Unstable Particles in Quantum Mechanics, Analytic S-matrix Theory and Quantum Field Theory

Zoltan Kunszt, ETH, Zurich

The case of a large-mass Higgs, Cern May 14, 2012

Unstable particles in quantum mechanics

* The energy spectrum of an unstable system is continuous:
  when it decays the outgoing particle goes to infinity.

* In case the decay probability is very small, we get quasi-stationary states, the particles are localized for long time, the energy spectrum is quasi-discrete. Smeared discrete energy levels.

* Boundary condition: at infinity only outgoing spherical waves.
  This boundary condition involves complex quantities,
  the energy eigen-values in general are also complex.

$$E = E_0 - \frac{i\Gamma}{2}$$

$$\exp(-iEt) = \exp(-iE_t)\exp\left(-\frac{\Gamma}{2}\right)$$

$$\Gamma > 0$$

Elastic scattering from a three dimensional square-well potential

Schrödinger equation:

$$\left[-\frac{1}{2m}\Delta - V\Theta(a-r)\right]\Phi(\vec{r}) = E\Phi(\vec{r})$$

l=0 solution:

$$\Phi_E(r) = \begin{cases} \frac{A(E)}{r}\sin Kr & , r \leq a \\ -\frac{B(E)}{r}\left[e^{-ikr} - \eta(E)e^{ikr}\right], & r \geq a \end{cases}$$

where $k = (2mE)^{\frac{1}{2}}$ and $K = [2m(E+V)]^{\frac{1}{2}}$

A, B and η are determined by continuity of Φ_E and its derivative at r=a and normalization.

H. A. Weldon, Phys.RevD14,2030(1976)

Scattering amplitude: $$\langle p_2|S|p_1\rangle = (2\pi)^3 \frac{\delta(p_2 - p_1)}{4\pi p_1 p_2} \sum_{l=0}^{\infty} (2l+1) e^{2i\delta_l} P_l(\cos\Theta)$$

Phase shift may be evaluated from asymptotic behavior: $$\Phi(\vec{r})_{\to\infty} \sim e^{i\vec{k}\vec{r}} + f(\Theta)\frac{e^{ikr}}{r}$$

where $$f(\Theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\Theta)$$

Comparing the asymptotic forms we get $$\eta(E) = e^{2i\delta_0}$$

$\eta(E)$ is determined by the boundary condition $$\frac{\Phi(r)'_E}{\Phi(r)_E}\Big|_{r<a} = \frac{\Phi(r)'_E}{\Phi(r)_E}\Big|_{r>a}$$

It gives the exact S-matrix : $$\eta(E)_{\mathrm{I}} = \frac{K\cot Ka + ik}{K\cot Ka - ik} e^{-2ika}$$

where $$k = (2mE)^{\frac{1}{2}}, \quad K = [2m(E+V)]^{\frac{1}{2}}$$

The S-matrix is two sheeted since it has a square-root cut in E

Assuming the cut is along the positive real axis, for the physical sheet we get

$$0 \le \arg\sqrt{E} \le \pi\,, \quad \mathrm{Im}k \ge 0\,, \quad \mathrm{Im}K \ge 0$$

With analytic continuation of $\eta_{\mathrm{I}}(E)$ we get

$$\eta(E)_{\mathrm{II}} = \frac{K\cot Ka - ik}{K\cot Ka + ik}e^{2ika}$$

$$\eta(E)_{\mathrm{I}} = \frac{K\cot Ka + ik}{K\cot Ka - ik}e^{-2ika}$$

$$k = (2mE)^{\frac{1}{2}}\,, \quad K = [2m(E+V)]^{\frac{1}{2}}$$

$$|\eta(E)_{\mathrm{I}}|^2 = 1 \quad \text{E is positive real}$$

$$\eta(E)_{\mathrm{I}}\eta(E)_{\mathrm{II}} = 1 \quad \text{E complex}$$

$$(\sqrt{E})^* = -\sqrt{E^*}$$

$$\eta(E)_{\mathrm{I}}^* = \eta(E^*)_{\mathrm{I}}$$

Unitarity, Hermitian analyticity

Location of the poles on the second Riemann sheet

The S-matrix has a second-sheet pole only if

$$K \cot Ka = -ik$$

The poles occur at a complex energy $\tilde{E}$

$$x = a[2m(\tilde{E}+V)]^{\frac{1}{2}} = x_1 + ix_2$$

$$y = a(2m\tilde{E})^{\frac{1}{2}} = y_1 + iy_2$$

$$x_2 \geq 0\,,\ y_2 \geq 0$$

$$x \cot x = -iy$$
$$x^2 - y^2 = 2mVa^2$$

Since x,y are complex we have four equations for four variables. We can easily see that we have infinitely many solutions which fulfill the conditions

$$\frac{\tan x_1}{x_1} = -\frac{\tanh x_2}{x_2}$$

$$x_2 = \pm \cos x_1 \left[\left(\frac{x_1}{\sin x_1}\right)^2 - 2mVa^2\right]^2$$

$$n\pi < x_1 < (n+\frac{1}{2})\pi \qquad y_2 > 0$$

$$\tilde{E} = E_P + \frac{i}{2}\gamma_P \quad , \tilde{E} = E_P - \frac{i}{2}\gamma_P$$

Schrödinger wave functions

The wave functions for an unstable is the analytic continuation in E of $\Phi(E,r)$ to the point $\tilde{E}$ on the second sheet. At this value

$$\psi(E,r) = \begin{cases} \frac{A(E)}{r}\sin Kr, & r \le a \\ \frac{B(E)\eta(E)}{r} e^{ikr} & r \ge a \end{cases}$$

$$\eta = \infty \qquad B = 0$$

$$B\eta = \text{finite}$$

At large r only outgoing spherical waves

$$\psi_{\mathrm{II}}(\tilde{E},r) = \Phi_{\mathrm{II}}(\tilde{E},r)$$

$$\psi(E,r)_{\mathrm{II}} = \begin{cases} -\frac{A(E)_{\mathrm{II}}}{r}\sin Kr, & r \le a \\ \frac{B(E)_{\mathrm{II}}\eta(E)_{\mathrm{II}}}{r} e^{-ikr} & r \ge a \end{cases}$$

$$\langle\psi(E)|\psi(E)\rangle \rightarrow \text{analytically continue to } \tilde{\mathrm{E}}$$

For large r diverges exponentially

$$\langle\psi_{\mathrm{II}}(\tilde{E})|\psi_{\mathrm{II}}(\tilde{E})\rangle = 0$$

$$\langle\psi_{\mathrm{II}}(E^*)|\psi_{\mathrm{II}}(E)\rangle \neq 0$$

At large r only outgoing spherical waves

Comments

From S-matrix theory to QFT:

Unstable states lie in a natural extension of the usual Hilbert space of stable particles. It corresponds to the second sheet of the S-matrix.

They are zero norm states and therefore Hamiltonian remain hermitian even with complex energy

$$\langle E|E\rangle E = \langle E|H|E\rangle = E^* \langle E|E\rangle$$

Similarly to the assumption of S-matrix theory, Green's function involving unstable particles should smoothly approach the value for stable ones as the interactions goes to zero.