Loop Quantum Gravity

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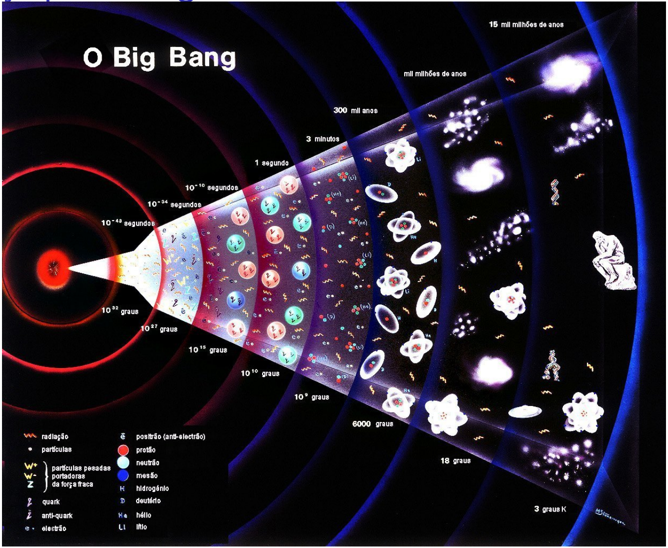
Abstract

The purpose of this talk is to give a short general introduction to Loop Quantum Gravity (LQG), beginning with some motivations for quantizing General Relativity, listing various attempts and then focusing on the case of LQG.

Summary

- Why quantize gravitation?
- Einstein Theory
- Perturbative Quantum Gravity
- Loop Quantum Gravity
- A simple exemple
- Prospects

Why quantize gravitation?



Planck's length $l_{\rm P} = \sqrt{\hbar G/c^3} \approx 1.6 \times 10^{-35}$ m

Planck's time $t_{\rm P} = l_{\rm P}/c \approx 0.5 \times 10^{-43}$ s

Planck's mass $m_{\rm P} = \hbar / l_{\rm P} c \approx 10^{19}$ GeV$/c^2$

At Planck's time $t_{\mathrm{P}} = \sqrt{\hbar G / c^5} \approx 0.5 \times 10^{-43}$,
or Planck's scale $l_{\mathrm{P}} = \sqrt{\hbar G / c^3} \approx 1.6 \times 10^{-35}$,
enter in interplay:

- Gravity (G);
- Relativity (c)

  $\Rightarrow$ *Relativistic gravity* = *General Relativity* (GR);
- Quantum Mechanics ($\hbar$)

  $\Rightarrow$ *Quantum General Relativity = "Quantum Gravity".*

Note: Planck's energy $E_{\mathrm{P}} \approx 10^{19}\,\mathrm{GeV} \approx 10^{15} E_{\,\mathrm{(LHC)}}$
$\approx 10^7 E_{\,\mathrm{(Ultra\ High\ Energy\ Cosmic\ Rays)}}$

Einstein Theory

Space-time Geometry defined by the metric $g_{\mu\nu}(x)$ $(\mu = 0, \cdots, D-1)$:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Einstein equation with cosmological constant Λ:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \,,$$

- $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ (defined from the Riemann curvature tensor) are expressions in derivatives of the gravitational field $g_{\mu\nu}$, up to the second order;
- $T_{\mu\nu}$ is the energy-momentum tensor, source of the gravitational field.

Perturbative Gravity

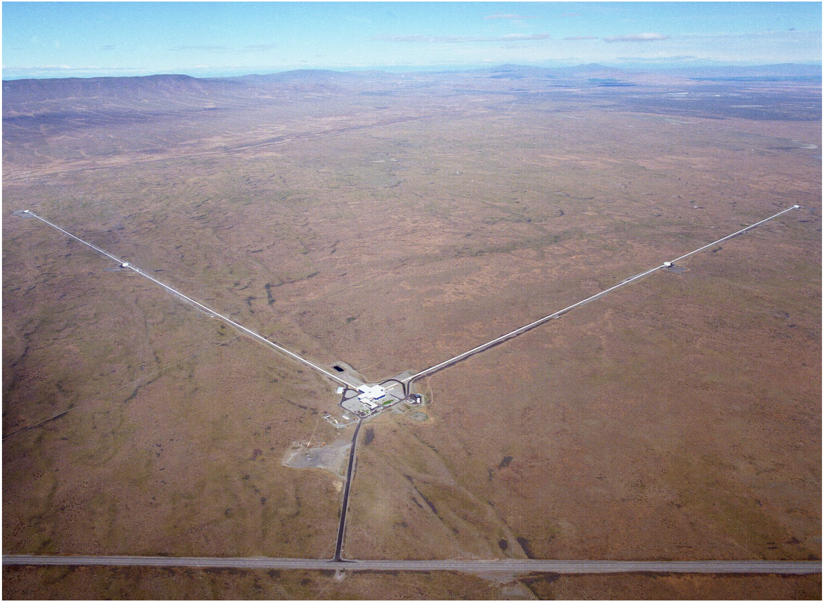
1. Consider a small perturbation $h_{\mu\nu}$ around the flat (Minkovsky) metric $\eta_{\mu\nu} = \text{diagonal}\,(-1, 1, 1, 1)$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad |h| << 1\,,$$

and introduce it in the action, or in the field equations:

$$\Box h_{\mu\nu} + \text{non-linear interactions}\,(h_{\mu\nu}, \partial_\rho h_{\mu\nu}, \partial_\rho \partial_\sigma h_{\mu\nu}) \propto T_{\mu\nu}\,.$$

At zeroth order, without matter: $\Box h_{\mu\nu} = 0 \to$ Gravitational waves.



LIGO detector of gravitational waves (2 sites, in Louisiana and Washington states, USA)

2. Try to quantize this theory perturbatively (Feynmann graphs expansion), considering $h_{\mu\nu}$ as just another field (but of spin 2), like the electromagnic field A_μ, etc., propagating in flat Minkowsky space.

Result: Non-renormalizable theory!

O que fazer?

Proposal 1 Look for a more fundamental theory, the Standard Model and General Relativity being 'low" energy approximations of it. Exemple: String Theory.

Proposal 2 Find a non-perturbative framework, hopefully free of ultraviolet singularities, too. Exemple: Loop Quantum Gravity.

proposals Delta Gravity (Alfaro); Horava Gravity; etc.

lternative: Gravity as an "emergent" phenomenon – Induced Gravity. (From Andrei Sakharov to AdS-CFT correspondence)

First Order Formalism of Einstein Theory

Tangent space at point x:
Vector basis $e_I = e_I^\mu \partial_\mu, \quad (I = 0, \cdots, D-1)$,
Pseudo-orthogonal: $e_I \cdot e_j := g_{\mu\nu} e_I^\mu e_j^\nu = \eta_{IJ} = \mathrm{diag}(-1, 1, \cdots, 1)$.
Dual basis (vielbein): 1-forms $e^I = e_\mu^I dx^\mu$, with $(e_\mu^I) = (e_I^\mu)^{-1}$ and $\eta_{IJ} e_\mu^I e_\nu^J = g_{\mu\nu}$.
Lorentz connection: $\omega^{IJ} = -\omega^{JI} = \omega_\mu^{IJ} dx^\mu$.
Curvature: $R^{IJ} = \frac{1}{2} R^{IJ}{}_{\mu\nu} dx^\mu \wedge dx^\nu = d\omega^{IJ} + \omega^{IK} \wedge \omega_K{}^J$.
Torsion: $T^I = de^I + \omega^I{}_K \wedge e^K$.
Palatini-Holst action (for pure $D = 4$ gravity):

$$S = \frac{1}{8\pi G} \int \left(\varepsilon_{IJKL} e^I \wedge e^J \wedge R^{KL} - \frac{1}{\gamma} e^I \wedge e^J \wedge R_{IJ} \right) .$$

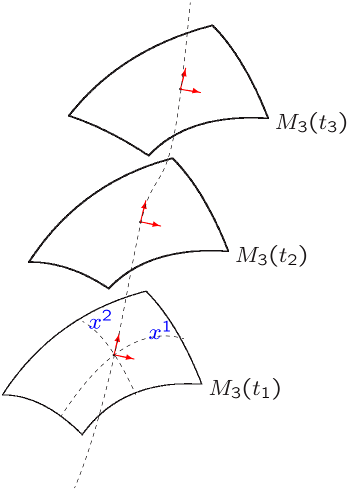
Field equations: $T^I = 0\,, \quad \mathcal{R}_{\mu\nu} := e_\mu^J e_I^\rho R^I{}_{J\rho\nu} = 0$.
Independent of the Barbero-Immirzi parameter γ!

Loop Quantum Gravity (LQG): The Main Issues

- Rewrite classical General Relativity as a Yang-Mills gauge theory with connection (gauge field) $\mathcal{A}^i(x)$ for a compact Lie group G (usually SU(2)) (Ashtekar, 1986). (GR gauge group is Lorentz SO(1,3), non-compact. Do partial gauge fixing $\rightarrow$ residual symmetry SO(3) or SU(2).)

  $\mathcal{A}^i$ is the Yang-Mills gauge vector field for the group $G =$ SO(3) or SU(2).

- Dirac canonical formalism:
  - Assume space-time manifold splitted as $M_4 = \mathbb{R} \times M_3 =$ times $\times$ space. No background metric!



Coordinates chosen such that
$T(P) = x^0 \equiv t$
and
$(x^a,\ a = 1, 2, 3)$ be the
coordinates of $M_3(t)$.
(in order to simplify)

1

- Dirac canonical formalism:
  - Hamiltonian $H[\mathcal{A},\mathcal{P}] = \sum_{\alpha=1}^{3} \lambda^\alpha C_\alpha[\mathcal{A},\mathcal{P}]$

    $\mathcal{P}_i(x)$ is the conjugate momentum of $\mathcal{A}^i(x)$.

    Vierbein e^I and Lorentz connection ω^{IJ} are functions of components of $\mathcal{A}$ and $\mathcal{P}$.

  - Constraints $C_\alpha \approx 0$ ($\Rightarrow H \approx 0$)

- Quantization – formal: Wave functional $\Psi[\mathcal{A}]$

  Operators $\hat{\mathcal{A}}$, $\hat{\mathcal{P}}$: $\hat{\mathcal{A}}^i(\mathbf{x})\Psi = \mathcal{A}^i(\mathbf{x})\Psi$, $\hat{\mathcal{P}}_j(\mathbf{x})\Psi = -i\hbar\frac{\delta}{\delta\mathcal{A}^j(\mathbf{x})}\Psi$.

- Apply the Constraints: $\hat{C}_\alpha\Psi[\mathcal{A}] = 0$.

Note: Constraints $\leftrightarrow$ Gauge invariance:

- Yang-Mills G-invariance $\sim$ local SU(2)($\subset$ Lorentz) invariance.
- Space diffeomorphisms
- Time diffeomorphisms ("coordinate time" evolution is a gauge transformation!)

Hamiltoniano $H \approx 0$:

Where time has gone?

Compare with QED (free theory)

1. Defined on a metric space.
2. Hamiltonian is:

$$H = \int d^3x \left(\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) + A_t \nabla \mathbf{E} \right)$$
$$= \text{energy} + \text{constraint}\,,$$

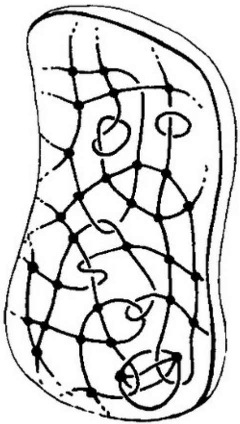
where the electric potencial $A_t \leftrightarrow$ Lagrange multiplier for the Gauss constraint $\nabla \mathbf{E} \approx 0$.

"Loop" Quantization (Ashtekar, Rovelli, Smolin, etc.)

Configuration space = space of connections $\mathcal{A}$
is very big and difficult to handle.
How to define an integration measure $\mathcal{D}\mathcal{A}$ $\longrightarrow$ a scalar product

$$< \Psi_1 | \Psi_2 >= \int \mathcal{D}\mathcal{A}\, \overline{\Psi_1[\mathcal{A}]} \Psi_2[\mathcal{A}] \quad \textbf{?}$$

Instead, **restrict on the values of $\mathcal{A}(x)$ on finite sets of lines (graphs)**:



(Roger Penrose)

Graph Γ

On each line γ_n: an holonomy $h_{\gamma_n} = P\, e^{\int_{\gamma_n} \mathcal{A}}$;
P means path ordering.

$h_{\gamma_n} \in G$ is an element of the gauge group.

Gauge transformation $g(x) \in G$:
$h'_{\gamma_n} = g(x_{\rm fin}) h_{\gamma_n} g^{-1}(x_{\rm in})$
$x_{\rm in}$, $x_{\rm fin}$: initial and final points of γ_n.

Consider all possible graphs!

$$\Psi[\mathcal{A}] \rightarrow \psi_\Gamma(h_{\gamma_1}, \cdots, h_{\gamma_N}) \quad (\text{"holonomy functions"}),$$

where $\Gamma = \{\gamma_1, \cdots, \gamma_N\}$ ("graph"), γ_n = curve in space Σ, and

$$h_{\gamma_n} = P\, e^{\int_{\gamma_n} \mathcal{A}} \quad (\text{"holonomy"});\ P \text{ means path ordering}.$$

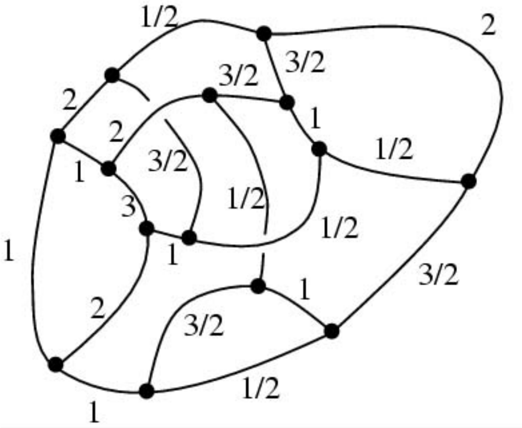
Scalar product:

$$\langle \psi_\Gamma | \psi'_{\Gamma'} \rangle = \delta_{\Gamma\Gamma'} \int d\mu(g_1) \cdots \int d\mu(g_N) \overline{\psi\,(g_1, \cdots g_N)} \psi'\,(g_1, \cdots g_N)\ ,$$

where $g_n \in G$, $d\mu(g_n)$ is the Haar integration measure on the group G – which must be compact!.

$\longrightarrow$ Kinematical Hilbert Space $\mathcal{H}_{\rm kin}$ (non-separable!)
Orthonormal basis: $\psi_{\Gamma, j_1, \cdots j_N}$ = gauge invariant combinations of products $R^{j_1}_{\alpha_1\beta_1} \cdots R^{j_N}_{\alpha_N\beta_N}$, where $R^{j_n}_{\alpha_n\beta_n}$ = matrix elements of the spin j_n representation of $h_{\gamma_n} \in G$ $\longrightarrow$

$\longrightarrow$ Spin networks, represented by "coloured" graphs:



Basis vectors of Hilbert space $\mathcal{H}_{\rm kin}$.

Finally, apply the constraints (the difficult part!)
$\rightarrow$ (separable) Physical Hilbert space.

... and construct the Observables of the theory.

1. (2+1) – Gravity as a Chern-Simons theory with a Barbero-Immirzi parameter

(Model due to Valentin Bonzom, Etera R. Livine, CQG 25 (2008)
Work by R.M.S. Barbosa, C.P. Constantinidis, Z. Oporto, O. Piguet, CQG 29 (2012))

$$e^I = e^I_\mu dx^\mu \text{ (Triad)}, \quad \omega_I = \tfrac{1}{2}\varepsilon_{IJK}\,\omega^{JK} = \omega_{I\mu} dx^\mu \text{ (Spin connection)},$$

$$\left(\text{metric } g_{\mu\nu}(x) = \eta_{IJ} e^I_\mu(x) e^J_\nu(x)\right) ,$$

$$\eta_{IJ} = \text{diag}(-1,1,1) \text{ (Minkowski metric)},$$

$$I, J, \cdots = 0,1,2\,, \quad \mu,\nu,\cdots = t,x,y\,.$$

Cosmological constant $\boldsymbol{\Lambda > 0}$ $\rightarrow$ de Sitter gauge group **SO(3,1)**
Generators: J^I (Lorentz), P_I (“translations”)

$$[J^I, J^J] = \varepsilon^{IJ}{}_K J^K\,, \quad [J^I, P_J] = \varepsilon^I{}_J{}^K P_K\,, \quad [P_I, P_J] = \Lambda\varepsilon_{IJK} J^K\,.$$

SO(3,1) – connection: $A = \omega_I J^I + e^I P_I$.

$D = 2 + 1$ de Sitter gravity described by SO(3,1) Chern-Simons action:

$$S_{\mathrm{CS}}(A) = -\frac{\kappa}{2} \int_{\mathcal{M}=\mathbb{R}\times\Sigma} \left\langle A,\, dA + \frac{2}{3} A\, A \right\rangle . \qquad (\wedge \text{ symbols omitted})$$

2 quadratic Casimir operators:

$$C_{(1)} = P_I J^I \,, \quad C_{(2)} = \eta_{IJ} \left(\frac{1}{\Lambda} P^I P^J - J^I J^J \right)$$

$\rightarrow$ 2 invariant quadratic forms $\langle\ ,\ \rangle$ $\rightarrow$ 2 actions: $S_{(1)}$ & $S_{(2)}$

Total action: $\quad S = S_{(1)} - \frac{1}{\gamma} S_{(2)}\,.$

γ**: Barbero-Immirzi-like parameter.**

N.B.: *Classical field equations are independent of* γ:

$F(A) = dA + A^2 = 0\,,\quad$ or: $\quad \mathbf{R} - \frac{\Lambda}{2} \mathbf{e} \times \mathbf{e} = 0\,, \quad \mathbf{T} \equiv \mathbf{de} + \boldsymbol{\omega} \times \mathbf{e} = 0\,.$

A well chosen partial gauge fixing $\to$ The theory is equivalent to a simple SU(2) Chern-Simons theory:

$$S_{\rm CS}(A) = -\frac{\kappa}{\gamma}\int_{\mathcal{M}=\mathbb{R}\times\Sigma}\left(\mathcal{A}^i,\, d\mathcal{A}^i + \frac{1}{3}\varepsilon_{ijk}\mathcal{A}^j\,\mathcal{A}^k\right)\,,$$

where the dynamical components of the connection are:

$$(\mathcal{A}^i_x,\ i=1,2,3) = (\sqrt{\Lambda}e^2_x - \gamma\omega^2_x\,,\ -\sqrt{\Lambda}e^1_x + \gamma\omega^1_x\,,\ -\omega^0_x - \gamma\sqrt{\Lambda}e^0_x)\,,$$
$$(\mathcal{A}^i_y,\ i=1,2,3) = \left(\sqrt{\Lambda}e^2_y\qquad,\ -\sqrt{\Lambda}e^1_y\qquad,\ -\omega^0_y\qquad\right).$$

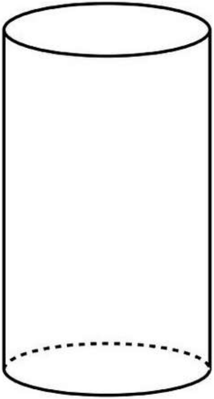
Poisson brackets: $\{\mathcal{A}^i_x(\mathbf{x}),\, \mathcal{A}^j_y(\mathbf{x}')\} = \frac{\gamma}{\kappa}\delta^{ij}\delta^2(\mathbf{x}-\mathbf{x}')\,,$
N.B. The Hamiltonian is completely constrained, with

$\mathcal{F}^i_{xy}(\mathcal{A}) \approx 0$ being the only constraint.

(Curvature constraint – the analog of the Gauss constraint of QED.)

Loop quantization of such a theory is known.
(C.P. Constantinidis, G. Luchini and O. Piguet, CQG27 (2010) 065009
R.M.S. Barbosa, C.P. Constantinidis, O. Piguet and Zui Oporto, CQG29 (2012))

Special case: space Σ is a cylinder



Coordenadas

$$x\,,\;\; y\,, \quad 0 \leq x < 2\pi\,,\; -\infty < y < +\infty$$

Commutation relations: $\left[\hat{\mathcal{A}}^i_x(\mathbf{x}), \hat{\mathcal{A}}^j_y(\mathbf{x}')\right] = -\frac{i\hbar\gamma}{\kappa}\delta^{ij}\delta^2(\mathbf{x}-\mathbf{x}')\,,$

Field operators act on wave functionals $\Psi[\mathcal{A}_x]$ as

$$\hat{\mathcal{A}}^i_x(\mathbf{x})\Psi[\mathcal{A}_x] = \mathcal{A}^i_x(\mathbf{x})\Psi[\mathcal{A}_x]\,, \quad \hat{\mathcal{A}}^i_y(\mathbf{x})\Psi[\mathcal{A}_x] = -\frac{i\hbar\gamma}{\kappa}\frac{\delta}{\delta\mathcal{A}^i_x(\mathbf{x})}\Psi[\mathcal{A}_x]\,.$$

$\downarrow$

Dunne, Jackiw and Trugenberger, Ann.Phys. 194 (1989) 197:
General solution of the (curvature) constraint is of the form:

$$\Psi[\mathcal{A}_x] = e^{2i\pi\alpha_0[\mathcal{A}_x]}\psi^{\rm inv}[\mathcal{A}_x]\,,$$

the phase $e^{2i\pi\alpha_0[\mathcal{A}_x]}$ is a special solution of the constraint, and $\psi^{\rm inv}$ is invariant under the "x-gauge" transformations

$$\delta\mathcal{A}_x = \partial_x\varepsilon + \mathcal{A}_x\times\varepsilon\,.$$

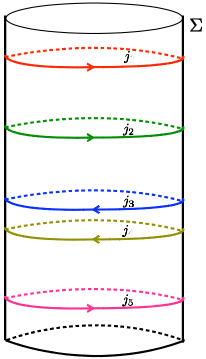
Restriction to "Holonomy functions":

$$\Psi_{\Gamma,f}[\mathcal{A}_x] = e^{2i\pi\alpha_0} f\left(h_1[\mathcal{A}_x], \cdots h_N[\mathcal{A}_x]\right) \,,$$

where:

"graph" $\Gamma = \{\gamma_1, \cdots, \gamma_N\}\,, \quad \gamma_n =$ closed curve at $y = y_n =$ const.

"holonomy" $h_n[\mathcal{A}_x] = \mathrm{P}\exp\left(\oint_{\gamma_n} dx \mathcal{A}_x^i(x, y_n) J_i\right) \in \mathrm{SU}(2)\,.$


Σ
j_1
j_2
j_3
j_4
j_5

Restriction to "Holonomy functions":

$$\Psi_{\Gamma,f}[\mathcal{A}_x] = e^{2i\pi\alpha_0[\mathcal{A}_x]} f\left(h_1[\mathcal{A}_x], \cdots h_N[\mathcal{A}_x]\right) \; ,$$

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Consider all graphs Γ
and all functions f: $G \times G \times \cdots G \to$ complex numbers.

$\longrightarrow$ Hilbert space $\mathcal{H}_{\rm kin}$,

with internal product

$$\left\langle \Psi_{\Gamma,f} | \Psi_{\Gamma',g} \right\rangle = \delta_{\Gamma\Gamma'} \int d\mu_{\rm Haar} \overline{f\left(h_1, \cdots h_N\right)} g\left(h_1, \cdots h_N\right) ,$$

where $d\mu_{\rm Haar}$ is the (multi) Haar integration measure on SU(2)$\times \cdots \times$ SU(2).

Implementation of diffeomorphism invriance:

Perform the group average on the diffeomorphisms in the z direction.

Result:

A separable Hilbert space $\mathcal{H}_{\rm phys}$ with an orthonormal basis:

$$\left\{ |0\rangle \right\} \oplus \left\{ |j_1, j_2, \cdots j_N\rangle \, ; \quad N = 0, 1, 2, \cdots \, ; \quad j_n = \frac{1}{2}, 1, \frac{3}{2}, \cdots \right\} .$$

Observable L

The classical gauge invariant observable L_{class}

$$L_{\mathrm{class}} = \int_{-\infty}^{+\infty} dy \sqrt{(\mathcal{A}_y - \Lambda^{-1}\partial_y\Lambda)^i\,(\mathcal{A}_y - \Lambda^{-1}\partial_y\Lambda)_i}\,,$$

$\Lambda = \Lambda[\mathcal{A}_x]$ defined as solution of: $\Lambda^{-1}\partial_x\Lambda = 0$

Consequence of the zero curvature constraint:

$$\mathbf{L}_{\mathrm{class}} = \mathbf{0}\ !$$

Observable L

The quantum gauge invariant observable $\hat{L}$

$$\hat{L} = \int_{-\infty}^{+\infty} dy \sqrt{\left(\hat{\mathcal{A}}_y - \Lambda^{-1}\partial_y\Lambda\right)^i \left(\hat{\mathcal{A}}_y - \Lambda^{-1}\partial_y\Lambda\right)_i}$$

as above, $\Lambda = \Lambda[\mathcal{A}_x]$ defined as solution of: $\Lambda^{-1}\partial_x\Lambda = 0$.

Result:

$$\hat{L}\,|0\rangle = 0\,,$$

$$\hat{L}\,|j_1, j_2, \cdots j_N\rangle = \frac{\gamma}{\kappa}\left(\sum_{n=1}^{N}\sqrt{j_n(j_n+1)}\right)|j_1, j_2, \cdots j_N\rangle$$

(Needs a regularization; but the result is finite– and non-zero! Analogy with the Area operator in D=4 LQG.)

Prospects

- Include matter: begin with "topological" matter (form fields coupled to the geometry fields e^I and ω^I). Work in progress, with Clisthenis and Zui.
- Dimensions > 3...

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