2}{\lambda_g e E_0}\frac{1-\cos\alpha}{\sin^2\alpha}d\alpha$$

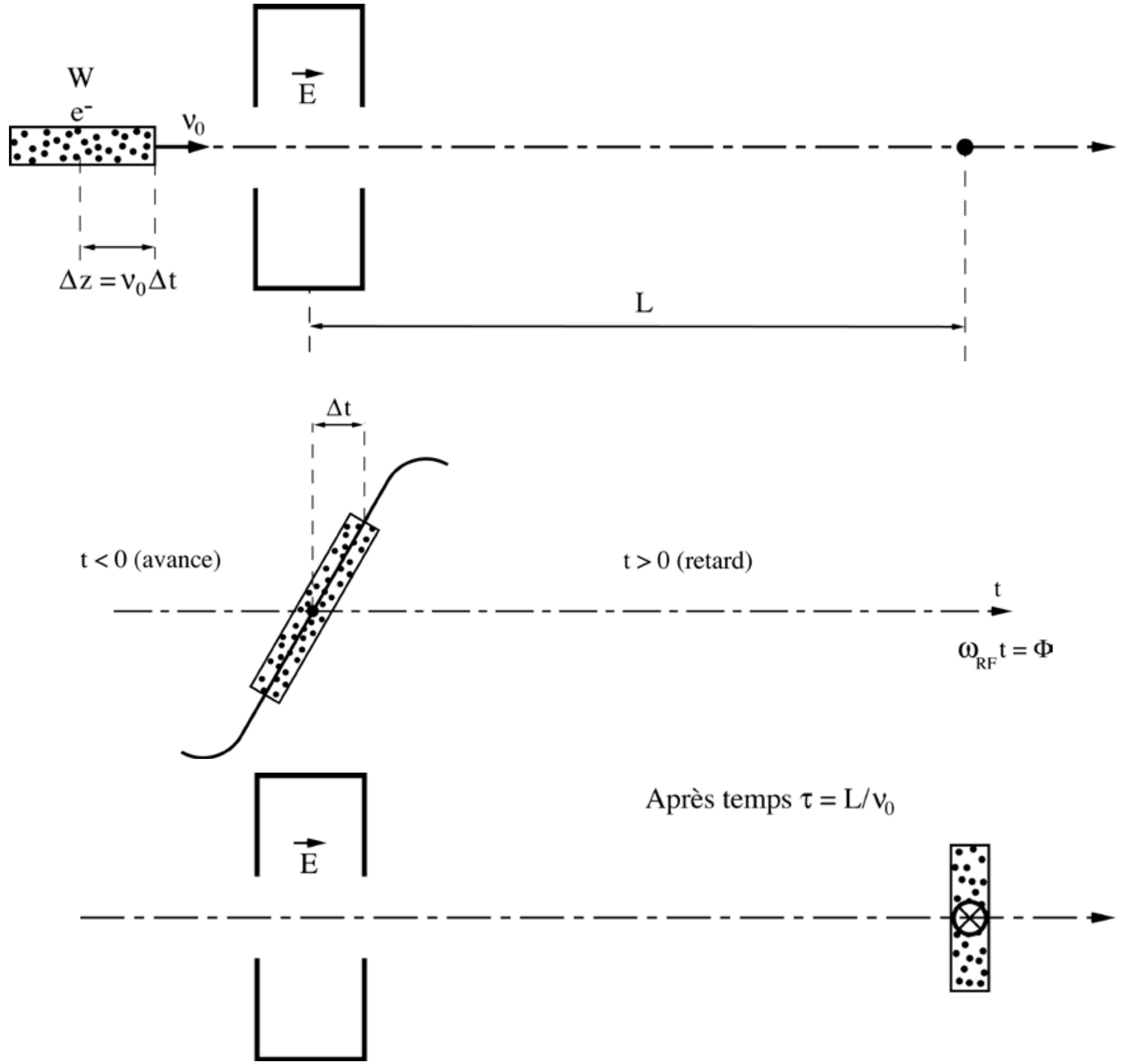
Integrating from t_0 to t (from $\beta=\beta_0$ to $\beta=1$) $\longrightarrow$ $$\cos\phi_0 - \cos\phi = \frac{2\pi m_0 c^2}{e\lambda_g E_0}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}} \le 2$$

Capture condition $\longrightarrow$ $$E_0 \ge \frac{\pi m_0 c^2}{e\lambda_g}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}}$$

Bunching with a Pre-buncher

A long bunch coming from the gun enters an RF cavity.
The reference particle is the one which has no velocity change. The others get accelerated or decelerated, so the bunch gets an energy and velocity modulation.

After a distance L bunch gets shorter: bunching effect.
This short bunch can now be captured more efficiently by a TW structure (v_φ=c).



Bunching with a Pre-buncher (2)

The bunching effect is a space modulation caused by a velocity modulation, similar to the phase stability phenomenon. Let's look at the particles in the vicinity of the reference and use a classical approach.

Energy gain as a function of cavity crossing time:

$$\Delta W = \Delta\left(\frac{1}{2}m_0 v^2\right) = m_0 v_0 \Delta v = eV_0 \sin\phi \approx eV_0\phi \qquad \Delta v = \frac{eV_0\phi}{m_0 v_0}$$

Perfect linear bunching will occur after a time delay τ, corresponding to a distance L, when the path difference is compensated between a particle and the reference one:

$$\Delta v\, \tau = \Delta z = v_0 \Delta t = v_0 \frac{\phi}{\omega_{RF}}$$

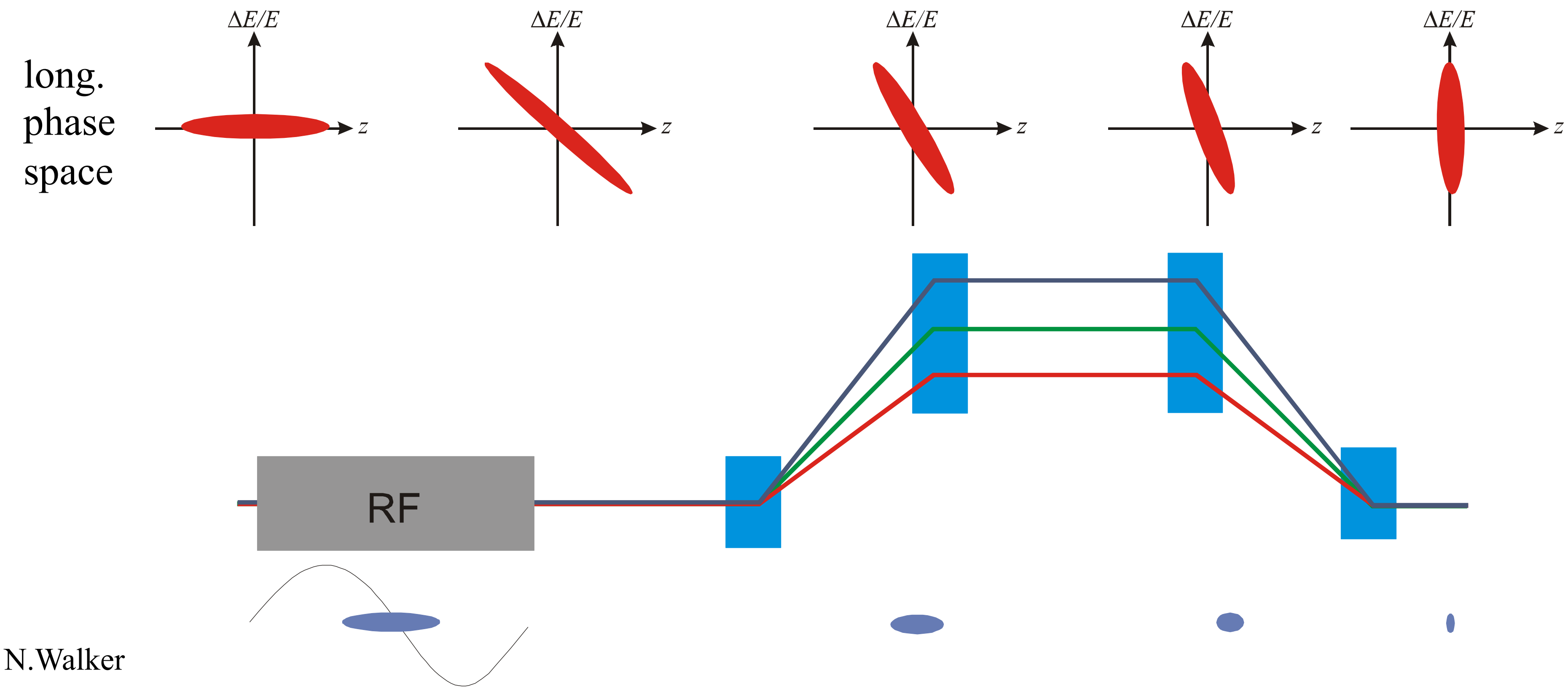
(assuming the reference particle enters the cavity at time t=0)

Since $L = v_0 \tau$ one gets:

$$\boxed{L = \frac{2 v_0 W}{eV_0 \omega_{RF}}}$$

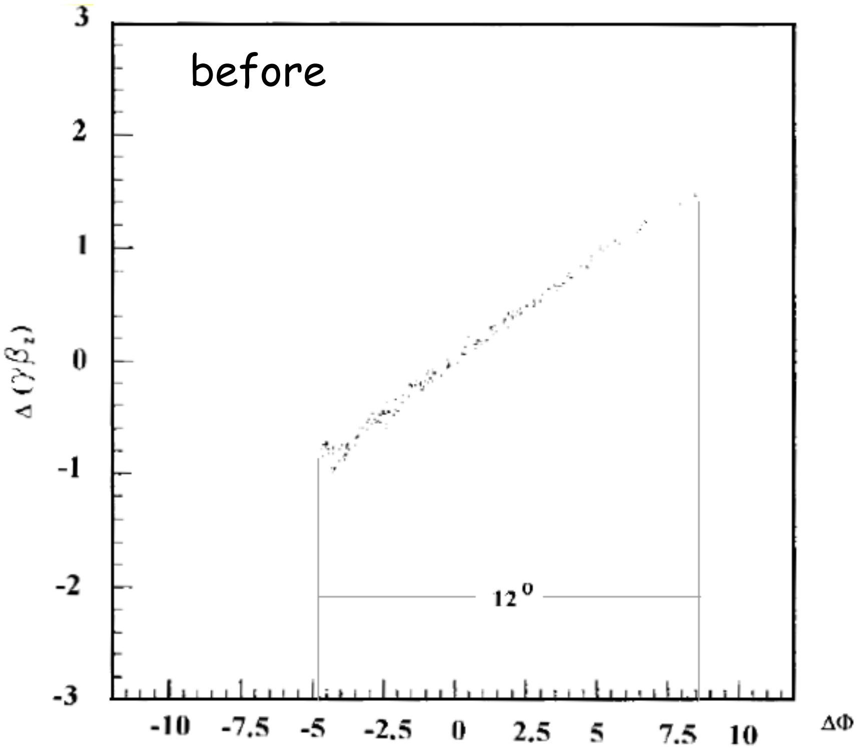
Bunch compression

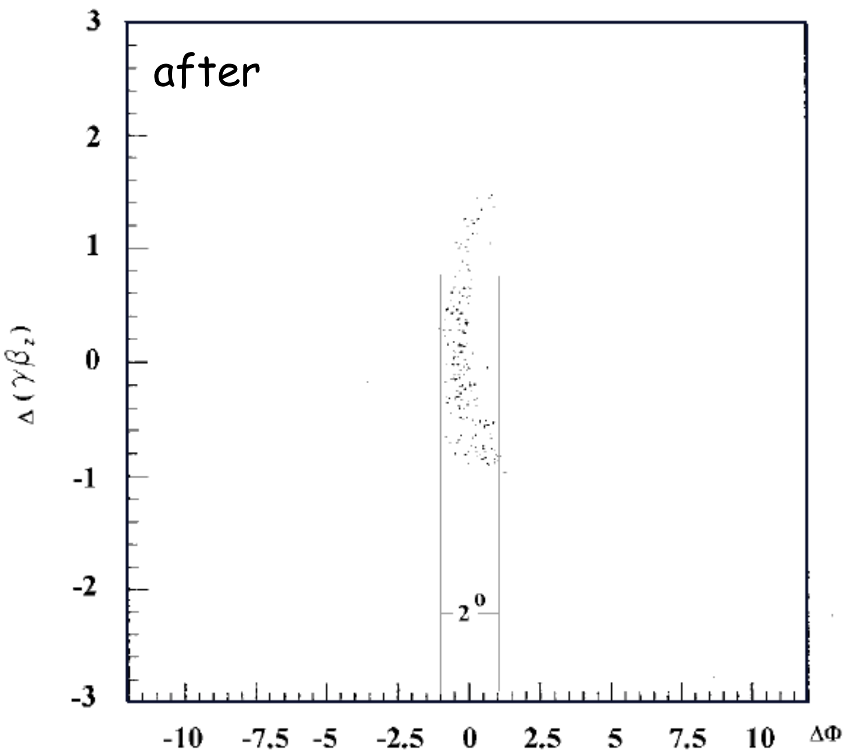
At ultra-relativistic energies ($\gamma \gg 1$) the longitudinal motion is frozen. This is rapidly the case for electrons.
For example for linear colliders, you need very short bunches (few 100-50μm).
Solution: introduce energy/time correlation with a magnetic chicane.



N.Walker

Bunch compression (2)





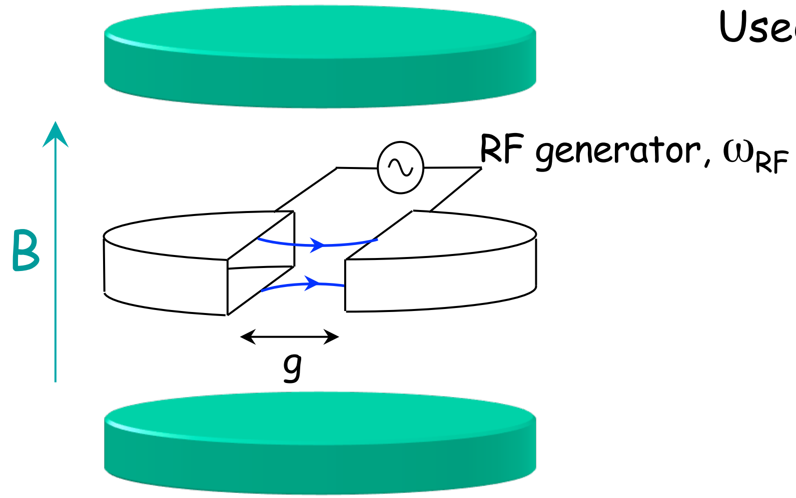
Longitudinal phase space evolution for a bunch compressor (PARMELA code simulations)

Introducing correlated energy spread increases total energy spread in the bunch. => chromatic effects (depend on relative energy spread $\Delta E/E$)

Solution: compress at low energy before further acceleration

=> absolute energy spread constant but relative is decreased

Circular accelerators: Cyclotron

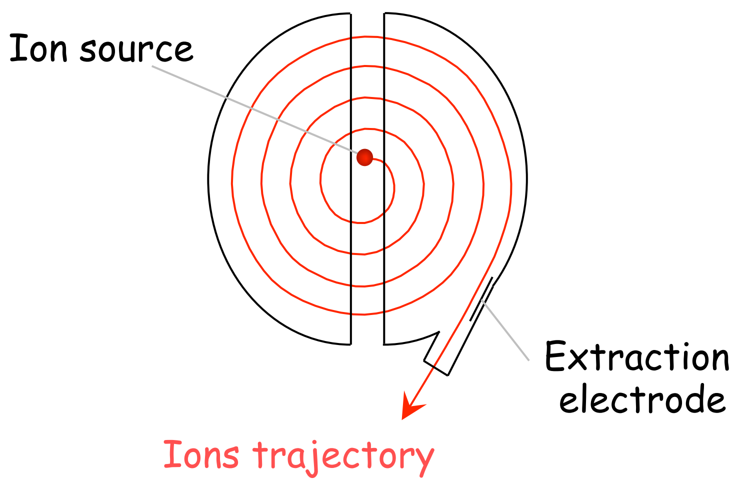


Used for protons, ions

B = constant
ω_{RF} = constant

Synchronism condition

$$\omega_s = \omega_{RF}$$
$$2\pi\,\rho = v_s\,T_{RF}$$



Cyclotron frequency $$\omega = \frac{q\,B}{m_0\,\gamma}$$

1. γ increases with the energy
   $\Rightarrow$ no exact synchronism

2. if $v \ll c \;\Rightarrow\; \gamma \cong 1$

Cyclotron / Synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma\,\omega_{RF}$ = constant

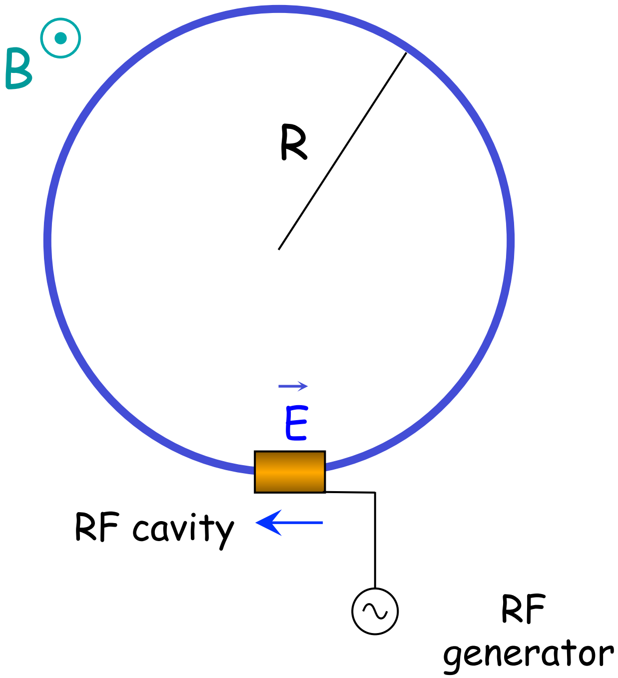
ω_{RF} decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q\,B}{m_0\,\gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: The Synchrotron



Synchronism condition

$$T_s = h\,T_{RF}$$

$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

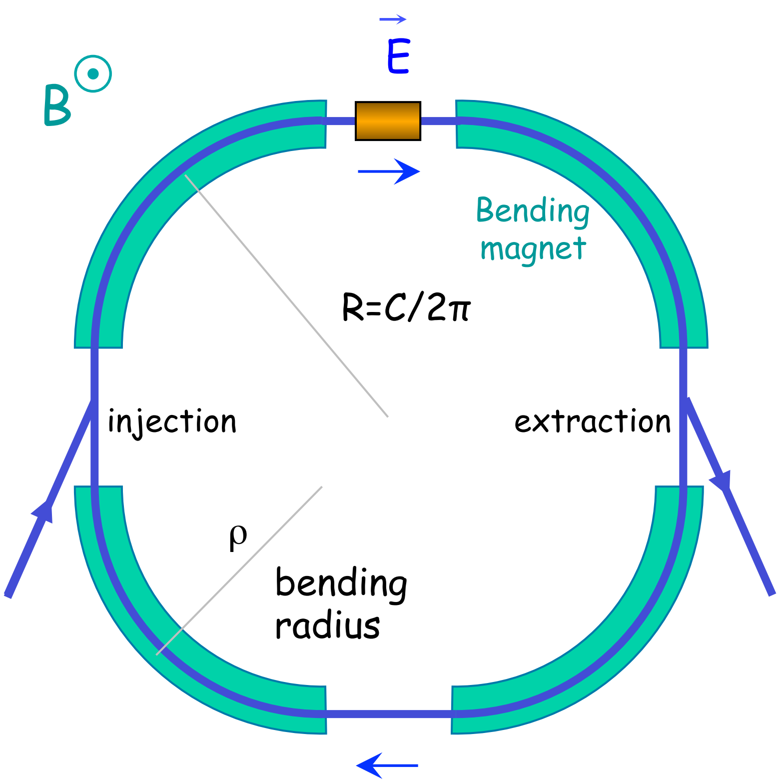
h integer,
harmonic number:
number of RF cycles per revolution

1. ω_{RF} and ω increase with energy

2. To keep particles on the closed orbit, B should increase with time

Circular accelerators: The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s = cte \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism (h - harmonic number)

$\rho = cte \quad R = cte \longrightarrow$ Constant orbit

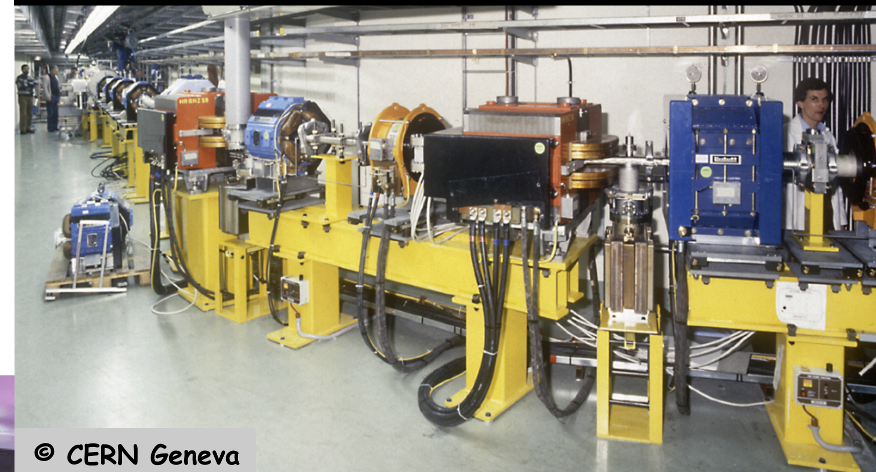
$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v≈c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

Circular accelerators: The Synchrotron

LEAR (CERN)
Low Energy Antiproton Ring

EPA (CERN)
Electron Positron Accumulator



PS (CERN)
Proton Synchrotron

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Examples of different proton and electron synchrotrons at CERN

The Synchrotron

Energy ramping is simply obtained by varying the B field (frequency follows v):

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\,\dot{B} \quad \Rightarrow (\Delta p)_{turn} = e\rho\,\dot{B}T_r = \frac{2\pi\,e\rho\,R\dot{B}}{v}$$

Since: $$E^2 = E_0^2 + p^2c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$\boxed{(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s}$$

Stable phase φ_s changes during energy ramping

$$\sin\phi_s = 2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \boxed{\phi_s = \arcsin\left(2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)}$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satisfies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

The Synchrotron

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

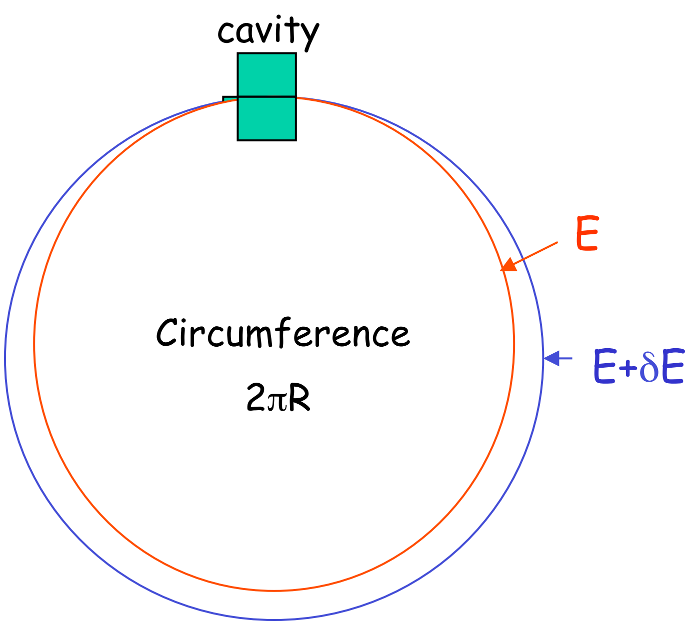
Hence: $$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{ec^2}{E_s(t)} \frac{\rho}{R_s} B(t)$$ (using $p(t) = eB(t)\rho, \quad E = mc^2$)

Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0c^2 / ec\rho)^2 + B(t)^2} \right\}^{1/2}$$

This asymptotically tends towards $f_r \to \dfrac{c}{2\pi R_s}$ when B becomes large compared to $m_0c^2 / (ec\rho)$ which corresponds to $v \to c$

Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit and the length is different.

The "momentum compaction factor" is defined as:

$$\alpha = \frac{dL/L}{dp/p} \Rightarrow \boxed{\alpha = \frac{p}{L}\frac{dL}{dp}}$$

If the particle is shifted in momentum it will have also a different velocity.
As a result of both effects the revolution frequency changes:

$$\eta = \frac{\mathrm{d}f_r/f_r}{\mathrm{d}p/p} \Rightarrow \boxed{\eta = \frac{p}{f_r}\frac{df_r}{dp}}$$

p=particle momentum

R=synchrotron physical radius

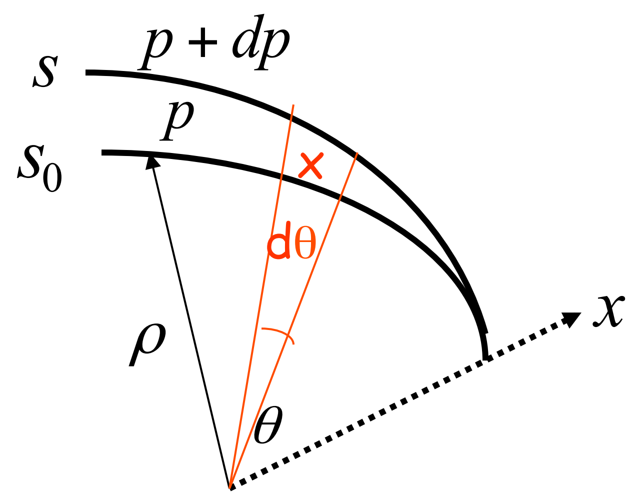
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{L}\frac{dL}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = \left(\rho + x\right)d\theta$$



The elementary path difference from the two orbits is:

definition of dispersion D_x

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} = \frac{D_x}{\rho}\frac{dp}{p}$$

leading to the total change in the circumference:

$$dL = \int_C dl = \int \frac{x}{\rho} ds_0 = \int \frac{D_x}{\rho}\frac{dp}{p} ds_0$$

$$\alpha = \frac{1}{L}\int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With ρ=∞ in straight sections we get:

$$\alpha = \frac{\langle D_x \rangle_m}{R}$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Dispersion Effects in a Synchrotron (3)

$$f_r = \frac{\beta c}{2\pi R} \quad \Rightarrow \quad \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha \frac{dp}{p}$$

definition of momentum compaction factor

$$p = mv = \beta\gamma \frac{E_0}{c} \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-1/2}}{\left(1-\beta^2\right)^{-1/2}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$$

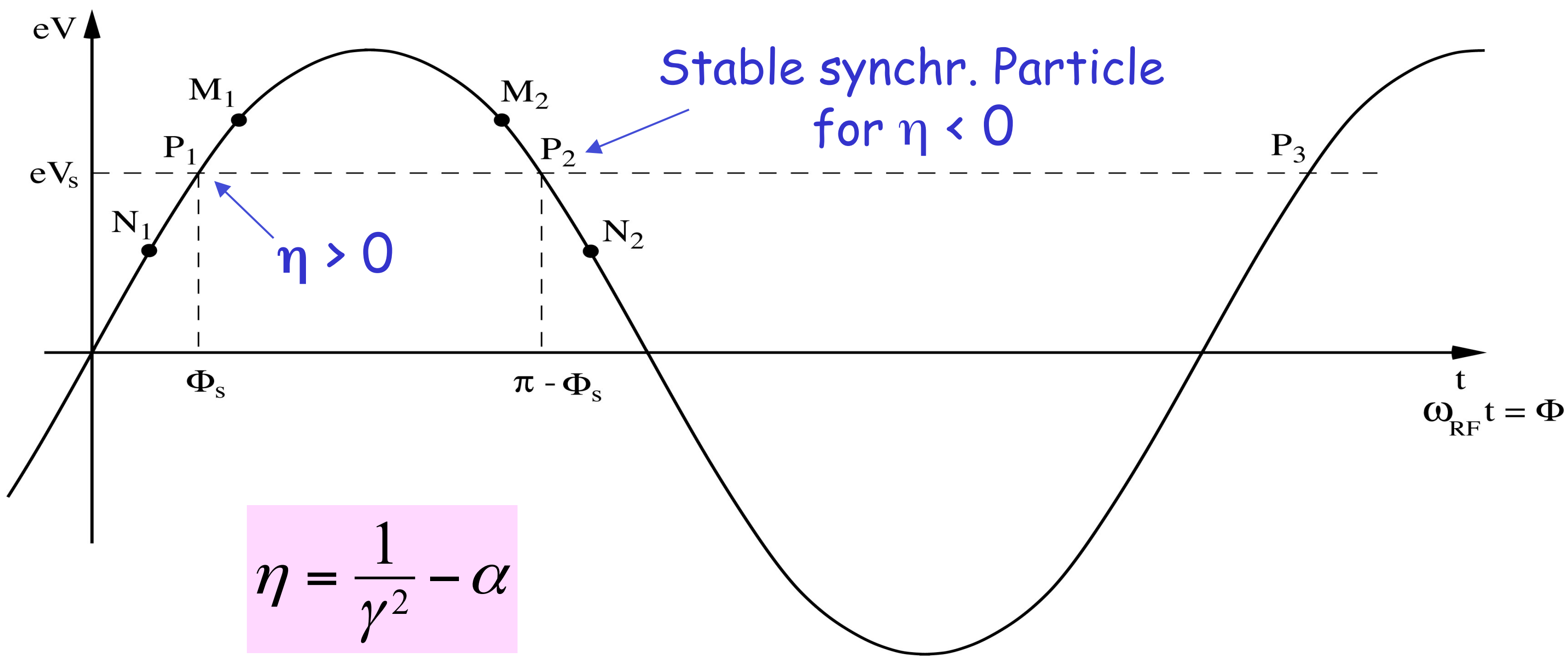
$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \xrightarrow{\frac{df_r}{f_r} = \eta \frac{dp}{p}} \quad \eta = \frac{1}{\gamma^2} - \alpha$$

η=0 at the transition energy $\quad \gamma_{tr} = \frac{1}{\sqrt{\alpha}}$

Phase Stability in a Synchrotron

From the definition of η it is clear that an increase in momentum gives
- below transition ($\eta > 0$) a higher revolution frequency (increase in velocity dominates) while

- above transition ($\eta < 0$) a lower revolution frequency ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.

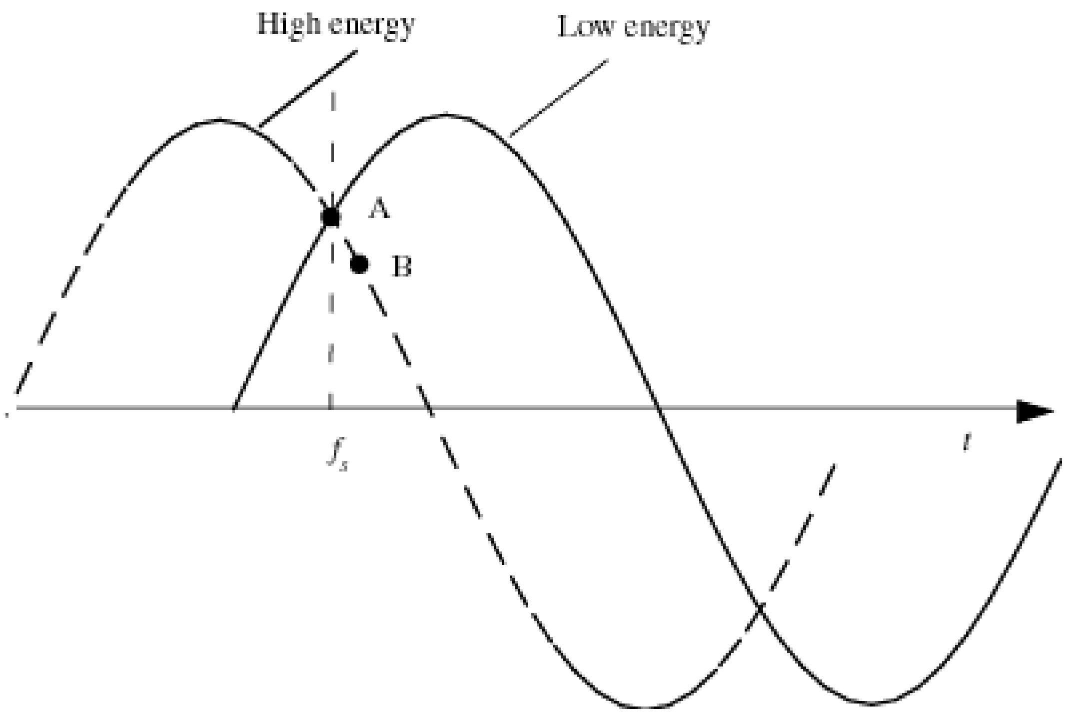


$$\eta = \frac{1}{\gamma^2} - \alpha$$

Crossing Transition

At transition, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a 'phase jump'.

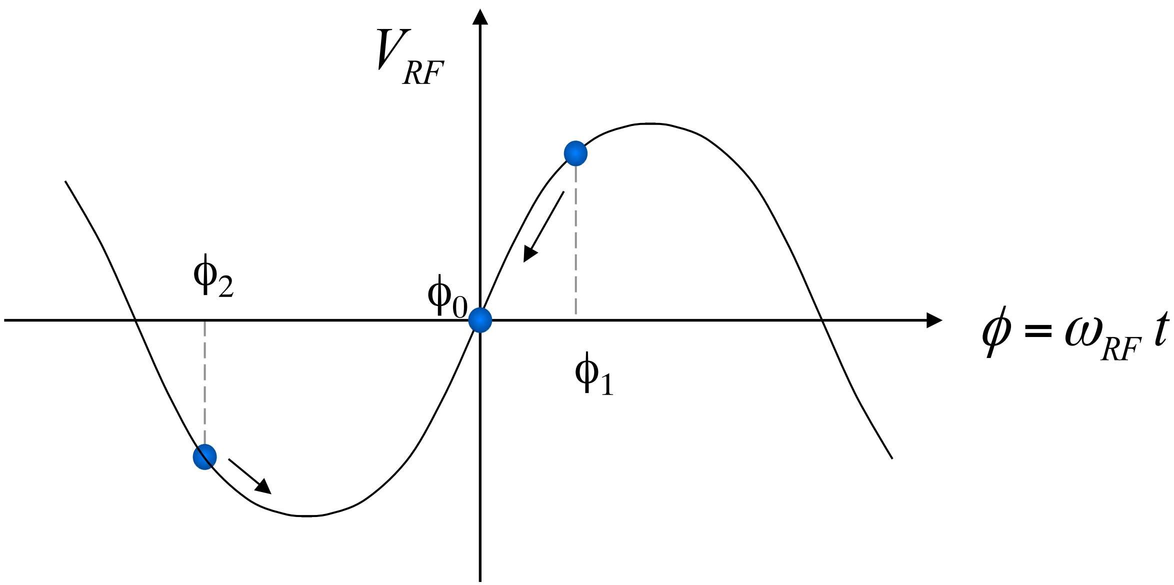


Synchrotron oscillations

Simple case (no accel.): B = const., below transition $\gamma < \gamma_{tr}$

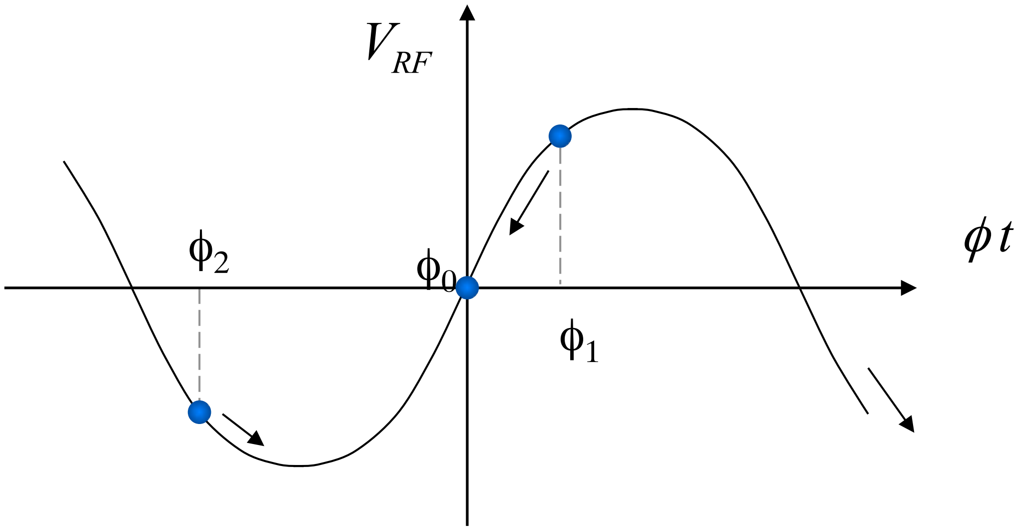
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

ϕ_1
- The particle is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier – tends toward ϕ_0

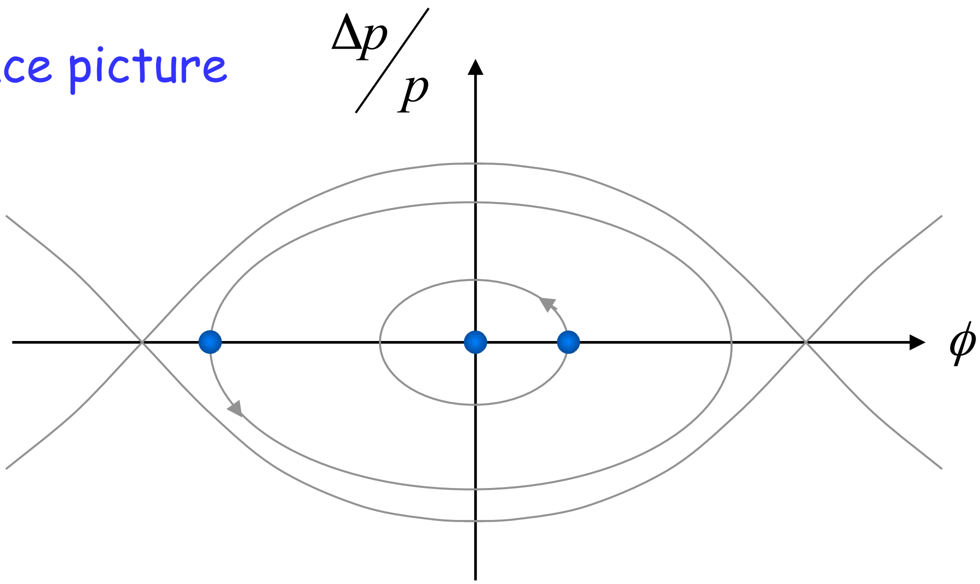


ϕ_2
- The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward ϕ_0

Synchrotron oscillations (2)

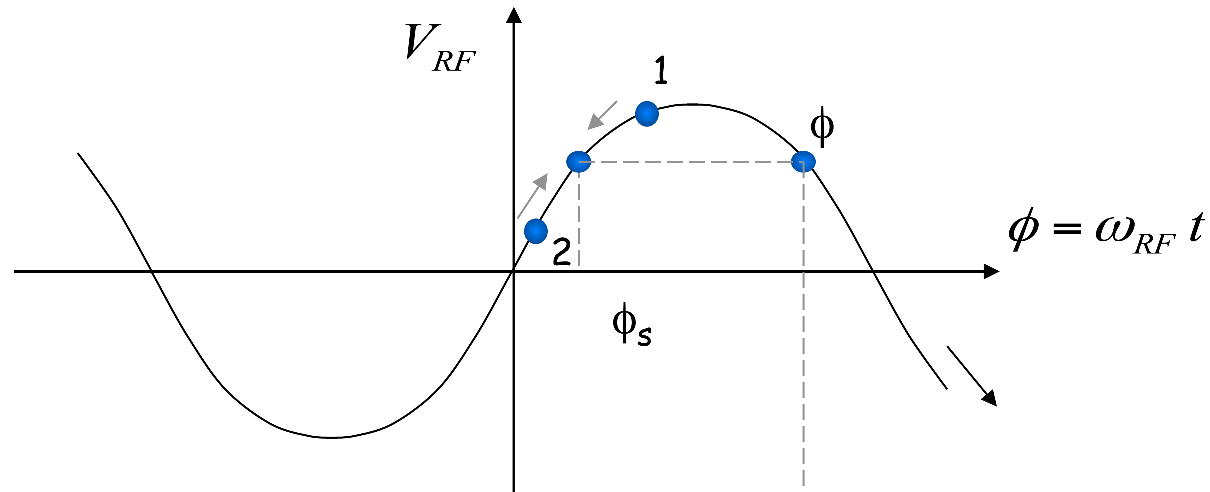


Phase space picture



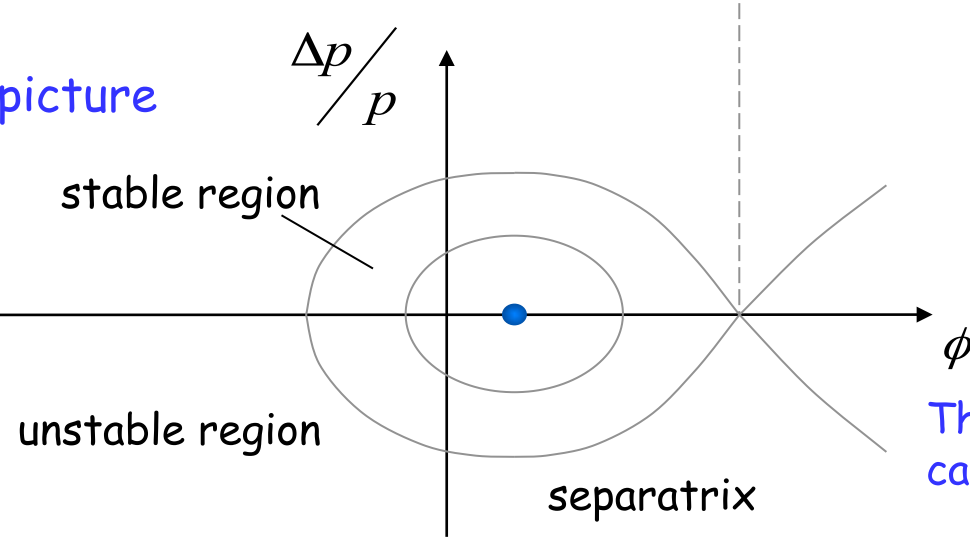
Synchrotron oscillations (3)

Case with acceleration *B increasing* $\gamma < \gamma_{tr}$



$$\phi_s < \phi < \pi - \phi_s$$

Phase space picture



The symmetry of the case B = const. is lost

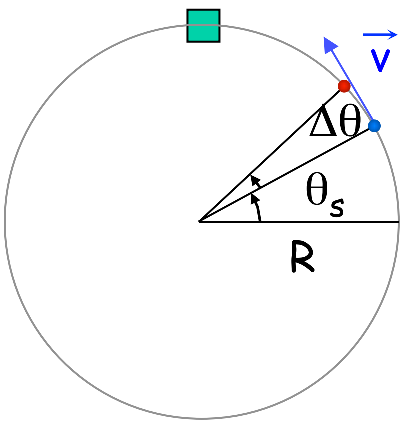
Longitudinal Dynamics

It is also often called "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle.
So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\,\Delta\theta \quad with \quad \theta = \int \omega_r \, dt$$

particle ahead arrives earlier
=> smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}\left(\Delta\theta\right) = -\frac{1}{h}\frac{d}{dt}\left(\Delta\phi\right) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$ and

$$E^2 = E_0^2 + p^2c^2$$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is: $$\frac{dE}{dt} = e\hat{V}\sin\phi\,\frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\Delta E\right)$$

leads to the second energy-phase equation:

$$\boxed{2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)}$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}} \dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}} \frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases later…

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with

$$\Omega_s^2 = \frac{h\eta\omega_{rs} e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \cong \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

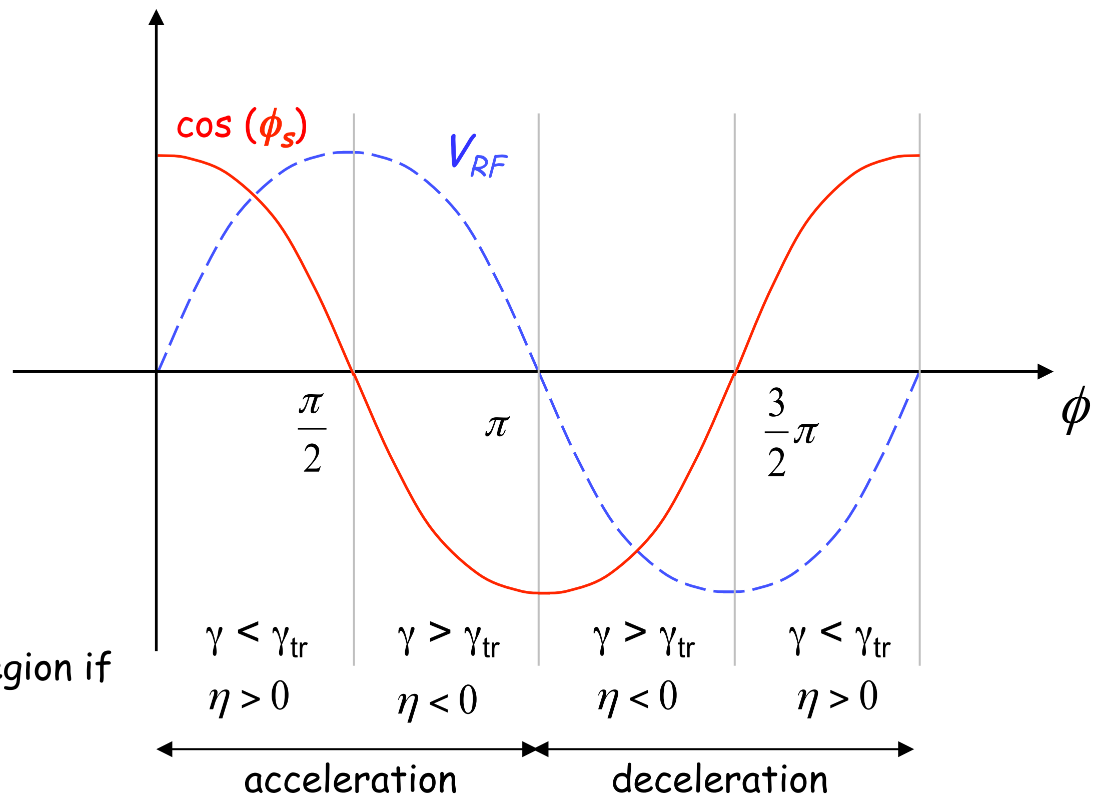
$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

where Ω_s is the synchrotron angular frequency

Stability condition for ϕ_s

Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\boxed{\Omega_s^2 = \frac{e\hat{V}_{RF}\eta h\omega_s}{2\pi R_s p_s}\cos\phi_s} \quad\Rightarrow\quad \Omega_s^2 > 0 \quad\Leftrightarrow\quad \boxed{\eta\cos\phi_s > 0}$$



Stable in the region if

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

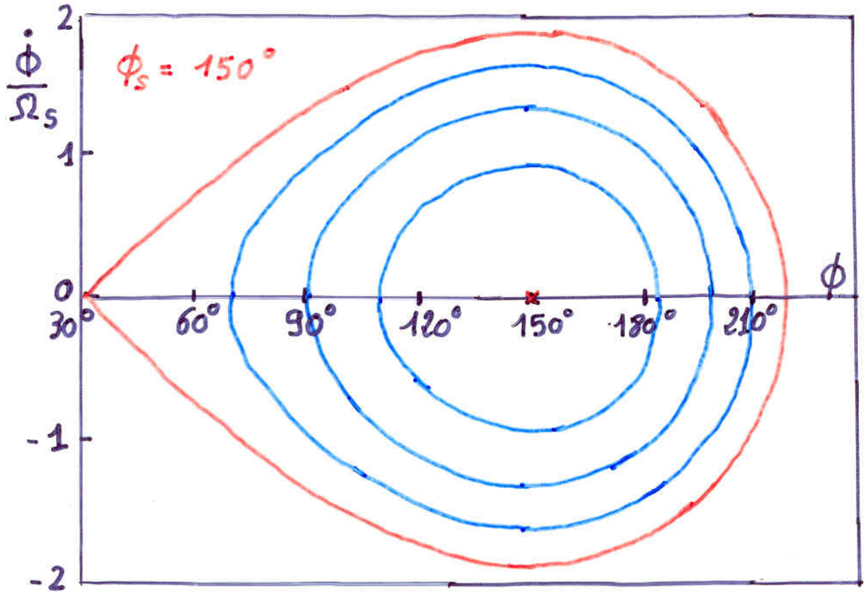
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{\left(\Delta\phi\right)^2}{2} = I'$$

(the variable is $\Delta\phi$, and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches π-ϕ_s the force goes to zero and beyond it becomes non restoring.
Hence π-ϕ_s is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s)\sin\phi_s$$

Area within this separatrix is called "RF bucket".

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi \right) \tan\phi_s \right\}$$

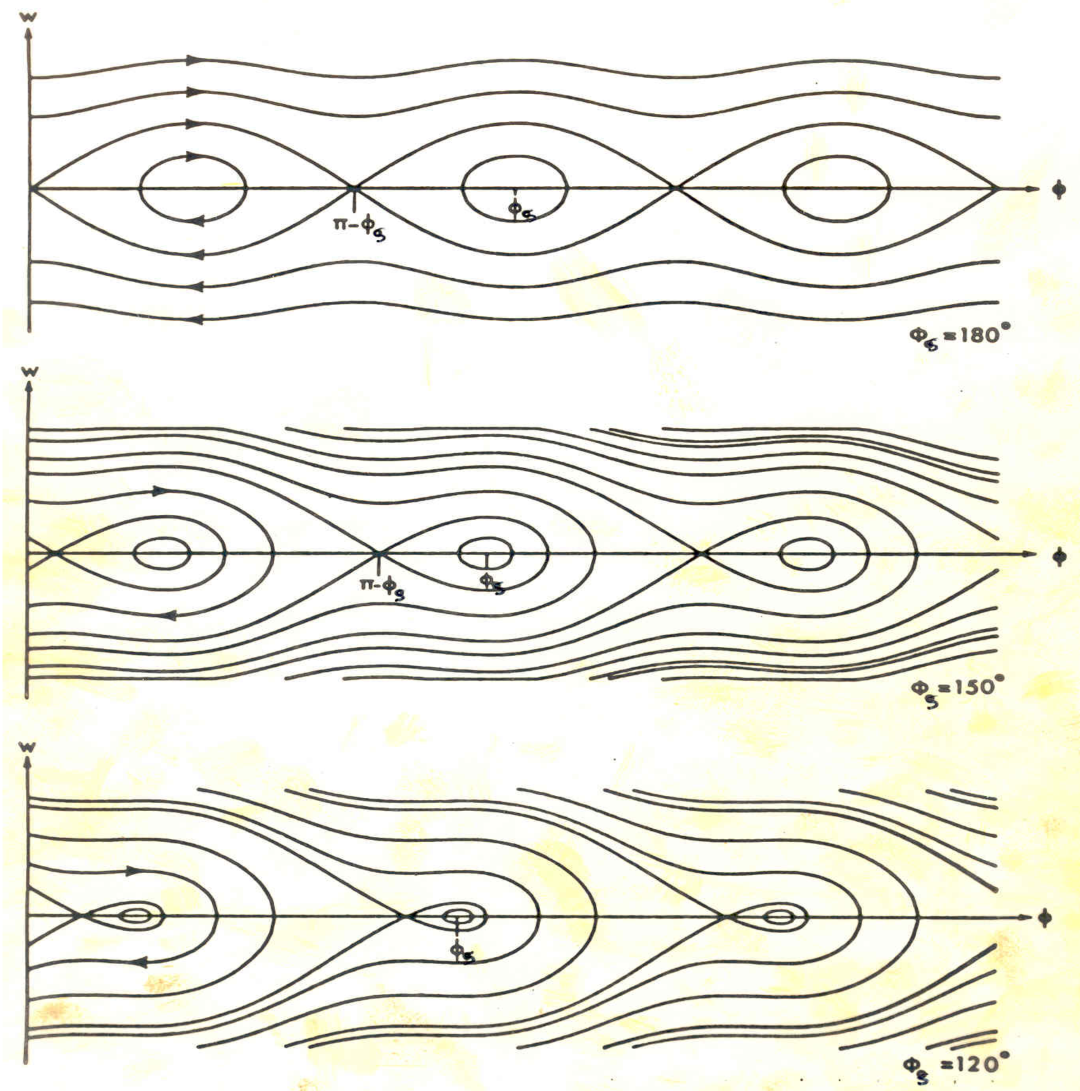
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s} \right)_{\max} = \mp\beta \sqrt{ -\frac{e\hat{V}}{\pi h \eta E_s} G\left(\phi_s\right) }$$

$$G\left(\phi_s\right) = \left[2\cos\phi_s + \left(2\phi_s - \pi \right) \sin\phi_s \right]$$

This “RF acceptance” depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



The areas of stable motion (closed trajectories) are called "BUCKET".

As the synchronous phase gets closer to 90° the buckets gets smaller.

The number of circulating buckets is equal to "h".

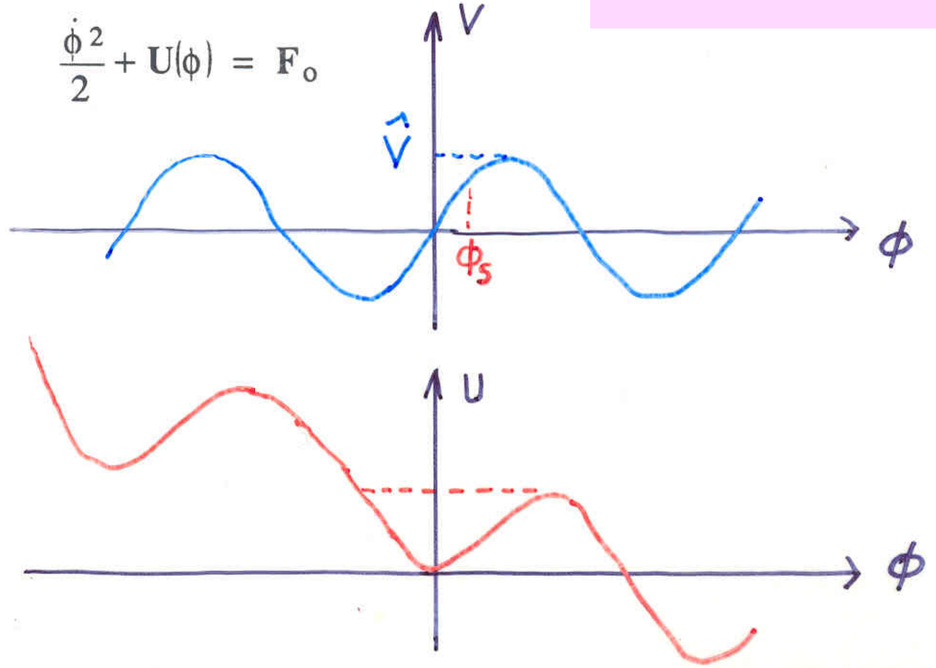
The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^{\phi} F(\phi)\, d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W, leads to the 1st order equations:

$$W = 2\pi\left(\frac{\Delta E}{\omega_{rs}}\right) = 2\pi R_s \Delta p \longrightarrow \begin{aligned} \frac{d\phi}{dt} &= -\frac{1}{2\pi}\frac{h\eta\omega_{rs}}{p_s R_s} W \\ \frac{dW}{dt} &= e\hat{V}\left(\sin\phi - \sin\phi_s\right) \end{aligned}$$

The two variables ϕ,W are canonical since these equations of motion can be derived from a Hamiltonian H(ϕ,W,t):

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W, t) = e\hat{V}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right] - \frac{1}{4\pi}\frac{h\eta\omega_{rs}}{R_s p_s} W^2$$

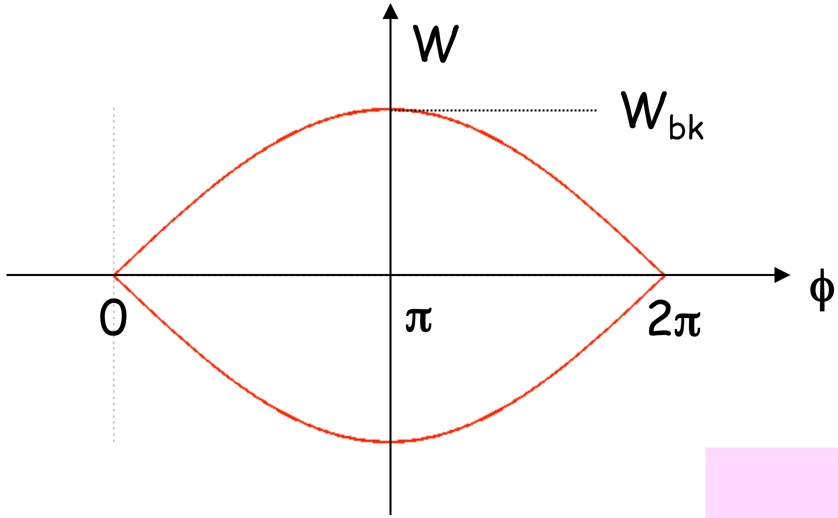
Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s= \pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2}+\Omega_s^2\cos\phi=\Omega_s^2$$

$$\frac{\dot{\phi}^2}{2}=2\Omega_s^2\sin^2\frac{\phi}{2}$$

Replacing the phase derivative by the canonical variable W:



$$W=2\pi\frac{\Delta E}{\omega_{rs}}=-2\pi\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

with $C=2\pi R_s$

$$W=\pm 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}\sin\frac{\phi}{2}=\pm W_{bk}\sin\frac{\phi}{2}$$

Stationnary Bucket (2)

Setting $\phi=\pi$ in the previous equation gives the height of the bucket:

$$W_{bk} = 2\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}}$$

This results in the maximum energy acceptance:

$$\Delta E_{\max} = \frac{\omega_{rs}}{2\pi}W_{bk} = \beta_s\sqrt{2\frac{-e\hat{V}_{RF}E_s}{\pi\eta h}}$$

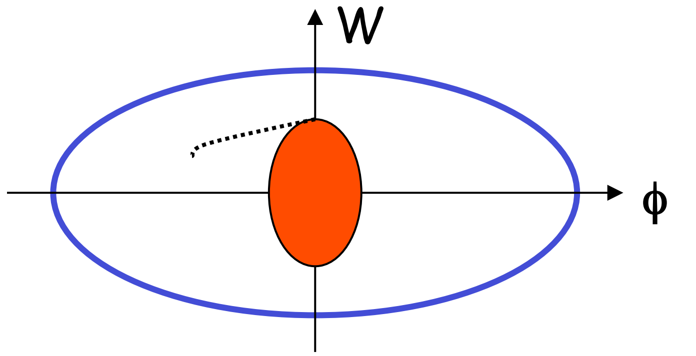
The area of the bucket is: $A_{bk} = 2\int_0^{2\pi} W\,d\phi$

Since: $\int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi = 4$

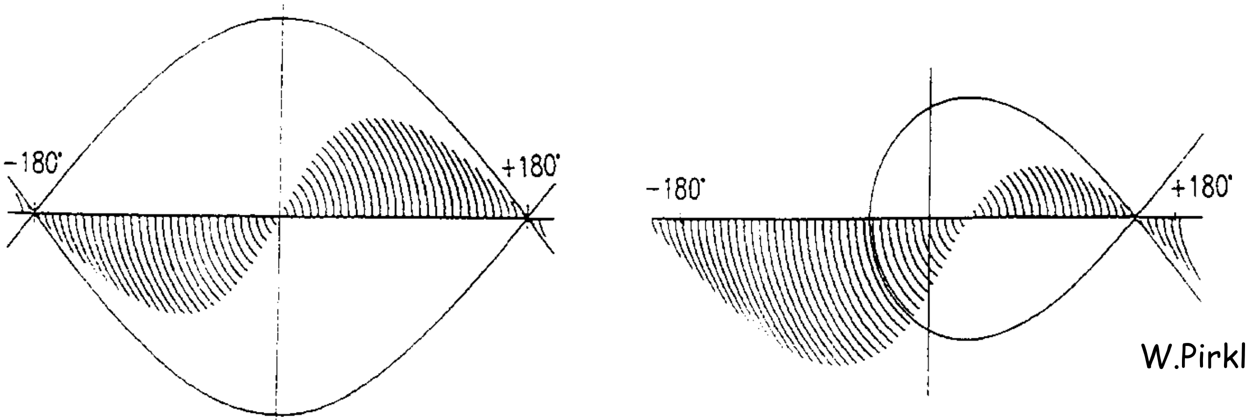
one gets: $$A_{bk} = 8W_{bk} = 16\frac{C}{c}\sqrt{\frac{-e\hat{V}E_s}{2\pi h\eta}} \longrightarrow W_{bk} = \frac{A_{bk}}{8}$$

Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below)   => can lead to filamentation and emittance growth

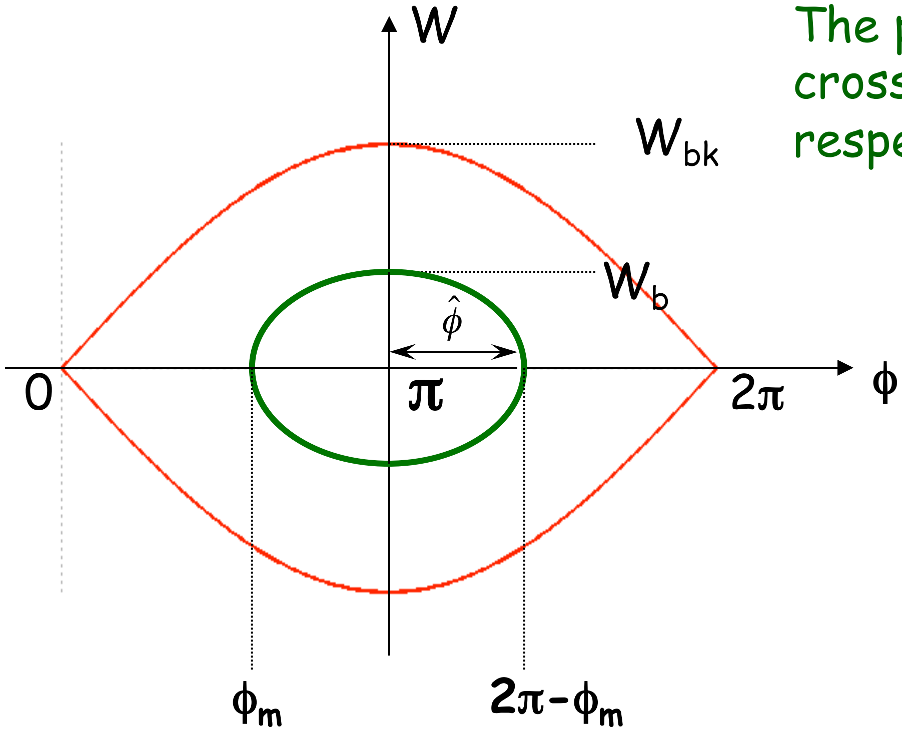


stationary bucket   accelerating bucket

Bunch Matching into a Stationnary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I \quad \xrightarrow{\phi_s = \pi} \quad \boxed{\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = I}$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\cos\phi = \Omega_s^2\cos\phi_m$$

$$\dot{\phi} = \pm\Omega_s\sqrt{2\left(\cos\phi_m - \cos\phi\right)}$$

$$\boxed{W = \pm W_{bk}\sqrt{\cos^2\frac{\varphi_m}{2} - \cos^2\frac{\varphi}{2}}}$$

$$\cos(\phi) = 2\cos^2\frac{\phi}{2} - 1$$

Bunch Matching into a Stationnary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk}\cos\frac{\phi_m}{2} = W_{bk}\sin\frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8}\cos\frac{\phi_m}{2}$$

$$\longrightarrow \quad \left(\frac{\Delta E}{E_s}\right)_b = \left(\frac{\Delta E}{E_s}\right)_{RF}\cos\frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s}\right)_{RF}\sin\frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a shorter bunch (ϕ_m close to π, $\hat{\phi}$ small) will require a bigger RF acceptance, hence a higher voltage

For small oscillation amplitudes the equation of the ellipse reduces to:

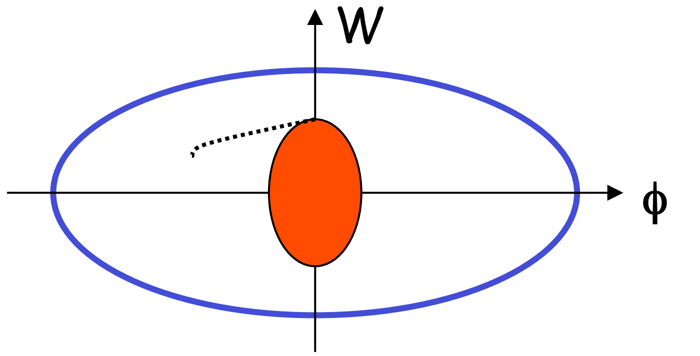
$$W = \frac{A_{bk}}{16}\sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \quad \longrightarrow \quad \left(\frac{16W}{A_{bk}\hat{\phi}}\right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}}\right)^2 = 1$$

Ellipse area is called longitudinal emittance

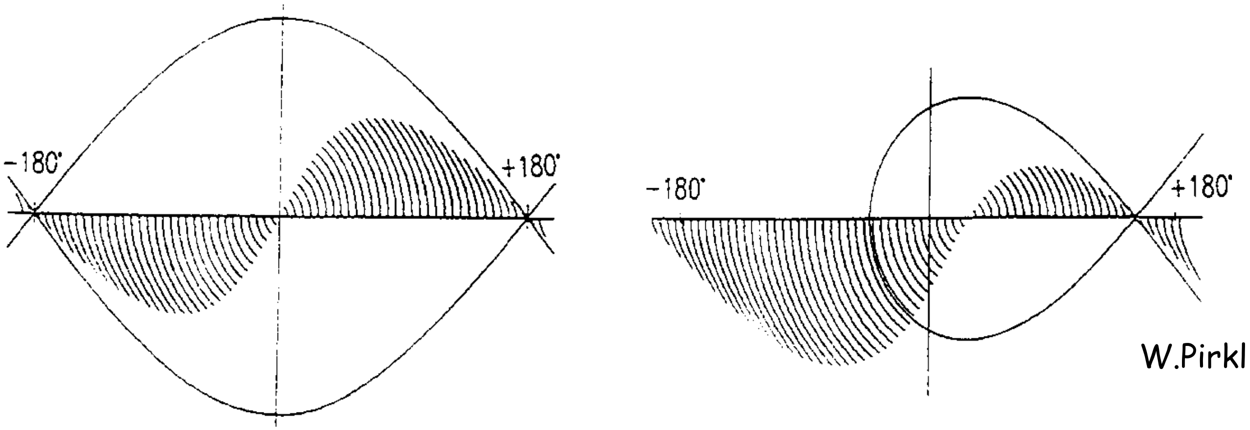
$$A_b = \frac{\pi}{16}A_{bk}\hat{\phi}^2$$

Effect of a Mismatch

Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.

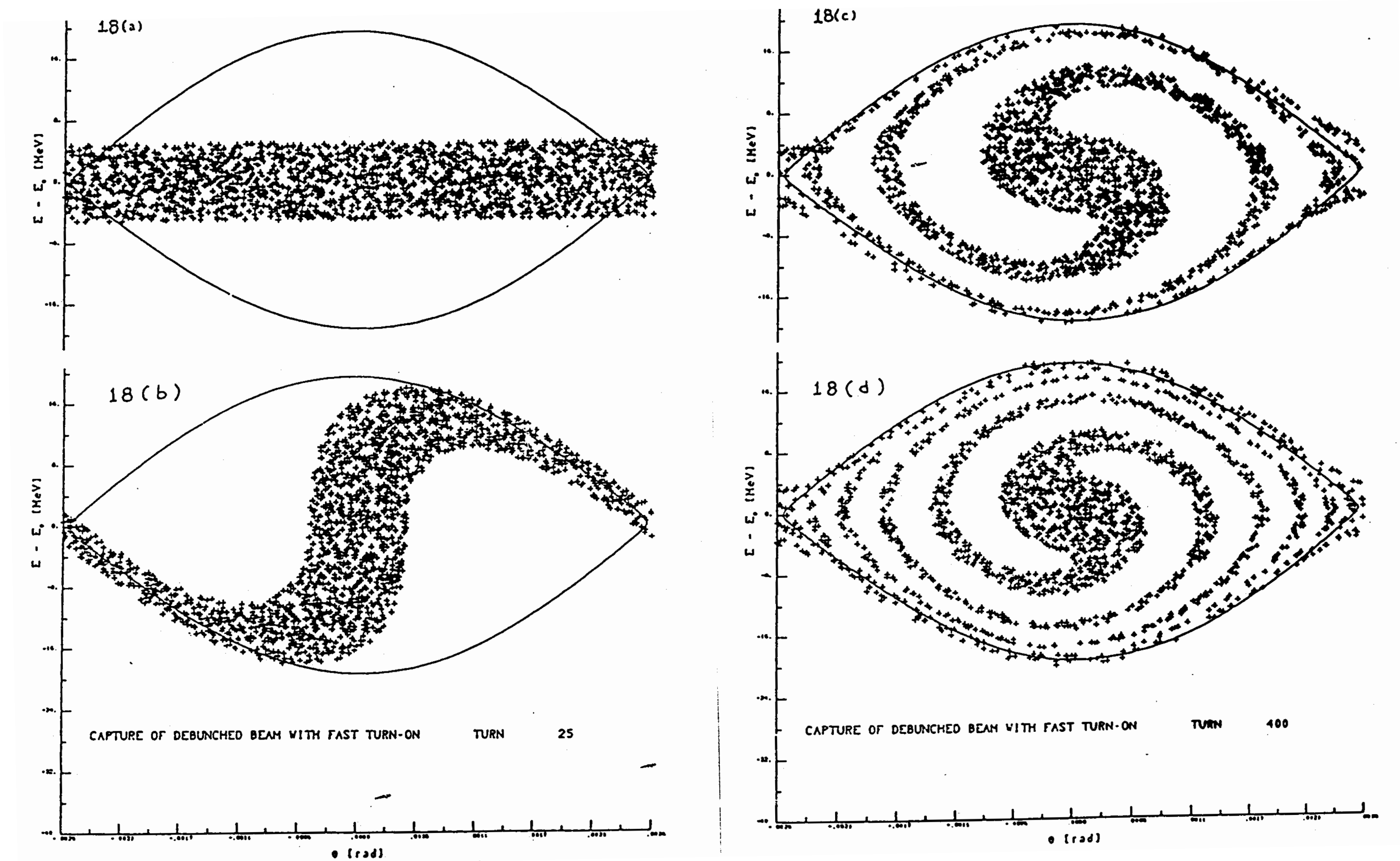


For larger amplitudes, the angular phase space motion is slower
(1/8 period shown below)   => can lead to filamentation and emittance growth



stationary bucket   accelerating bucket

Capture of a Debunched Beam with Fast Turn-On



Capture of a Debunched Beam with Adiabatic Turn-On