Introduction to Transverse Beam Dynamics

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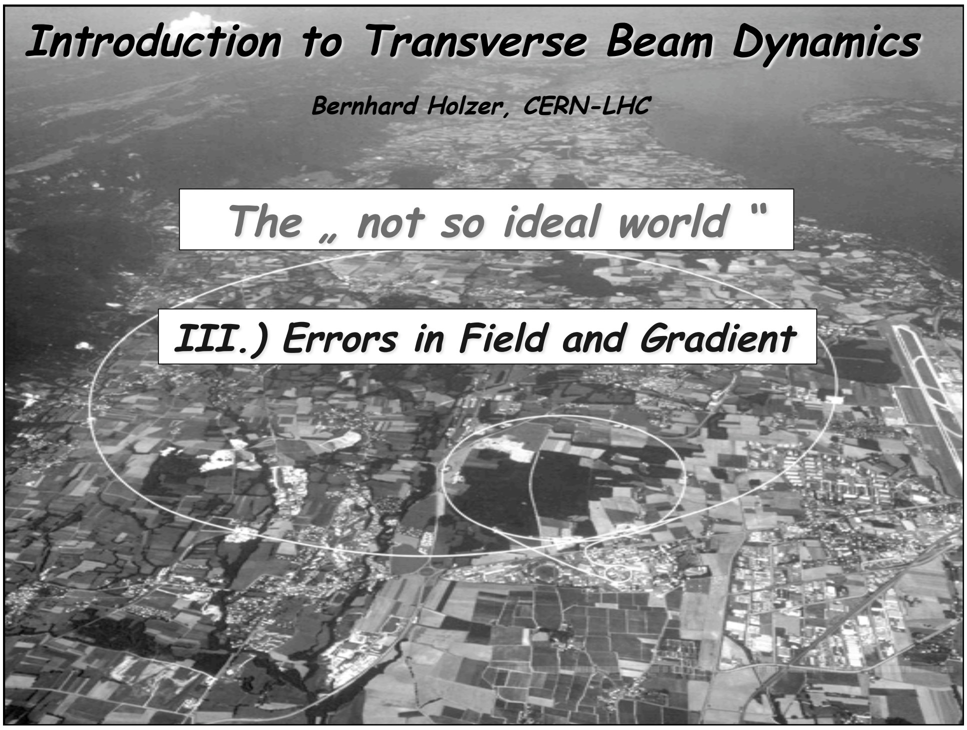
The „ not so ideal world "

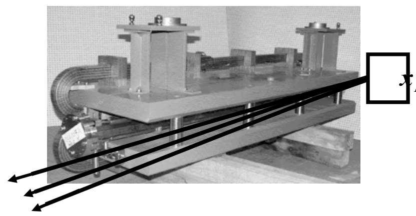
III.) Errors in Field and Gradient



19.) Chromaticity: A Quadrupole Error for Δp/p ≠ 0

Influence of external fields on the beam: prop. to magn. field & prop. zu 1/p

dipole magnet $\alpha = \frac{\int B\, dl}{p/e}$



$x_D(s) = D(s)\,\frac{\Delta p}{p}$

focusing lens $k = \frac{g}{p/e}$

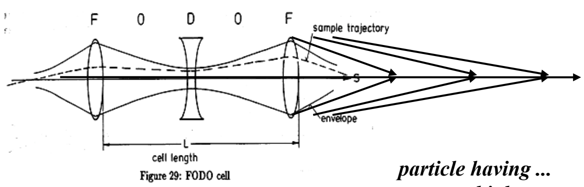


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

Chromaticity: Q'

$$k = \frac{g}{p/e} \qquad\qquad p = p_0 + \Delta p$$

in case of a momentum spread:

$$\boldsymbol{k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0}(1 - \frac{\Delta p}{p_0})\, g = k_0 + \Delta k}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

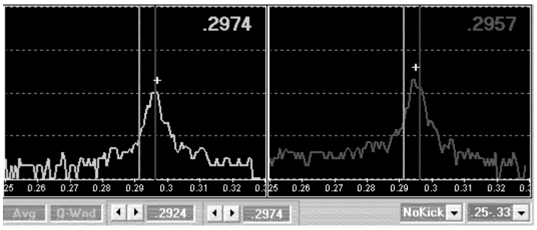
$$\boldsymbol{\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0} k_0 \beta(s) ds}$$

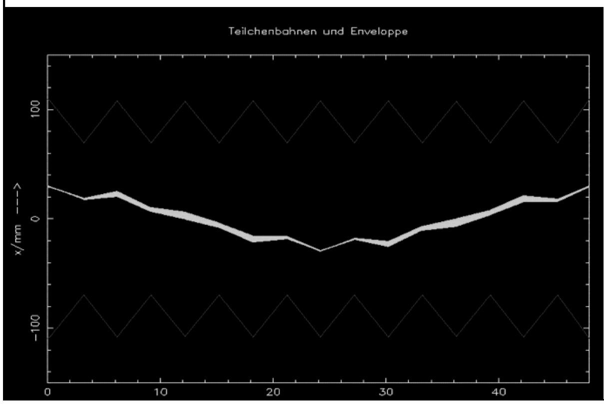
definition of chromaticity:

$$\Delta Q = Q' \frac{\Delta p}{p} \quad ; \qquad Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

Where is the Problem ?

Tunes and Resonances





avoid resonance conditions:

$$m\,Q_x + n\,Q_y + l\,Q_s = \text{integer}$$

… for example: 1 Q_x=1

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

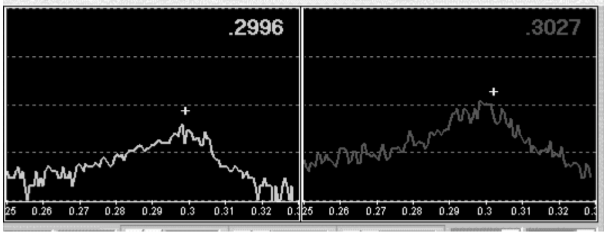
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

Q' = 250
$\Delta p/p$ = +/- 0.2 *10^{-3}
ΔQ = 0.256 … 0.36

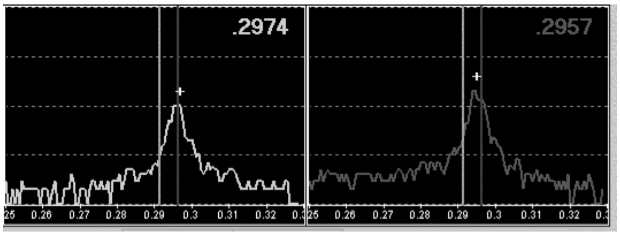
→Some particles get very close to resonances and are lost

in other words: the tune is not a point it is a pancake



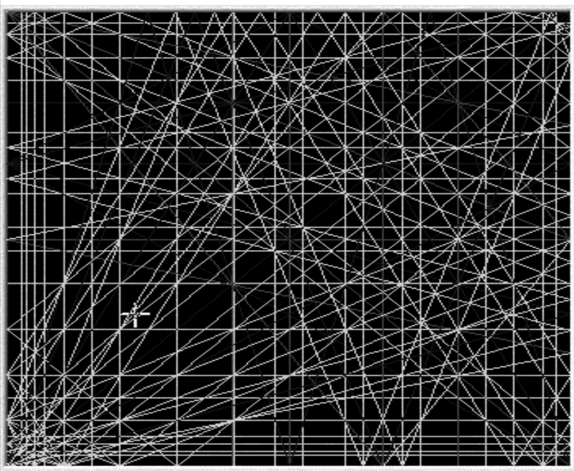
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$



RA e Tune diagram up to 3rd order

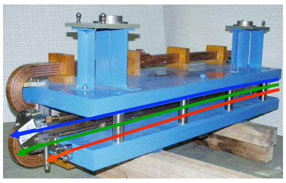
… and up to 7th order

Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

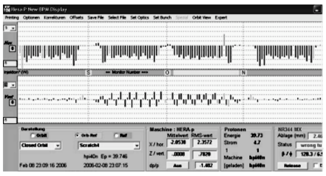
Correction of Q':

Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum $x_D(s) = D(s)\frac{\Delta p}{p}$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \qquad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

linear rising „gradient":

Correction of Q':

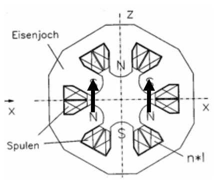
k_1 normalised quadrupole strength

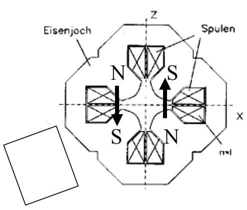
k_2 normalised sextupole strength

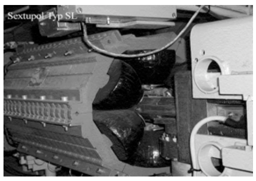
Sextupole Magnets:

$$k_1(sext) = \frac{\tilde{g}\,x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$







Combined effect of „natural chromaticity" and Sextupole Magnets:

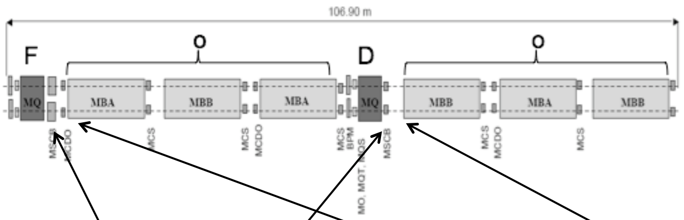
$$Q' = -\frac{1}{4\pi}\left\{\int k_1(s)\beta(s)ds + \int k_2 * D(s)\beta(s)ds\right\}$$

You only should not forget to correct Q' in both planes ...

and take into account the contribution from quadrupoles of both polarities.

corrected chromaticity

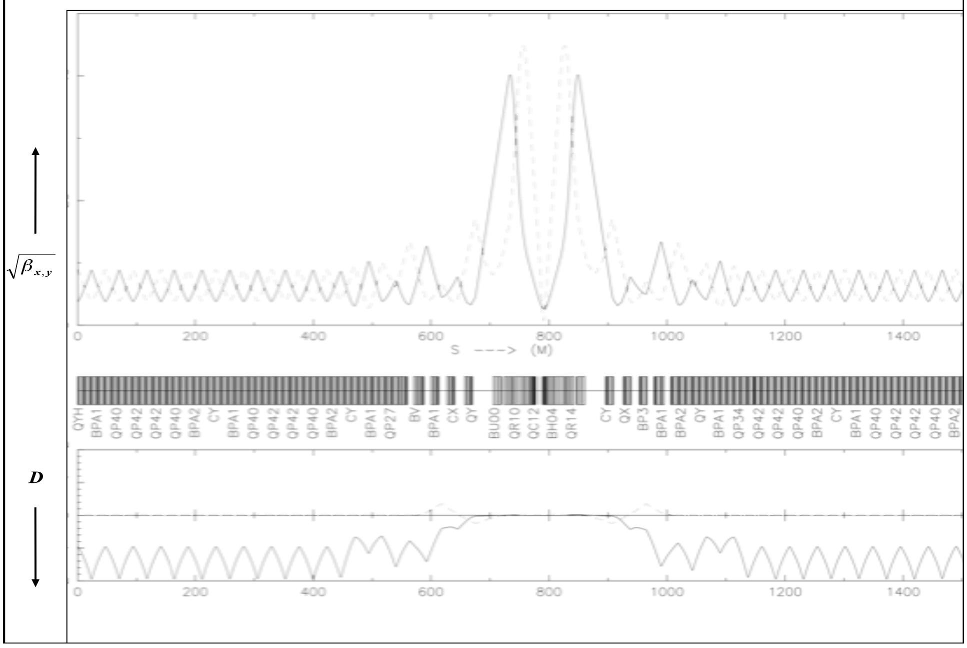
considering an arc built out of single cells:



$$Q'_x = -\frac{1}{4\pi}\left\{\sum_{F\,quad} k_{qf}\hat{\beta}_x l_{qf} - \sum_{D\,quad} k_{qd}\bar{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_y = -\frac{1}{4\pi}\left\{-\sum_{F\,quad} k_{qf}\bar{\beta}_y l_{qf} + \sum_{D\,quad} k_{qd}\hat{\beta}_y l_{qd}\right\} - \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F + \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

20.) Insertions



Insertions

... the most complicated one: the drift space

Question to the audience: what will happen to the beam parameters α, β, γ if we stop focusing for a while …?

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_S = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_0$$

transfer matrix for a drift:

$$M = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{aligned} \beta(s) &= \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \\ \alpha(s) &= \alpha_0 - \gamma_0 s \\ \gamma(s) &= \gamma_0 \end{aligned}$$

β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

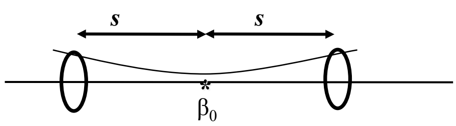
$$as \quad \alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$$

and we get for the β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0} \qquad !!!$$

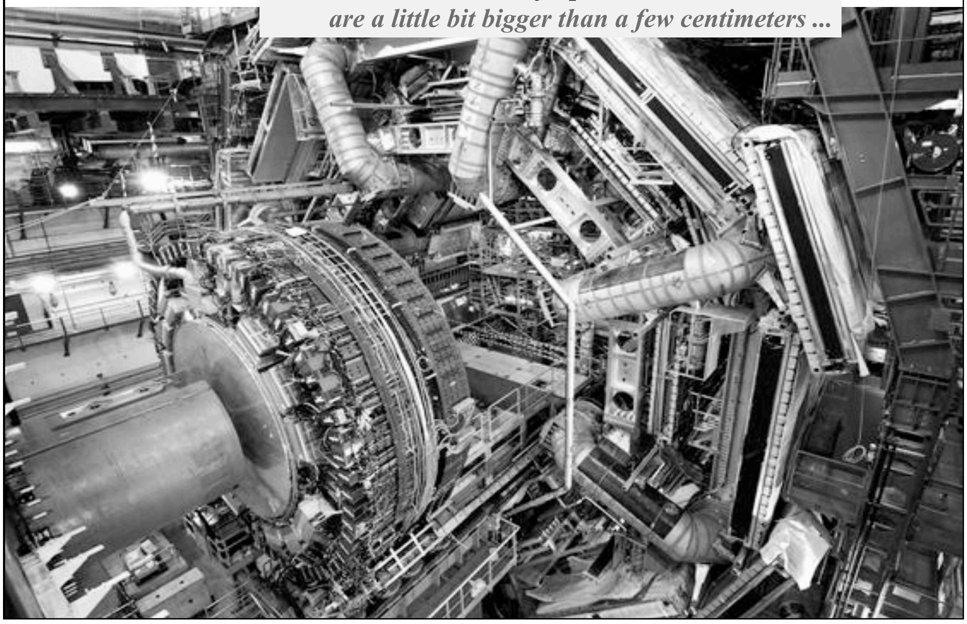
At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice. -> here we get the largest beam dimension.

-> keep l as small as possible

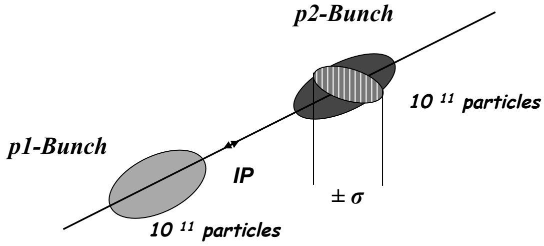


... clearly there is an

But: ... unfortunately ... in general
high energy detectors ***that are***
installed in that drift spaces
are a little bit bigger than a few centimeters ...



21.) Luminosity



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$ $\qquad f_0 = 11.245\ kHz$

$\varepsilon_{x,y} = 5*10^{-10}\ rad\ m$ $\qquad n_b = 2808$

$\sigma_{x,y} = 17\ \mu m$

$I_p = 584\ mA$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

$$L = 1.0*10^{34}\ {}^{1}/_{cm^2 s}$$

Mini-β Insertions: some guide lines

**** calculate the periodic solution in the arc***

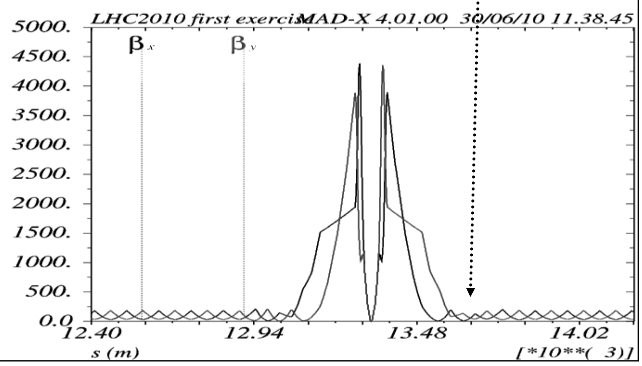
**** introduce the drift space needed for the insertion device (detector ...)***

**** put a quadrupole doublet (triplet ?) as close as possible***

**** introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure***

parameters to be optimised & matched to the periodic solution: $\alpha_x,\ \beta_x \quad D_x,\ D_x'$ / $\alpha_y,\ \beta_y \quad Q_x,\ Q_y$

8 individually powered quad magnets are needed to match the insertion (... at least)



beam sizes in the order of my cat's hair !!

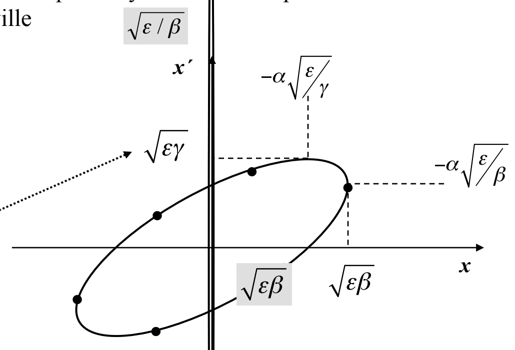
Mini-β Insertions: Betafunctions

A mini-β insertion is always a kind of special symmetric drift space.
→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

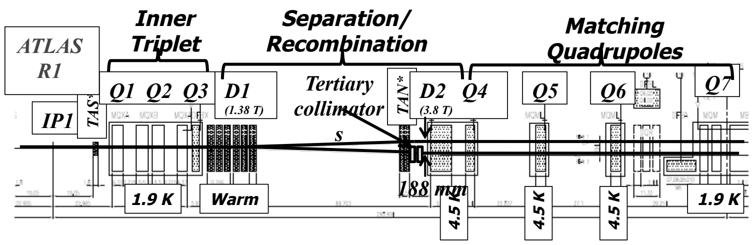
$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$

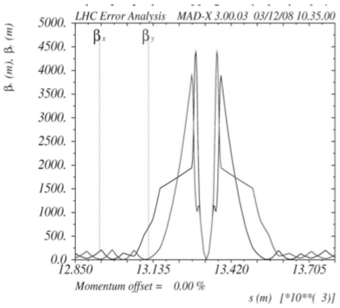


$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

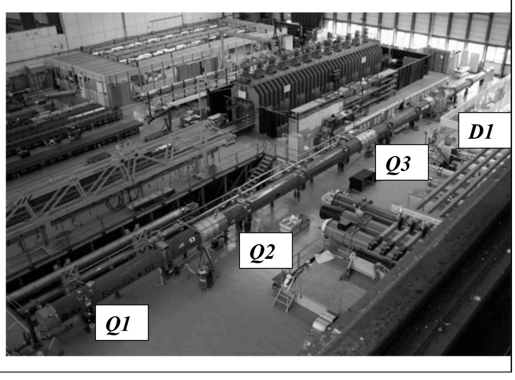
at a symmetry point β is just the ratio of beam dimension and beam divergence.

The LHC Insertions





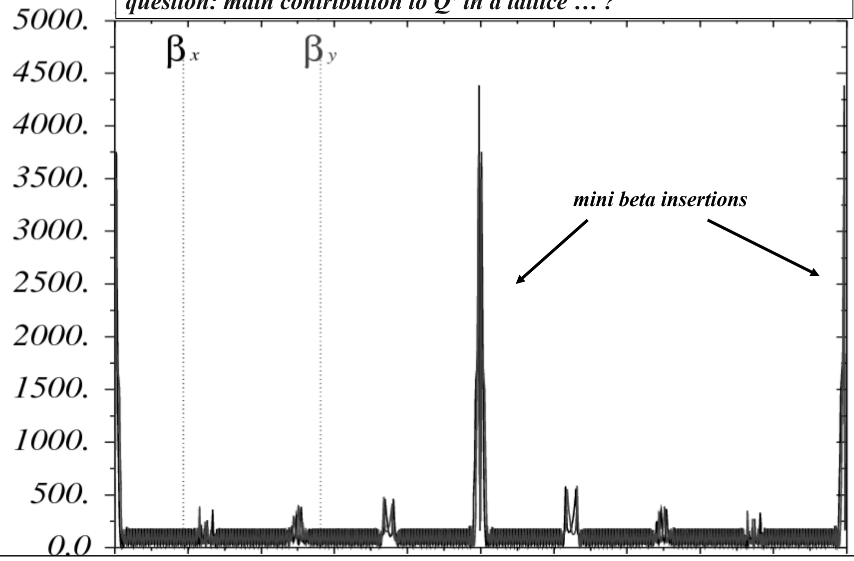
mini β optics



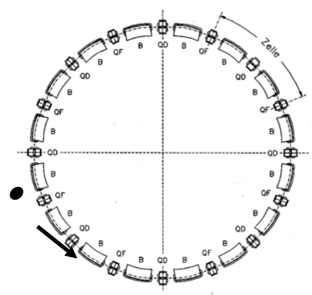
… and now back to the Chromaticity

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

question: main contribution to Q' in a lattice … ?



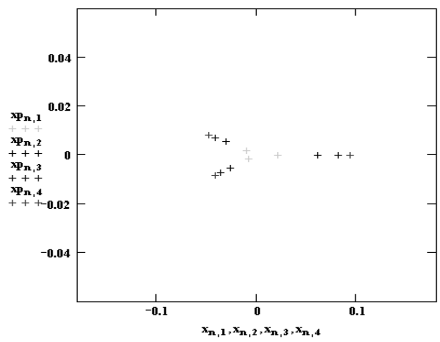
Clearly there is another problem ...
... if it were easy everybody could do it



Again: the phase space ellipse

for each turn write down – at a given position „s" in the ring – the single partilce amplitude x and the angle x´... and plot it.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{turn} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



A beam of 4 particles
– each having a slightly different emittance:

25.) Particle Tracking Calculations

particle vector: $\begin{pmatrix} x \\ x' \end{pmatrix}$

Idea: ***calculate the particle coordiantes x, x´ through the linear lattice***
… using the matrix formalism.
if you encounter a nonlinear element (e.g. sextupole): stop
calculate explicitly the magnetic field at the particles coordinate

$$B = \begin{pmatrix} g'xz \\ \frac{1}{2} g'(x^2 - z^2) \end{pmatrix}$$

calculate kick on the particle

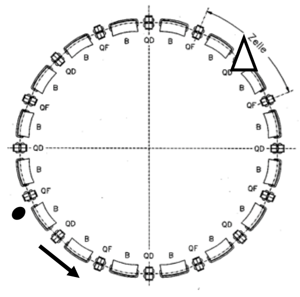
$$\Delta x_1' = \frac{B_z l}{p/e} = \frac{1}{2}\frac{g'}{p/e} l(x_1^2 - z_1^2) = \frac{1}{2} m_{sext} l(x_1^2 - z_1^2) \qquad \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_1' + \Delta x' \end{pmatrix}$$

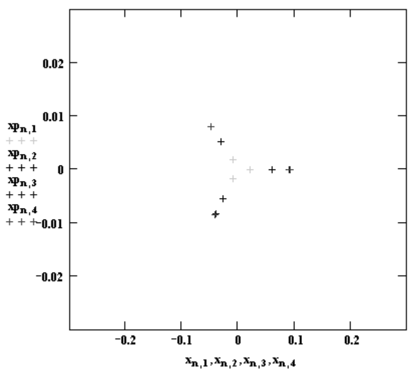
$$\Delta z_1' = \frac{B_x l}{p/e} = \frac{g' x_1 z_1}{p/e} l = m_{sext} l x_1 z_1 \qquad \begin{pmatrix} z_1 \\ z_1' \end{pmatrix} \rightarrow \begin{pmatrix} z_1 \\ z_1' + \Delta z_1' \end{pmatrix}$$

and continue with the linear matrix transformations

Installation of a weak (!!!) sextupole magnet

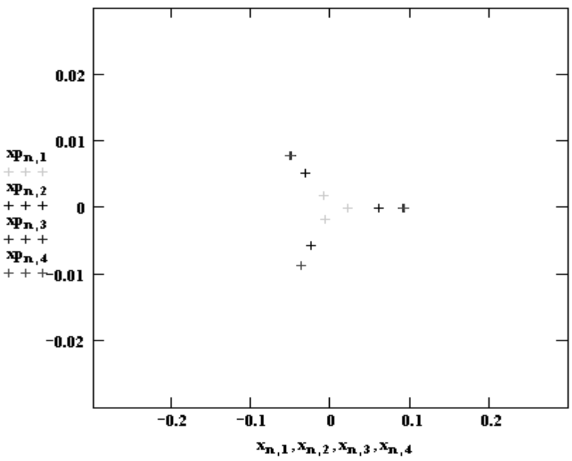
The good news: sextupole fields in accelerators cannot be treated analytically anymore.
→ no equatiuons; instead: Computer simulation
„ particle tracking "

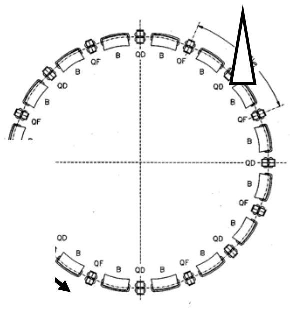




Effect of a strong (!!!) Sextupole …

→ *Catastrophy !*





„dynamic aperture"

Resume´:

quadrupole error: *tune shift*

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi}\,ds \approx \frac{\Delta k(s)\ l_{quad}\ \overline{\beta}}{4\pi}$$

beta beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta k \cos\big(2(\psi_{s1}-\psi_{s0}) - 2\pi Q\big)ds$$

chromaticity

$$\Delta Q = Q'\ \frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

momentum compaction

$$\frac{\delta\, l_{\varepsilon}}{L} = \alpha_p \frac{\Delta p}{p}$$

$$\alpha_p \approx \frac{2\pi}{L}\ \langle D\rangle \approx \frac{\langle D\rangle}{R}$$

beta function in a symmateric drift

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

Appendix I:

Dispersion: Solution of the inhomogenious equation of motion

Ansatz: $$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

$$D'(s) = S' * \int \frac{1}{\rho} C\, dt + S \frac{1}{\rho} C - C' * \int \frac{1}{\rho} S\, dt - C \frac{1}{\rho} S$$

$$D'(s) = S' * \int \frac{C}{\rho}\, dt \; - \; C' * \int \frac{S}{\rho}\, dt$$

$$D''(s) = S'' * \int \frac{C}{\rho}\, d\tilde{s} + S' \frac{C}{\rho} \; - \; C'' * \int \frac{S}{\rho}\, d\tilde{s} \; - \; C' \frac{S}{\rho}$$

$$= S'' * \int \frac{C}{\rho}\, d\tilde{s} - \; C'' * \int \frac{S}{\rho}\, d\tilde{s} + \frac{1}{\rho} \underbrace{\left(CS' \; - S\, C' \right)}_{= \det M = 1}$$

---

remember: for Cs) and S(s) to be independent solutions the Wronski determinant has to meet the condition $W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$

and as it is independent of the variable „s“ $$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC'' = -K(CS - SC) = 0$$

we get for the inititial conditions that we had chosen ... $C_0 = 1,\; C_0' = 0$; $S_0 = 0,\; S_0' = 1$ $\Rightarrow W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} = 1$

$$D'' = S'' * \int \frac{C}{\rho}\, d\tilde{s} - \; C'' * \int \frac{S}{\rho}\, d\tilde{s} + \frac{1}{\rho}$$

remember: S & C are solutions of the homog. equation of motion:

$$S'' + K * S = 0$$
$$C'' + K * C = 0$$

$$D'' = -K * S * \int \frac{C}{\rho}\, d\tilde{s} + \; K * C * \int \frac{S}{\rho}\, d\tilde{s} + \frac{1}{\rho}$$

$$D'' = -K * \underbrace{\left\{ S \int \frac{C}{\rho}\, d\tilde{s} + C \int \frac{S}{\rho}\, d\tilde{s} \right\}}_{=D(s)} + \frac{1}{\rho}$$

$$D'' = -K * D + \frac{1}{\rho}$$ ***... or*** $$D'' + K * D = \frac{1}{\rho}$$

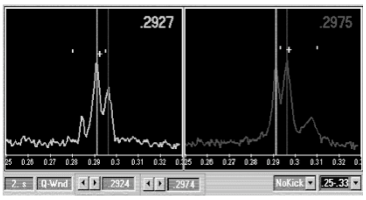
qed

Appendix II:

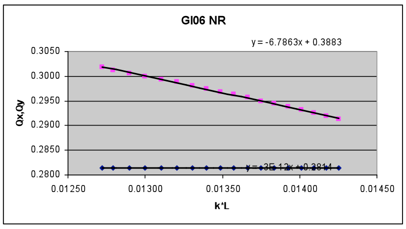
Quadrupole Error and Beta Function

a change of quadrupole strength in a synchrotron leads to tune sift:

$$\Delta Q \approx \int_{s0}^{s0+l} \frac{\Delta k(s)\,\beta(s)}{4\pi} ds \approx \frac{\Delta k(s) * l_{quad} * \overline{\beta}}{4\pi}$$



tune spectrum ...



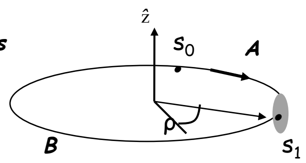
tune shift as a function of a gradient change

But we should expect an error in the β-function as well …
… shouldn´t we ???

Quadrupole Errors and Beta Function

a quadrupole error will not only influence the oscillation frequency … „tune"
… but also the amplitude … „beta function"

split the ring into 2 parts, described by two matrices A and B



$$M_{turn} = B * A \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

matrix of a quad error between A and B

$$M_{dist} = \begin{pmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{pmatrix} = B \begin{pmatrix} 1 & 0 \\ -\Delta kds & 1 \end{pmatrix} A$$

$$M_{dist} = B \begin{pmatrix} a_{11} & a_{12} \\ -\Delta kdsa_{11} + a_{12} & -\Delta kdsa_{12} + a_{22} \end{pmatrix}$$

$$M_{dist} = \begin{pmatrix} \sim & b_{11}a_{12} + b_{12}(-\Delta kdsa_{12} + a_{22}) \\ \sim & \sim \end{pmatrix}$$

the beta function is usually obtained via the matrix element „m12", which is in Twiss form for the undistorted case

$$m_{12} = \beta_0 \sin 2\pi Q$$

and including the error:

$$m_{12}^* = \underbrace{b_{11}a_{12} + b_{12}a_{22}}_{m_{12} = \beta_0 \sin 2\pi Q} - b_{12}a_{12}\Delta k ds$$

$$(1) \quad m_{12}^* = \beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds$$

As M* is still a matrix for one complete turn we still can express the element m_{12} in twiss form:

$$(2) \quad m_{12}^* = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

Equalising (1) and (2) and assuming a small error

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi (Q + dQ)$$

$$\beta_0 \sin 2\pi Q - a_{12}b_{12}\Delta k ds = (\beta_0 + d\beta) * \sin 2\pi Q \underbrace{\cos 2\pi dQ}_{\approx 1} + \cos 2\pi Q \underbrace{\sin 2\pi dQ}_{\approx 2\pi dQ}$$

$$\beta_0 \cancel{\sin 2\pi Q} - a_{12}b_{12}\Delta k ds = \beta_0 \cancel{\sin 2\pi Q} + \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q + \cancel{d\beta_0 2\pi dQ} \cos 2\pi Q$$

ignoring second order terms

$$- a_{12}b_{12}\Delta k ds = \beta_0 2\pi dQ \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

remember: tune shift dQ due to quadrupole error: $dQ = \dfrac{\Delta k \beta_1 ds}{4\pi}$
(index „1" refers to location of the error)

$$- a_{12}b_{12}\Delta k ds = \frac{\beta_0 \Delta k \beta_1 ds}{2} \cos 2\pi Q + d\beta_0 \sin 2\pi Q$$

solve for dβ

$$d\beta_0 = \frac{-1}{2 \sin 2\pi Q} \left\{ 2a_{12}b_{12} + \beta_0 \beta_1 \cos 2\pi Q \right\} \Delta k ds$$

express the matrix elements a_{12}, b_{12} in Twiss form

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} \left(\cos\psi_s + \alpha_0 \sin\psi_s \right) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta s}} \left(\cos\psi_s - \alpha_s \sin\psi_s \right) \end{pmatrix}$$

$$d\beta_0 = \frac{-1}{2\sin 2\pi Q}\left\{2a_{12}b_{12} + \beta_0\beta_1 \cos 2\pi Q\right\}\Delta k ds$$

$$\boldsymbol{a}_{12} = \sqrt{\beta_0\beta_1}\,\sin\Delta\psi_{0\to 1}$$
$$\boldsymbol{b}_{12} = \sqrt{\beta_1\beta_0}\,\sin(2\pi\boldsymbol{Q} - \Delta\psi_{0\to 1})$$

$$d\beta_0 = \frac{-\beta_0\beta_1}{2\sin 2\pi Q}\underbrace{\left\{2\sin\Delta\psi_{12}\sin(2\pi Q - \Delta\psi_{12}) + \cos 2\pi Q\right\}}\Delta k ds$$

… after some TLC transformations … $= \cos(2\Delta\psi_{01} - 2\pi Q)$

$$\Delta\beta(s_0) = \frac{-\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta k\cos\left(2(\psi_{s1} - \psi_{s0}) - 2\pi Q\right)ds$$

Nota bene: ***! the beta beat is proportional to the strength of the error Δk***

!! and to the β function at the place of the error ,

!!! and to the β function at the observation point, (… remember orbit distortion !!!)

!!!! there is a resonance denominator