Averaging on inhomogenous spacetimes

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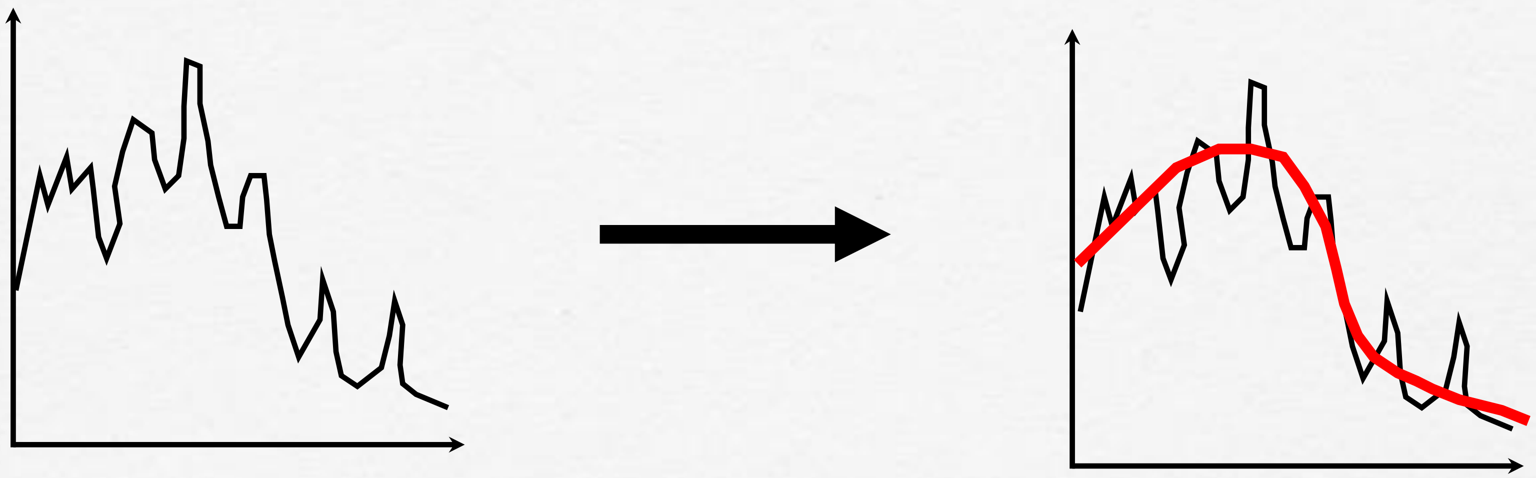
STARS 2013
La Habana, Cuba, 3-7 May 2013

Averaging in GR: why it matters?

- There is always some process of “averaging” when smoothing out “real” discrete matter--energy sources as part of “modeling”

$$G^{ab} = \frac{8\pi G}{c^4} T^{ab}, \qquad T^{ab} \sim \langle T^{ab}_{\text{real}} \rangle$$



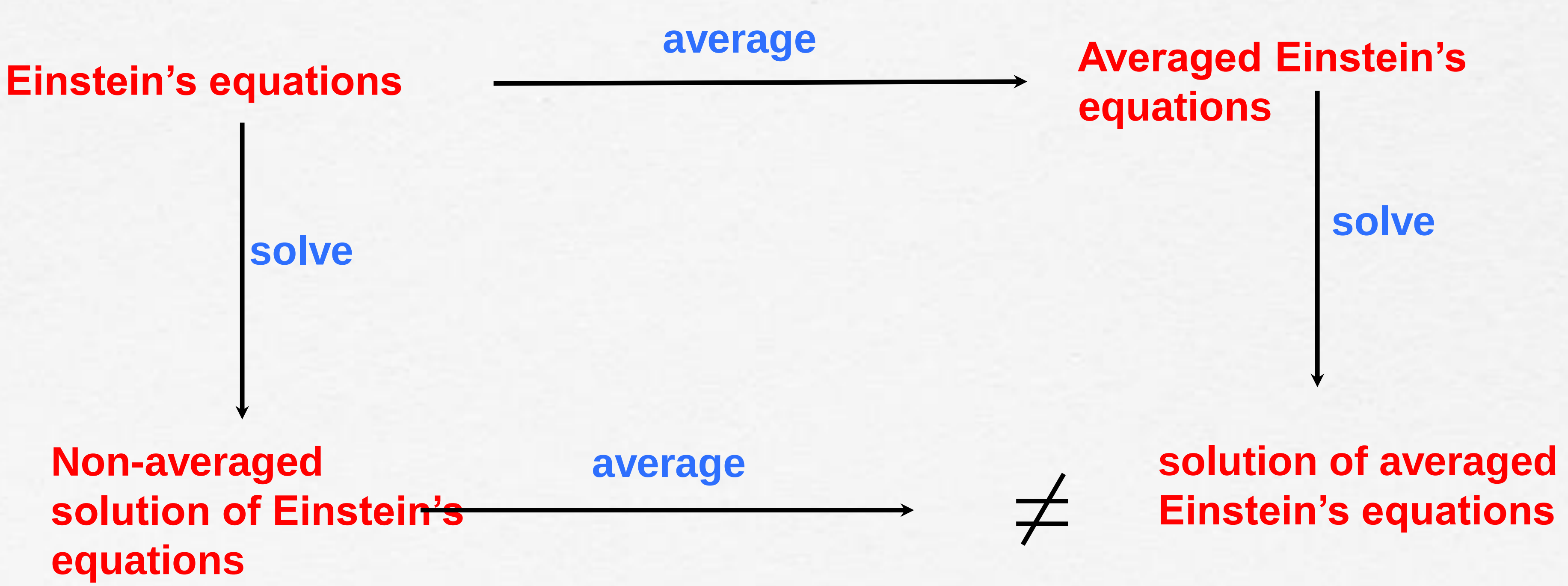
$$\langle \rho_{\text{real}} \rangle = \left\langle \Sigma \left(\frac{m_i c^2}{r_i^3} \right) \right\rangle \sim \rho_{\text{cont.}} \sim \frac{M_{\text{cont.}}}{\ell^3}$$

$$\langle p_{\text{real}} \rangle = \left\langle \Sigma \left(\frac{m_i v_i^2}{r_i^3} \right) \right\rangle \sim p_{\text{cont.}} \sim \rho_{\text{cont.}} T_{\text{cont.}}$$

Averaging exact solutions of Einstein's equations NOT EQUAL TO solving the averaged Einstein's equations:

$$\langle G_{ab}[g_{ab}]\rangle \neq \langle G_{ab}\rangle[\langle g_{ab}\rangle],$$

$$\langle \mathcal{R}_{ab}\rangle - \frac{1}{2}\langle g_{ab}\mathcal{R}\rangle = \langle \mathcal{R}_{ab}\rangle - \frac{1}{2}\langle g_{ab}\rangle\langle \mathcal{R}\rangle - \mathbf{corr}(g_{ab},\mathcal{R})$$



Best attempt so far by R Zalaletdinov's "Macroscopic Gravity:

Zalaletdinov R M, *Averaging Problem in Cosmology and Macroscopic Gravity*, Online Proceedings of the Atlantic Regional Meeting on General Relativity and Gravitation, Fredericton, NB, Canada, May 2006, ed. R.J. McKellar (*Preprint* arXiv:gr-qc/0701116)

Coley A A and Pelavas N 2007 Phys.Rev. D **75** 043506; Coley A A, Pelavas N and Zalaletdinov R M 2005 *Phys.Rev.Lett.* **95** 151102

=> Macroscopic gravity is a non-perturbative geometrical approach (Zalaletdinov - 1992-2005) to resolve the Averaging Problem: a reformulation in a broader context as the problem of macroscopic description of gravitation

▲ Classical physical phenomena possess two levels of description (Lorentz, 1897, 1916):

The microscopic description ⟺ The discrete matter model

↓ by a suitable averaging procedure ↓

The macroscopic description ⟺ The continuous matter model

Lorentz 'theory of electrons — Maxwell's electrodynamics

$$F^{\mu\nu}_{\ ,\nu} = \frac{4\pi}{c} j^\mu = 4\pi \sum_i q_i u^\mu(t_i) \quad \rightarrow \langle\text{averaging}\rangle \rightarrow \quad H^{\mu\nu}{}_{,\nu} = \frac{4\pi}{c}\langle j\rangle^\mu = \frac{4\pi}{c}(J^\mu - cP^{\mu\nu}_{\ ,\nu})$$

$$F_{[\alpha\beta,\gamma]} = 0 \quad \rightarrow \langle\text{averaging}\rangle \rightarrow \quad \langle F\rangle_{[\alpha\beta,\gamma]} = 0, \ H^{\mu\nu} = \langle F\rangle^{\mu\nu} + 4\pi P^{\mu\nu}$$

Problem: Macroscopic Gravity is INTRACTABLE.

ALSO: we should take these analogies with a big "grain of salt", as they often fail ...

- **XIX century analogy between elastic waves and electromagnetic waves --- gave rise to the notion of "ether" as fixed reference frame**

Failed !!

- **Canonical Quantization (Wheeler De Witt) 1970-1980's attempts to quantize gravity by analogy with quantization of electromagnetism: Hamiltonian formalism + "canonical variables" (p,q) + Poisson Parenthesis ---- Classical Commutators become Quantum Operators (Klein-Gordon equation in "super space")**

Failed !!

Consolation: we know how to do "spatial" average of covariant scalars along 3-dimensional "time slices"
Problem: only applicable to spacetimes for which Einstein Equations can be reduced to purely scalar modes (LRS spacetimes):

- 3-dimensional Lie groups with 2-dimensional orbits (spherical symmetry
- Most Bianchi models

Best known formalism by Thomas Buchert.

Buchert T, 2000 *Gen. Rel. Grav.* **32** 105; Buchert T, 2000 *Gen.Rel.Grav.* **32** 306-321; Buchert T 2001 *Gen. Rel. Grav.* **33** 1381-1405; Ellis G F R and Buchert T 2005 *Phys.Lett.* A**347** 38-46; Buchert T and Carfora M 2002 *Class.Quant.Grav.* **19** 6109-6145; Buchert T 2006 *Class. Quantum Grav.* 23 819; Buchert T, Larena J and Alimi J M 2006 *Class. Quantum Grav.* 23 6379; Buchert T 2005 *Class. Quantum Grav.* **22** L113–L119; Buchert T 2006 *Class. Quantum Grav.* **23** 817–844 (*Preprint* arXiv:gr-qc/0509124)

Good things: it is a tractable formalism, it provides a non-trivial modification of the dynamics by emergence of "back-reaction" terms (the statistical correlation functions from non-linearity)

Problem: does not yield a closed self-consistent set of dynamical equations unless we make "ad hoc" assumptions on the back-reaction terms

Examine scalar averaging by means of Szekeres dust models

Why Szekeres models?

Szekeres models offer a much better description of cosmological structure than spherical LTB models

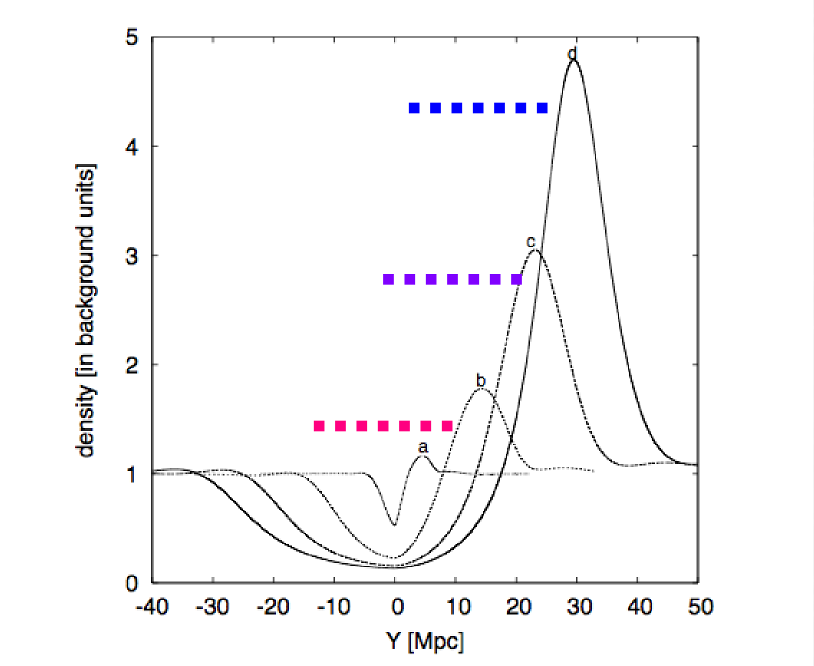
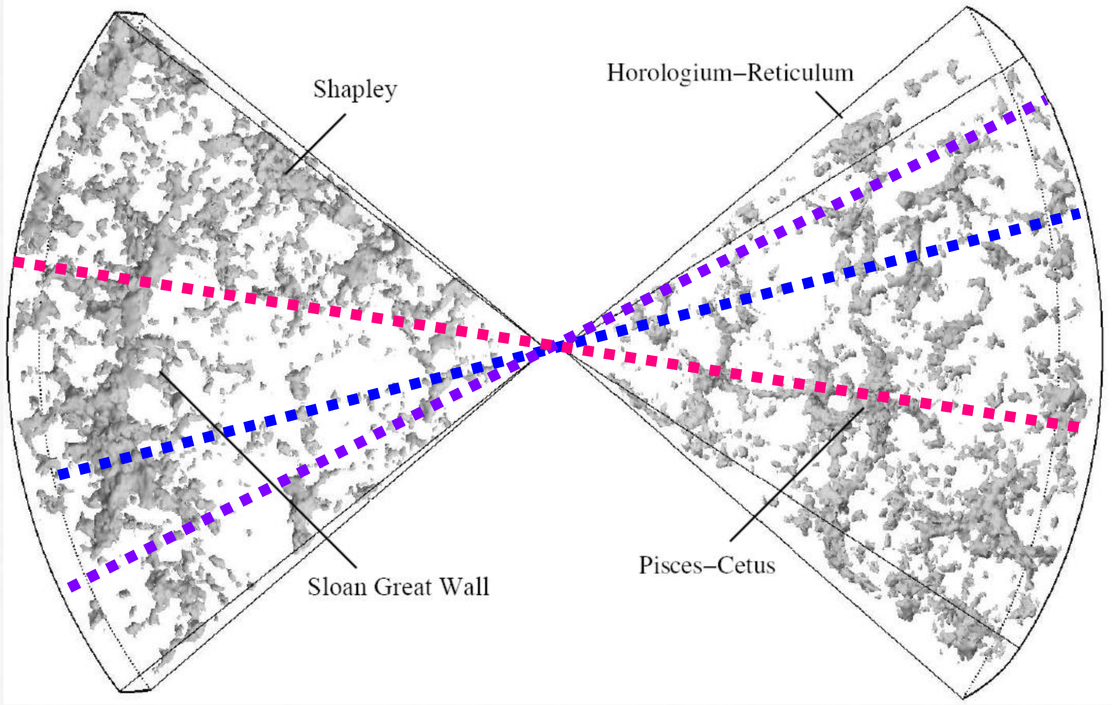
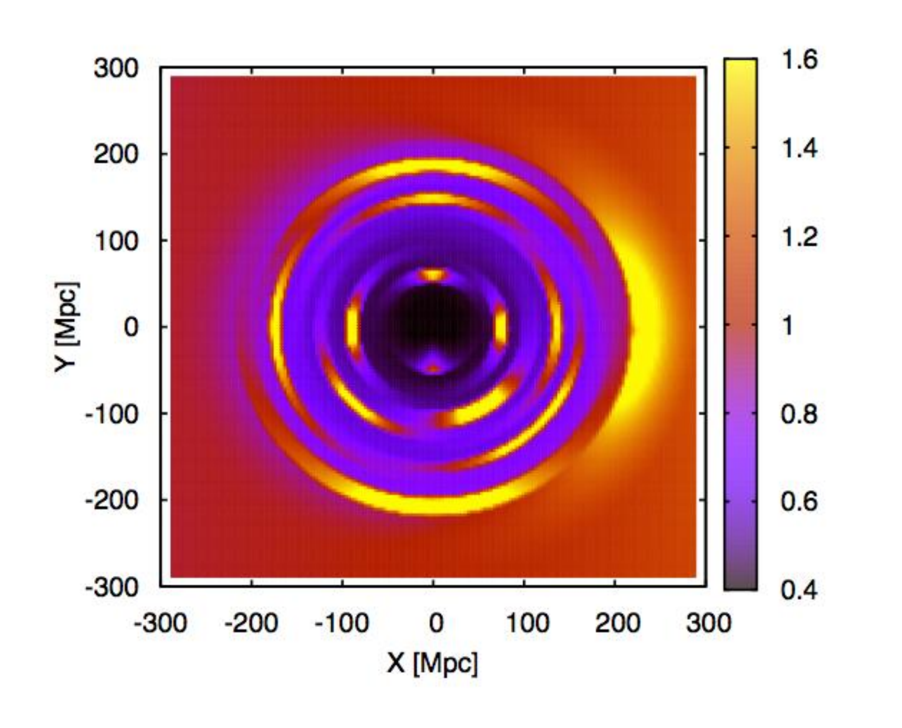


Figure 5. The density profile for diffrent time instants: a — 1 Gy after the Big Bang, b — 5.5 Gy, c — 10 Gy, d — present instant.





But observations are not the full story

Szekeres models are theoretically interesting !!

★ **They are among the less idealized inhomogeneous & anisotropic geometries: do not admit isometries (in general). NOT spherically nor axially symmetric.**

→ **mathematically interesting candidates to test theoretical issues not (necessarily) related to fitting observations.**

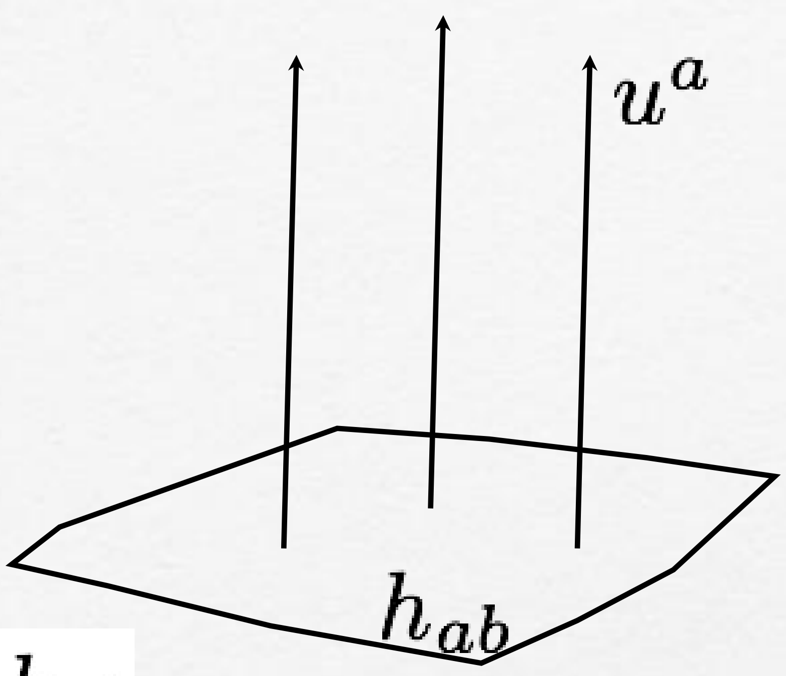
→ **Averaging: connection to Perturbation theory and Statistical Mechanics:**

FLRW background ↔ averaged scalars

Perturbations ↔ local scalars

Gravitational entropy ↔ inhomogeneity (disorder)

Dynamics through 1+3 formalism



Covariant objects:

density	$\rho,$
expansion	$H \equiv \frac{\theta}{3}$
shear	$\sigma_{ab} = \tilde{\nabla}_{(a} u_{b)} - H\, h_{ab}$
electric Weyl	$E_{ab} = u^c u^d\, C_{acbd}$
spatial curvature	$K \equiv \frac{{}^3\mathcal{R}}{6}$

Zero vorticity &
zero 4-acceleration

Einstein's Equations = Dynamical System

$$\dot{\rho} = -3\,H\,\rho$$

$$\dot{H} = -H^2 - \frac{4\pi}{3}\rho - \frac{1}{3}\sigma_{ab}\sigma^{ab}$$

$$\dot{\sigma}_{\langle ab\rangle} = -2\,H\,\sigma_{ab} - \sigma_{\langle ac}\sigma^{c}_{b\rangle} - E_{ab}$$

$$\dot{E}_{\langle ab\rangle} = -3\,H\,E_{ab} - 4\pi\,\rho\,\sigma_{ab} + 3\,\sigma_{\langle ac}E^{c}_{b\rangle}$$

FLRW subset

$$\{\rho,\,H,\,K\}$$

Constraints

$$\tilde{\nabla}_b\,\sigma^b_a - 2\,\tilde{\nabla}_a H = 0$$

$$\tilde{\nabla}_b\,E^b_a - \frac{4\pi}{3}\,\tilde{\nabla}_a\rho = 0$$

$$H^2 = \frac{8\pi}{3}\rho - K - \sigma_{ab}\sigma^{ab}$$

Hamiltonian constraint

Are there dynamical effects from averaging ?

Yes because General Relativity is a NON-LINEAR theory:

$$\left\langle \frac{\partial A}{\partial t} \right\rangle \neq \frac{\partial}{\partial t} \langle A \rangle \qquad \langle AB \rangle \neq \langle A \rangle \langle B \rangle$$

FLRW Raychaudhury equation with Lambda:

$$\dot{\mathcal{H}} = -\mathcal{H}^2 - \frac{4\pi}{3}\rho + \Lambda,$$

Szekeres Raychaudhury equation without Lambda:

$$\dot{\mathcal{H}} = -\mathcal{H}^2 - \frac{4\pi}{3}\rho - \sigma_{ab}\sigma^{ab},$$

Let's average Sz equation:

$$\langle \mathcal{H} \rangle = \frac{\int \mathcal{H}\, dV_p}{\int dV_p}, \qquad \langle \rho \rangle = \frac{\int \rho\, dV_p}{\int dV_p}$$

The result is:

$$\langle \mathcal{H} \rangle^{\cdot} = -\langle \mathcal{H} \rangle^2 - \frac{4\pi}{3}\langle \rho \rangle + \mathcal{Q},$$

where the "backreaction" is

$$\mathcal{Q} \equiv \langle\, (\mathcal{H} - \langle \mathcal{H} \rangle)^2 \rangle - \langle \sigma_{ab}\sigma^{ab} \rangle,$$

Therefore, if: $\mathcal{Q} > 0$ IT CAN PLAY THE ROLE OF A COSMOLOGICAL CONSTANT (accelerating expansion of averages)

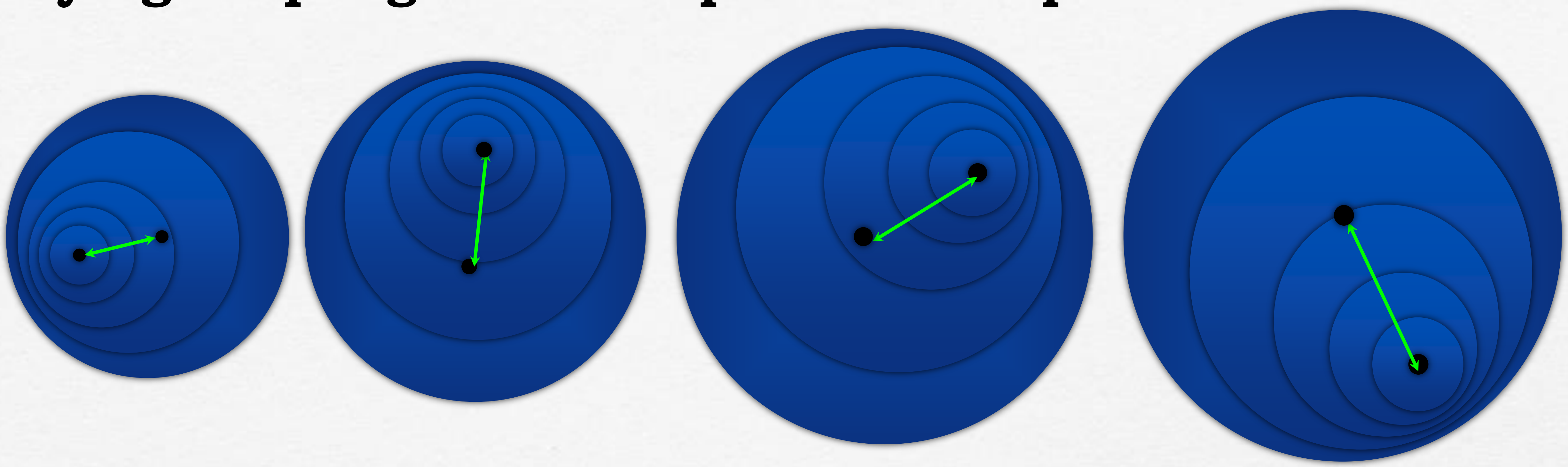
- Can we observed the effects of Back-Reaction?

  YES (IN PRINCIPLE)

- Can we use Back-Reaction to substitute the Cosmological Constant of the concordance LCDM model?

  Unfortunately NO

EFFECTS OF AVERAGING: **Szekeres models are a time varying coupling of a monopole and a dipole:**



scalars expressible as: $$A = A^{(m)}(t,r) + A^{(d)}(t,r,\theta,\phi)$$

but for every domain we have: $$\langle A^{(d)} \rangle_q = 0$$

Therefore: averages are spherically symmetric $$\langle A \rangle_q = \langle A^{(m)} \rangle_q (t,r)$$

Conjecture: averaging "smoothes out" non-spherical structure

Try the following task: weighed average

Construct averages ("q-averages") with WEIGHT factor F such that

★ **q-averages of scalars common to FLRW:** $\langle \rho \rangle_q,\ \langle H \rangle_q,\ \langle K \rangle_q,$
satisfy FLRW evolution laws

★ **Define dimensionless q-fluctuations**

$$\delta^{(\rho)} = \frac{\rho - \langle \rho \rangle_q}{\langle \rho \rangle_q}, \qquad \delta^{(H)} = \frac{H - \langle H \rangle_q}{\langle H \rangle_q}, \qquad \delta^{(K)} = \frac{K - \langle K \rangle_q}{\langle K \rangle_q},$$

DEMAND that the dynamics of Szekeres models is completely determined by q-averages and their q-fluctuations

The idea leads to a RIGOROUS perturbation formalism in which:

★ **FLRW "background" defined by q-averaged scalars**

★ **The "perturbations" are the q-fluctuations**

The result is encouraging: Szekeres = exact perturbations on FLRW

Evolution equations:

$$\langle\rho\rangle\dot{}_q = -3\langle\rho\rangle_q\langle H\rangle_q,$$
$$\langle H\rangle\dot{}_q = -\langle H\rangle_q^2 - \frac{4\pi}{3}\langle\rho\rangle_q,$$

FLRW evolution equations "background" evolution.

$$\dot{\delta}^{(\rho)} = -3(1+\delta^{(\rho)})\,\langle H\rangle_q\delta^{(H)},$$
$$\dot{\delta}^{(H)} = -(1+3\delta^{(H)})\,\langle H\rangle_q\delta^{(H)} + \frac{4\pi\,\langle\rho\rangle_q}{3\langle H\rangle_q}(\delta^{(H)}-\delta^{(\rho)}),$$

evolution of "perturbations"

Constraints are purely algebraic:

$$2\delta^{(H)} = \langle\Omega\rangle_q\,\delta^{(\rho)} + [1-\langle\Omega\rangle_q]\;\delta^{(K)},$$
$$\delta^{(\Omega)} = \delta^{(\rho)} - 2\delta^{(H)},$$
$$\langle\Omega\rangle_q = \frac{8\pi\,\langle\rho\rangle_q}{3\langle H\rangle_q^2},$$
$$\langle H\rangle_q^2 = \frac{8\pi}{3}\,\langle\rho\rangle_q - \langle K\rangle_q,$$

Friedman FLRW equation and FLRW Omega factor

The perturbations provide an invariant measure of inhomogeneity

$$\delta^{(\rho)} = \frac{\xi}{1-\xi}, \qquad \xi \equiv \frac{\psi_2}{\mathcal{R}}$$

Ratio of Weyl to Ricci curvature.

$$\delta^{(H)} = -\frac{\zeta}{1-\zeta}, \qquad \zeta \equiv \frac{\Sigma}{H}$$

where: Σ is the eigenvalue of σ_{ab}

Ratio of anisotropic to isotropic expansion.

Relation between fluctuations and invariants

$$\mathbf{D}(A) = A(t,r,x,y) - \langle A\rangle_q(t,r)$$

$$\mathcal{R}^{ab}_{cd} = \; = \; \frac{8\pi}{3}\rho\left(3\delta^{[a}_{[c}\delta^{b]}_{d]} + 6\delta^{[a}_{[c}u^{b]}u_{d]} - \delta^{a}_{[c}\delta^{b}_{d]}\right) - \frac{4\pi}{3}\mathbf{D}(\rho)\,\left(\mathrm{h}^{[a}_{[c} - 3\mathrm{u}_{[c}\mathrm{u}^{[a}\right)\,\mathbf{e}^{b]}_{d]}, \tag{19}$$

$$\mathcal{R}^{a}_{b} = 4\pi\rho\left(h^{a}_{b} + u^{a}u_{b}\right), \tag{20}$$

$$E_{ab} = -\frac{4\pi}{3}\mathbf{D}(\rho)\,\mathbf{e}_{\mathrm{ab}}, \qquad \mathrm{C}^{\mathrm{ab}}_{\mathrm{cd}} = -\frac{4\pi}{3}\mathbf{D}(\rho)\,\left(\mathrm{h}^{[a}_{[c} - 3\mathrm{u}_{[c}\mathrm{u}^{[a}\right)\,\mathbf{e}^{b]}_{d]}, \tag{21}$$

$$\sigma_{ab} = -\mathbf{D}(\mathcal{H})\,\mathbf{e}_{\mathrm{ab}}, \qquad \mathcal{H}_{\mathrm{ab}} = \mathcal{H}\mathrm{h}_{\mathrm{ab}} - \mathbf{D}(\mathcal{H})\,\mathbf{e}_{\mathrm{ab}}, \tag{22}$$

while their scalar contractions take the form:

$$\mathcal{R}_{abcd}\mathcal{R}^{abcd} = \frac{256\pi^2}{3}\left([\mathbf{D}(\rho)]^2 + \frac{5}{4}\rho^2\right), \qquad \mathcal{R}_{ab}\mathcal{R}^{ab} = 64\pi^2\rho^2, \tag{23}$$

$$C_{abcd}C^{abcd} = \frac{256\pi^2}{3}[\mathbf{D}(\rho)]^2 = 8\mathrm{E}_{\mathrm{ab}}\mathrm{E}^{\mathrm{ab}}, \tag{24}$$

$$\sigma_{ab}\sigma^{ab} = 6[\mathbf{D}(\mathcal{H})]^2, \qquad \sigma_{\mathrm{ab}}\mathrm{E}^{\mathrm{ab}} = \frac{4\pi}{3}\mathbf{D}(\rho)\mathbf{D}(\mathcal{H}). \tag{25}$$

Averages of quadratic invariants are statistical moments: variances & covariances of the density and Hubble scalar:

$$\mathbf{Var}_q(A) \equiv \langle A^2\rangle - \langle A\rangle^2 \qquad \mathbf{Cov}_q(A,B) \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$$

$$\begin{aligned}
\langle \sigma_{ab}\sigma^{ab}\rangle_q &= 6\langle \Sigma^2\rangle_q = 6\mathbf{Var}_q(\mathcal{H}),\\
\langle E_{ab}E^{ab}\rangle_q &= 6\langle \mathcal{E}^2\rangle_q = 6\langle(\Psi_2)^2\rangle_q = \frac{32\pi^2}{3}\mathbf{Var}_q(\rho),\\
\langle \sigma_{ab}E^{ab}\rangle_q &= 6\langle \Sigma\,\mathcal{E}\rangle_q = 8\pi\mathbf{Cov}_q(\rho,\mathcal{H}),
\end{aligned}$$

$$\langle \mathcal{R}_{abcd}\mathcal{R}^{abcd}\rangle_q = \frac{256\pi^2}{3}\left[\mathbf{Var}_q(\rho) + \frac{5}{4}\langle\rho^2\rangle_q\right] = \frac{4}{3}\left[\mathbf{Var}_q(\mathcal{R}) + \frac{5}{4}\langle\mathcal{R}^2\rangle_q\right], \qquad (32)$$

$$\langle C_{abcd}C^{abcd}\rangle_q = \frac{256\pi^2}{3}\mathbf{Var}_q(\rho) = \frac{4}{3}\mathbf{Var}_q(\mathcal{R}) = 8\langle E_{ab}E^{ab}\rangle_q, \qquad (33)$$

where we used the fact that $\langle\mathcal{R}\rangle_q = 8\pi\langle\rho\rangle_q$ and $\mathcal{R}_{ab}\mathcal{R}^{ab} = \mathcal{R}^2$.

The gravitational entropy functional of Morita & Buchert)

$$S - S_{\mathrm{eq}} = \gamma_0 \int_{\mathcal{D}} p_i \ln\left[\frac{p_i}{P}\right] \mathcal{F} dV_p = \gamma_0 \int_{\mathcal{D}} \rho \ln\left[\frac{\rho}{\langle\rho\rangle_q}\right] \mathcal{F} dV_p$$

$$\frac{\dot{S}}{\mathcal{V}_q} = -3\gamma_0 \mathbf{Cov}_q(\rho, \mathcal{H}) = -3\gamma_0 \langle \mathbf{D}(\rho)\, \mathbf{D}(\mathcal{H})\rangle_{\mathrm{q}} \geq 0, \qquad (37)$$

so that:

$$\mathbf{Cov}_q(\rho, \mathcal{H}) = \langle \mathbf{D}(\mathcal{H})\mathbf{D}(\rho)\rangle_{\mathrm{q}}[\mathrm{r}] < 0 \;\Rightarrow\; \dot{\mathrm{S}}(\mathrm{r}) > 0, \qquad (38)$$

This condition can also be given in terms of the q–average of a scalar invariant by:

$$\langle \sigma_{ab} E^{ab} \rangle_q [r] < 0 \;\Rightarrow\; \dot{S}(r) > 0, \qquad (39)$$

which is a very elegant way to connect (34) with an unequivocal and completely coordinate independent marker of inhomogeneity, as it contains contributions from density and velocity fluctuations. It remains to prove in

Connection and/or analogies to Statistical Mechanics

WARNING: **under construction, so expect lots of hand waiving !!!**

Phase space in Microcanonical ensamble

Take as phase space coordinates:

$$\{p, q\} = \{\rho,\, H\}$$

Allowed phase space states:

$$\{\Delta p, \Delta q\} = \{1 + \delta^{(\rho)},\, 1 + \delta^{(H)}\} = \{\tfrac{\rho}{\langle\rho\rangle_q}, \tfrac{H}{\langle H\rangle_q}\}$$

Phase space volume occupied by Szekeres model:

$$\omega = \left[1 + \delta^{(\rho)}\right]\left[1 + \delta^{(H)}\right]$$

Microcanonical entropy: available volume in phase space

$$S = k_B \ln \omega = k_B \left[\ln(1 + \delta^{(\rho)}) + \ln(1 + \delta^{(H)})\right]$$

Notice: **FLRW models are a point of phase space with zero entropy**

$$\delta^{(\rho)} = \delta^{(H)} = 0$$

Canonical Ensamble (part 2)

SYSTEM: **compact comoving domain,**

HEAT BATH: reamining "exterior" of manifold.

Partition function $Z(\beta, J_\rho, J_K) = \int_{\mathcal{D}} \exp(-\beta H - J_\rho \rho - J_K K)\, dV_p,$

★ **Ensamble averages are expectation values:**

$$-\left[\frac{\partial}{\partial \beta} \ln Z\right]_{J_\rho = J_K = 0} = \frac{\int H e^{-\beta H} dV_p}{\int e^{-\beta H} dV_p} = \langle H \rangle_q$$

$$-\left[\frac{\partial}{\partial J_\rho} \ln Z\right]_{J_\rho = J_K = 0} = \frac{\int \rho\, e^{-\beta H} dV_p}{\int e^{-\beta H} dV_p} = \langle \rho \rangle_q \qquad \Rightarrow \qquad \beta = -\frac{\ln F}{H}$$

$$-\left[\frac{\partial}{\partial J_K} \ln Z\right]_{J_\rho = J_K = 0} = \frac{\int K\, e^{-\beta H} dV_p}{\int e^{-\beta H} dV_p} = \langle K \rangle_q$$

★ **Entropy:**

$$S = k_B \left[\beta \langle H \rangle_q + \ln Z\right] = k_B \left[-\frac{\ln F}{1+\delta^{(H)}} + \ln \int_{\mathcal{D}} F \exp(-J_\rho\, \rho - J_K\, K)\, dV_p\right]$$

Must satisfy: $\dot{S} \geq 0, \qquad \nabla_a S^a \geq 0$ where $S^a = n\, S\, u^a$

THANKS FOR YOUR ATTENTION