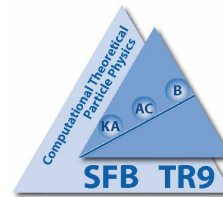


Recent Results on Three Loop Corrections to Heavy Quark Contributions to DIS Structure Functions

Johannes Blümlein, DESY

in collaboration : with J. Ablinger (JKU), A. De Freitas (DESY), A. Hasselhuhn (DESY),
C. Schneider (JKU), F. Wißbrock (DESY)

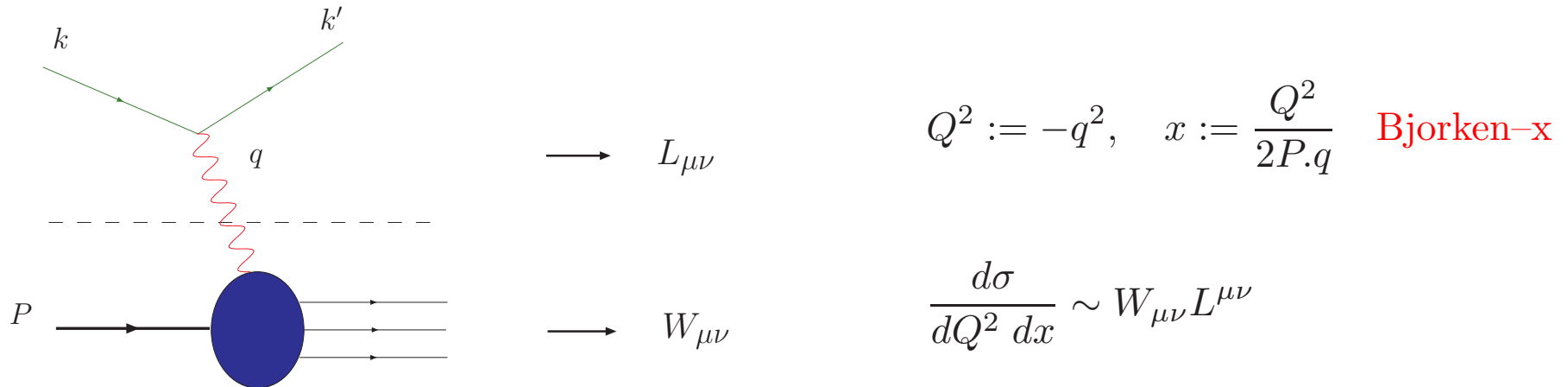


- Introduction
- 3-Loop Gluonic OMEs $O(n_f T_F^2 C_{F,A})$ and FVNS
- 3-Loop Graphs with m_c and m_b
- A General Method to Compute Non-Singular 3-Loop Massive Graphs
- Conclusions

arXiv:1205.4184, 1206.2252

Introduction

Unpolarized Deep-Inelastic Scattering (DIS):



$$\begin{aligned}
 W_{\mu\nu}(q, P, s) &= \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P, s | [J_\mu^{em}(\xi), J_\nu^{em}(0)] | P, s \rangle \\
 &= \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \frac{2x}{Q^2} \left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu} \right) F_2(x, Q^2) .
 \end{aligned}$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)} \left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right)}_{\text{perturbative}} \otimes \underbrace{f_j(x, \mu^2)}_{\text{nonpert.}}$$

into (pert.) **Wilson coefficients** and (nonpert.) **parton distribution functions (PDFs)**.

\otimes denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \delta(x - yz) f(y) g(z) .$$

The subsequent calculations are performed in Mellin space, where \otimes reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx x^{N-1} f(x) .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = C_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) + H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)} \left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i C_{i,(2,L)} \left(N, \frac{Q^2}{\mu^2} \right) A_{ij} \left(\frac{m^2}{\mu^2}, N \right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the **light flavor Wilson coefficients** C and the **massive operator matrix elements (OMEs)** of local operators O_i between partonic states j

$$A_{ij} \left(\frac{m^2}{\mu^2}, N \right) = \langle j | O_i | j \rangle .$$

→ additional **Feynman rules with local operator insertions** for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are **known up to NNLO**

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x, Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

The Heavy Flavor Wilson Coefficients

$$\begin{aligned}
L_{2,q}^{\text{NS}}(n_f) &= a_s^2 \left[A_{qq,Q}^{\text{NS},(2)}(n_f) + \hat{C}_{2,q}^{\text{NS},(2)}(n_f) \right] \\
&+ a_s^3 \left[A_{qq,Q}^{\text{NS},(3)}(n_f) + A_{qq,Q}^{\text{NS},(2)}(n_f) C_{2,q}^{\text{NS},(1)}(n_f) + \hat{C}_{2,q}^{\text{NS},(3)}(n_f) \right] \\
\tilde{L}_{2,q}^{\text{PS}}(n_f) &= a_s^3 \left[\tilde{A}_{qq,Q}^{\text{PS},(3)}(n_f) + A_{qq,Q}^{(2)}(n_f) \tilde{C}_{2,g}^{(1)}(n_f + 1) + \hat{C}_{2,q}^{\text{PS},(3)}(n_f) \right] \\
\tilde{L}_{2,g}^{\text{S}}(n_f) &= a_s^2 A_{gg,Q}^{(1)}(n_f) \tilde{C}_{2,g}^{(1)}(n_f + 1) \\
&+ a_s^3 \left[\tilde{A}_{gg,Q}^{(3)}(n_f) + A_{gg,Q}^{(1)}(n_f) \tilde{C}_{2,g}^{(2)}(n_f + 1) + A_{gg,Q}^{(2)}(n_f) \tilde{C}_{2,g}^{(1)}(n_f + 1) \right. \\
&\quad \left. + A_{Qg}^{(1)}(n_f) \tilde{C}_{2,q}^{\text{PS},(2)}(n_f + 1) + \hat{C}_{2,g}^{(3)}(n_f) \right] \\
H_{2,q}^{\text{PS}}(n_f) &= a_s^2 \left[A_{Qq}^{\text{PS},(2)}(n_f) + \tilde{C}_{2,q}^{\text{PS},(2)}(n_f + 1) \right] \\
&+ a_s^3 \left[A_{Qq}^{\text{PS},(3)}(n_f) + \tilde{C}_{2,q}^{\text{PS},(3)}(n_f + 1) + A_{qq,Q}^{(2)}(n_f) \tilde{C}_{2,g}^{(1)}(n_f + 1) \right. \\
&\quad \left. + A_{Qq}^{\text{PS},(2)}(n_f) C_{2,q}^{\text{NS},(1)}(n_f + 1) \right] \\
H_{2,g}^{\text{S}}(n_f) &= a_s \left[A_{Qg}^{(1)}(n_f) + \tilde{C}_{2,g}^{(1)}(n_f + 1) \right] \\
&+ a_s^2 \left[A_{Qg}^{(2)}(n_f) + A_{Qg}^{(1)}(n_f) C_{2,q}^{\text{NS},(1)}(n_f + 1) + A_{gg,Q}^{(1)}(n_f) \tilde{C}_{2,g}^{(1)}(n_f + 1) \right. \\
&\quad \left. + \tilde{C}_{2,g}^{(2)}(n_f + 1) \right] \\
&+ a_s^3 \left[A_{Qg}^{(3)}(n_f) + A_{Qg}^{(2)}(n_f) C_{2,q}^{\text{NS},(1)}(n_f + 1) + A_{gg,Q}^{(2)}(n_f) \tilde{C}_{2,g}^{(1)}(n_f + 1) \right. \\
&\quad \left. + A_{Qg}^{(1)}(n_f) \left[C_{2,q}^{\text{NS},(2)}(n_f + 1) + \tilde{C}_{2,q}^{\text{PS},(2)}(n_f + 1) \right] + A_{gg,Q}^{(1)}(n_f) \tilde{C}_{2,g}^{(2)}(n_f + 1) \right. \\
&\quad \left. + \tilde{C}_{2,g}^{(3)}(n_f + 1) \right].
\end{aligned}$$

Status of OME calculations

Leading Order: [Witten, 1976 Nucl.Phys.B; Babcock, Sivers, 1978 Phys.Rev.D; Shifman, Vainshtein, Zakharov, 1978 Nucl.Phys.B; Leveille, Weiler, 1979 Nucl.Phys.B; Glück, Reya, 1979 Phys.Lett.B; Glück, Hoffmann, Reya, 1982 Z.Phys.C.]

Next-to-Leading Order : [Laenen, van Neerven, Riemersma, Smith, 1993 Nucl. Phys. B]

[Large Q^2/m^2 : Buza, Matiounine, Smith, Migneron, van Neerven, 1996 Nucl.Phys.B] IBP

[Bierenbaum, Blümlein, Klein, 2007 Nucl.Phys.B] via $_pF_q$'s, more compact results

[Bierenbaum, Blümlein, Klein 2008 Nucl.Phys.B, 2009 Phys.Lett.B]: $O(\alpha_s^2 \varepsilon)$ contributions (all N)

NNLO:[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] Moments for F_2 : $N = 2 \dots 10(14)$

[Blümlein, Klein, Tödtli 2009 Phys. Rev. D] contrib. to transversity: $N = 1 \dots 13$

[Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl.Phys.B] contrib. $\propto n_f$ to F_2 (all N):

At 3-loop order known:

- $A_{qq,Q}^{\text{PS}}, A_{qg,Q}$: **complete**; $A_{Qg}, A_{Qq}^{\text{PS}}, A_{qq,Q}^{\text{NS}}, A_{qq,Q}^{\text{NS,TR}}$: all terms of $O(n_f T_F^2 C_{A/F})$
- $A_{Qq}^{\text{PS}}, A_{qq,Q}^{\text{NS}}, A_{qq,Q}^{\text{NS,TR}}$: all terms of $O(T_F^2 C_{A/F})$
- $A_{gq,Q}, A_{gg,Q}$: see [this talk](#) \longrightarrow all terms of $O(n_f T_F^2 C_{A/F})$
- Ladder and Benz topologies with a single massive line: first results [this talk](#).

VFNS Relations for PDFs

The matching conditions for the the VFNS:

[Buza, Matiounine, Smith, van Neerven 1998 Eur.Phys.J.C] → NLO

[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] → NNLO

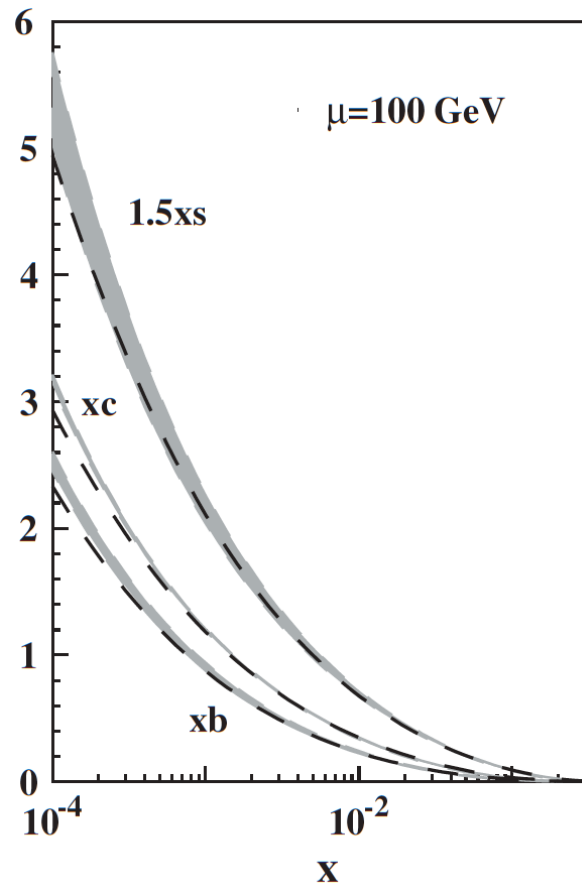
$$\begin{aligned}
 & f_k(N, n_f + 1, \mu^2, m^2) + f_{\bar{k}}(N, n_f + 1, \mu^2, m^2) \\
 &= A_{qq,Q}^{\text{NS}} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes [f_k(N, n_f, \mu^2, m^2) + f_{\bar{k}}(N, n_f, \mu^2, m^2)] \\
 &+ \frac{1}{n_f} A_{qq,Q}^{\text{PS}} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes \Sigma(N, n_f, \mu^2, x) + \frac{1}{n_f} A_{qg,Q} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes G(N, n_f, \mu^2, x)
 \end{aligned}$$

$$\begin{aligned}
 & f_Q(N, n_f + 1, \mu^2, m^2) + f_{\bar{Q}}(N, n_f + 1, \mu^2, m^2) \\
 &= A_{Qq}^{\text{PS}} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes \Sigma(N, n_f, \mu^2, m^2) + A_{Qg} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes G(N, n_f, \mu^2, m^2)
 \end{aligned}$$

$$\begin{aligned}
 & G(N, n_f + 1, \mu^2, m^2) \\
 &= A_{gq,Q} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes \Sigma(N, n_f, \mu^2, m^2) + A_{gg,Q} \left(N, n_f, \frac{\mu^2}{m^2} \right) \otimes G(N, n_f, \mu^2, m^2)
 \end{aligned}$$

where: $\left(\Sigma(N, n_f, \dots) = \sum_{k=1}^{n_f} (f_k + f_{\bar{k}}), \quad n_f = 3 \right)$

The ABKM09 heavy flavor PDFs: (grey band) compared to MSTW2008 NNLO (dashed line):
 [Alekhin, Blümlein, Klein, Moch 2010 Phys.Rev.D] [Martin, Stirling, Thorne, Watt 2010 Eur.Phys.J.C]



Influence on W^+ production at the LHC (e.g. $\sqrt{s} = 7$ TeV):

$$\text{ABM11}(n_f = 5): \sigma_{W^+} = 59.53^{+0.38}_{-0.23} {}^{+0.88}_{-0.88}$$

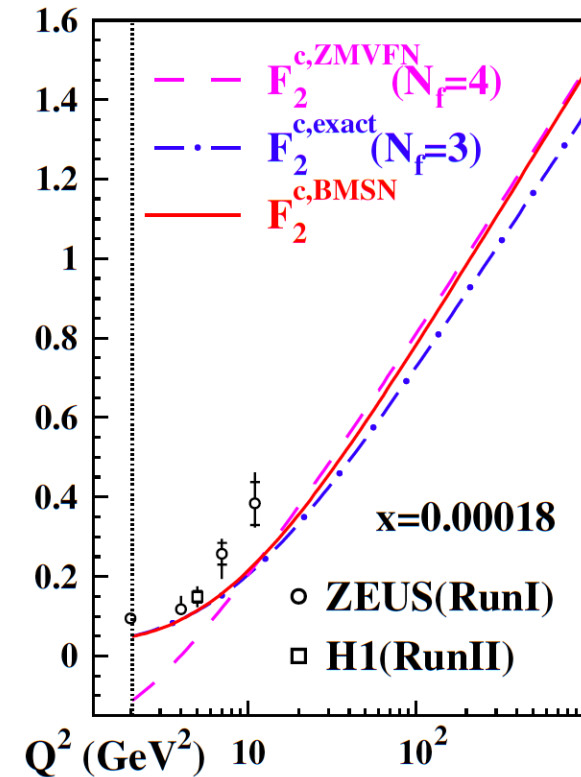
$$\text{ABM11}(n_f = 4): \sigma_{W^+} = 59.08^{+0.30}_{-0.14} {}^{+0.87}_{-0.87}$$

[Alekhin, Blümlein, Moch 2012]

BMSN-scheme for the heavy quark part F_2^h in F_2 :

[Buza, Matiounine, Smith, van Neerven 1998 Eur.Phys.J.C], [Alekhin, Blümlein, Klein, Moch 2010 Phys.Rev.D]

$$F_2^{h,\text{BMSN}}(n_f + 1, x, Q^2) = F_2^{h,\text{exact}}(n_f, x, Q^2) + F_2^{h,\text{ZMVFN}}(n_f + 1, x, Q^2) - F_2^{h,\text{asymp}}(n_f, x, Q^2)$$



The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gg,Q}$

calculation of the 1PI part of $A_{gg,Q}^{(3),n_f T_F^2}$

- generation of Diagrams with QGRAF [Nogueira 1993 J. Comput. Phys] → 76 Diagrams
- momentum integrals (regularized in $4 + \varepsilon$ dimensions) → Feynman parameterization → finite sums and hypergeometric functions
- All- ε representation: maximum nestedness 2, hypergeometric functions ${}_3F_2$

Moments were tested using earlier calculations based on MATAD by [M. Steinhauser, 2000 CPC].

Then the package Sigma [C. Schneider, 2005–] is used for:

- reducing the sums to a small number of key sums
- expanding the summands in ε
- simplifying by symbolic summation algorithms based on $\Pi\Sigma$ -fields [Karr 1981 J. ACM, Schneider 2005–]
- harmonic sums are algebraically reduced using the package HarmonicSums (Ablinger) [Ablinger, Blümlein, Schneider 2011]

→ single harmonic sums and ζ -values of max. weight 3 $S_i \equiv \sum_{j=1}^N \frac{1}{j^i}$

$$\begin{aligned}
A_{ggQ}^{n_f T_f^2 1PI} = & S_\epsilon^3 a_s^3 n_f T_F^2 \frac{1 + (-1)^N}{2} \left(\frac{m^2}{\mu^2} \right)^{\frac{3}{2}\epsilon} \left\{ \frac{1}{\epsilon^3} \left(\mathbf{C}_A \left[\frac{512}{27} S_1 - \frac{64(3N^4 + 6N^3 + 13N^2 + 10N + 16)}{27(N-1)N(N+1)(N+2)} \right] \right. \right. \\
& - \mathbf{C}_F \frac{512(N^2 + N + 2)^2}{9(N-1)N^2(N+1)^2(N+2)} \left. \right) + \frac{1}{\epsilon^2} \left(\mathbf{C}_A \left[\frac{1280}{81} S_1 - \frac{16P_1}{81(N-1)N^2(N+1)^2(N+2)} \right] \right. \\
& + \mathbf{C}_F \frac{1}{(N-1)(N+2)} \left[\frac{128(N^2 + N + 2)^2}{9N^2(N+1)^2} S_1 - \frac{128P_2}{27N^3(N+1)^3} \right] \left. \right) \\
& + \frac{1}{\epsilon} \left(\mathbf{C}_A \frac{1}{(N-1)(N+2)} \left[-\frac{4P_8}{81N^3(N+1)^3} - \frac{8(3N^4 + 6N^3 + 13N^2 + 10N + 16)}{9N(N+1)} \zeta_2 \right. \right. \\
& + \left. \frac{32P_9}{27N^2(N+1)^2} S_1 + \frac{64}{9}(N-1)(N+2)\zeta_2 S_1 \right] + \mathbf{C}_F \frac{1}{(N-1)(N+2)} \left[-\frac{160(N^2 + N + 2)^2}{9N^2(N+1)^2} S_1^2 \right. \\
& - \left. \frac{64(N^2 + N + 2)^2}{3N^2(N+1)^2} \zeta_2 + \frac{32(N^2 + N + 2)^2}{3N^2(N+1)^2} S_2 - \frac{64P_{10}}{81N^4(N+1)^4} + \frac{64P_{11}}{27N^3(N+1)^3} S_1 \right] \left. \right) \\
& + \mathbf{C}_A \frac{1}{(N-1)(N+2)} \left[\frac{4P_3}{27N^2(N+1)^2} S_1^2 + \frac{8P_4}{729N^3(N+1)^3} S_1 + \frac{160}{27}(N-1)(N+2)\zeta_2 S_1 \right. \\
& - \frac{448}{27}(N-1)(N+2) \zeta_3 S_1 + \frac{P_5}{729N^4(N+1)^4} - \frac{2P_6}{27N^2(N+1)^2} \zeta_2 - \frac{4P_7}{27N^2(N+1)^2} S_2 \\
& + \left. \frac{56(3N^4 + 6N^3 + 13N^2 + 10N + 16)}{27N(N+1)} \zeta_3 \right] + \mathbf{C}_F \frac{1}{(N-1)(N+2)} \left[\frac{112(N^2 + N + 2)^2}{27N^2(N+1)^2} S_1^3 \right. \\
& - \frac{16P_{12}}{27N^3(N+1)^3} S_1^2 + \frac{32P_{13}}{81N^4(N+1)^4} S_1 + \frac{16(N^2 + N + 2)^2}{3N^2(N+1)^2} \zeta_2 S_1 + \frac{16(N^2 + N + 2)^2}{3N^2(N+1)^2} S_2 S_1 \\
& - \left. \frac{32P_{14}}{243N^5(N+1)^5} - \frac{16P_2}{9N^3(N+1)^3} \zeta_2 + \frac{448(N^2 + N + 2)^2}{9N^2(N+1)^2} \zeta_3 + \frac{16P_{15}}{9N^3(N+1)^3} S_2 - \frac{160(N^2 + N + 2)^2}{27N^2(N+1)^2} S_3 \right] \left. \right\}
\end{aligned}$$

Renormalization of the OME:

[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

1. include contributions from **reducible** diagrams
2. perform on-shell **mass** renormalization
3. renormalize the **coupling in a MOM-scheme**, using the background field method
4. remove remaining UV singularities through the **Z-factors** of the local operators
5. remove **collinear singularities** via coll. factorization (being different from the former one)
6. transform **coupling constant to $\overline{\text{MS}}$**
7. choice: m on-shell or $m_{\overline{\text{MS}}}$

Combining the 1PI part with the gluon self energy contribution:

$$\hat{\Pi}_{\mu\nu}^{ab}(p^2, \hat{m}^2, \mu^2, \hat{a}_s^2) = i\delta^{ab} [-g_{\mu\nu}p^2 + p_\mu p_\nu] \sum_{k=1}^{\infty} \hat{a}_s^k \hat{\Pi}^{(k)}(p^2, \hat{m}^2, \mu^2)$$

$$\hat{\Pi}^{(k)} \equiv \hat{\Pi}^{(k)}(0, \hat{m}^2, \mu^2)$$

giving:

$$\begin{aligned} \hat{A}_{gg,Q}^{(3)} &= \hat{A}_{gg,Q}^{(3),1\text{PI}} - \hat{\Pi}^{(3)} - \hat{A}_{gg,Q}^{(2),1\text{PI}} \hat{\Pi}^{(1)} - 2\hat{A}_{gg,Q}^{(1)} \hat{\Pi}^{(2)} + \hat{A}_{gg,Q}^{(1)} \hat{\Pi}^{(1)} \hat{\Pi}^{(1)} \\ &\equiv \frac{a_{gg,Q}^{(3,0)}}{\varepsilon^3} + \frac{a_{gg,Q}^{(3,1)}}{\varepsilon^2} + \frac{a_{gg,Q}^{(3,2)}}{\varepsilon} + a_{gg,Q}^{(3)} \end{aligned}$$

The renormalization determines the **coefficients of the ε -poles** in terms of coefficients from the **mass renormalization**: $\delta m_1^{(-1)}$, $\delta m_1^{(0)}$, $\delta m_1^{(1)}$, $\delta m_2^{(-2)}$, $\delta m_2^{(-1)}$, $\delta m_2^{(0)}$ contributions β_i to the **$\overline{\text{MS}}$ - β -function** and

$\beta_{i,Q}$ to the **MOM-beta-function** in QCD,

splitting functions: $\gamma_{ij} \equiv \sum_{l=1}^{\infty} a_s^{\overline{\text{MS}}l} \gamma_{ij}^{(l)}$ with $\hat{\gamma}_{ij}^{(k)} \equiv \gamma_{ij}^{(k)}(n_f + 1) - \gamma_{ij}^{(k)}(n_f)$ and

lower order contributions to the unrenormalized OMEs:

$$\hat{A}_{gg,Q}^{(2)} \equiv \frac{a_{gg,Q}^{(2,0)}}{\varepsilon^2} + \frac{a_{gg,Q}^{(2,1)}}{\varepsilon} + a_{gg,Q}^{(2)} + \varepsilon \bar{a}_{gg,Q}^{(2)} + O(\varepsilon^2)$$

$$\hat{A}_{gq,Q}^{(2)} \equiv \frac{a_{gq,Q}^{(2,0)}}{\varepsilon^2} + \frac{a_{gq,Q}^{(2,1)}}{\varepsilon} + a_{gq,Q}^{(2)} + \varepsilon \bar{a}_{gq,Q}^{(2)} + O(\varepsilon^2)$$

$$\hat{A}_{qg,Q}^{(2)} \equiv \frac{a_{qg,Q}^{(2,0)}}{\varepsilon^2} + \frac{a_{qg,Q}^{(2,1)}}{\varepsilon} + a_{qg,Q}^{(2)} + \varepsilon \bar{a}_{qg,Q}^{(2)} + O(\varepsilon^2)$$

[Bierenbaum, Blümlein, Klein 2007,2008 Nucl.Phys.B, 2009 Phys.Lett.B]

The structure of the unrenormalized OME: [Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B]

$$\begin{aligned}
\hat{A}_{gg,Q}^{(3)} = & \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2} \left[\frac{1}{\varepsilon^3} \left(-\frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)}}{6} \left[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 4n_f \beta_{0,Q} + 10\beta_{0,Q} \right] - \frac{2\gamma_{gg}^{(0)} \beta_{0,Q}}{3} \left[2\beta_0 + 7\beta_{0,Q} \right] \right. \right. \\
& - \frac{4\beta_{0,Q}}{3} \left[2\beta_0^2 + 7\beta_{0,Q}\beta_0 + 6\beta_{0,Q}^2 \right] \left. \right) + \frac{1}{\varepsilon^2} \left(\frac{\hat{\gamma}_{qg}^{(0)}}{6} \left[\gamma_{gq}^{(1)} - (2n_f - 1)\hat{\gamma}_{gq}^{(1)} \right] + \frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(1)}}{3} - \frac{\hat{\gamma}_{gg}^{(1)}}{3} \left[4\beta_0 + 7\beta_{0,Q} \right] \right. \\
& + \frac{2\beta_{0,Q}}{3} \left[\gamma_{gg}^{(1)} + \beta_1 + \beta_{1,Q} \right] + \frac{2\gamma_{gg}^{(0)} \beta_{1,Q}}{3} + \delta m_1^{(-1)} \left[-\hat{\gamma}_{qg}^{(0)} \gamma_{gg}^{(0)} - 2\beta_{0,Q} \gamma_{gg}^{(0)} - 10\beta_{0,Q}^2 - 6\beta_{0,Q}\beta_0 \right] \left. \right) \\
& + \frac{1}{\varepsilon} \left(\frac{\hat{\gamma}_{gg}^{(2)}}{3} - 2(2\beta_0 + 3\beta_{0,Q}) \mathbf{a}_{gg,Q}^{(2)} - n_f \hat{\gamma}_{qg}^{(0)} \mathbf{a}_{gq,Q}^{(2)} + \gamma_{gq}^{(0)} \mathbf{a}_{Qg}^{(2)} + \beta_{1,Q}^{(1)} \gamma_{gg}^{(0)} + \frac{\gamma_{gq}^{(0)} \hat{\gamma}_{qg}^{(0)} \zeta_2}{16} \left[\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} \right. \right. \\
& + 2(2n_f + 1)\beta_{0,Q} + 6\beta_0 \left. \right] + \frac{\beta_{0,Q} \zeta_2}{4} \left[\gamma_{gg}^{(0)} \{2\beta_0 - \beta_{0,Q}\} + 4\beta_0^2 - 2\beta_{0,Q}\beta_0 - 12\beta_{0,Q}^2 \right] \\
& + \delta m_1^{(-1)} \left[-3\delta m_1^{(-1)} \beta_{0,Q} - 2\delta m_1^{(0)} \beta_{0,Q} - \hat{\gamma}_{gg}^{(1)} \right] + \delta m_1^{(0)} \left[-\hat{\gamma}_{qg}^{(0)} \gamma_{gg}^{(0)} - 2\gamma_{gg}^{(0)} \beta_{0,Q} - 4\beta_{0,Q}\beta_0 - 8\beta_{0,Q}^2 \right] \\
& \left. + 2\delta m_2^{(-1)} \beta_{0,Q} \right) + \mathbf{a}_{gg,Q}^{(3)} \left. \right].
\end{aligned}$$

→ use for **checking** the ε singular parts

We **confirm** the $n_f T_F^2$ part of the 3-Loop anomalous dimension:

[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$\hat{\gamma}_{gg}^{(2)} = n_f T_F^2 \mathbf{C}_A \left[-\frac{32(8N^6 + 24N^5 - 19N^4 - 78N^3 - 253N^2 - 210N - 96)}{27(N-1)N^2(N+1)^2(N+2)} S_1 \right. \\ \left. - \frac{8(87N^8 + 348N^7 + 848N^6 + 1326N^5 + 2609N^4 + 3414N^3 + 2632N^2 + 1088N + 192)}{27(N-1)N^3(N+1)^3(N+2)} \right] \\ + n_f T_F^2 \mathbf{C}_F \left[\frac{64(N^2 + N + 2)^2}{3(N-1)N^2(N+1)^2(N+2)} (S_1^2 - 3S_2) - \frac{16P_1}{27(N-1)N^4(N+1)^4(N+2)} \right. \\ \left. + \frac{128(4N^6 + 3N^5 - 50N^4 - 129N^3 - 100N^2 - 56N - 24)}{9(N-1)N^3(N+1)^3(N+2)} S_1 \right]$$

$$P_1 = 33N^{10} + 165N^9 + 256N^8 - 542N^7 - 3287N^6 - 8783N^5 - 11074N^4 - 9624N^3 \\ - 5960N^2 - 2112N - 288$$

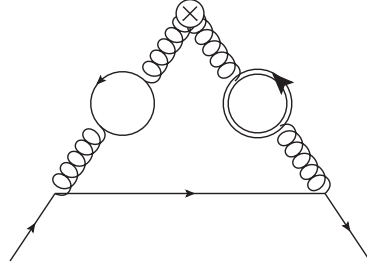
First diagrammatic recalculation

$$\begin{aligned}
A_{gg,Q}^{(3),\overline{\text{MS}}} &= \frac{1}{48} \left\{ \gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(0)} \left(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 6\beta_0 - 4n_f \beta_{0,Q} - 10\beta_{0,Q} \right) - 4 \left(\gamma_{gg}^{(0)} \left[2\beta_0 + 7\beta_{0,Q} \right] + 4\beta_0^2 + 14\beta_{0,Q}\beta_0 \right. \right. \\
&\quad \left. \left. + 12\beta_{0,Q}^2 \right) \beta_{0,Q} \right\} \ln^3 \left(\frac{\mathbf{m}^2}{\mu^2} \right) + \frac{1}{8} \left\{ \hat{\gamma}_{gg}^{(0)} \left(\gamma_{gg}^{(1)} + (1 - n_f) \hat{\gamma}_{gg}^{(1)} \right) + \gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(1)} - 4\hat{\gamma}_{gg}^{(1)} [\beta_0 + 2\beta_{0,Q}] \right. \\
&\quad \left. + 4\gamma_{gg}^{(1)} \beta_{0,Q} + 4[\beta_1 + \beta_{1,Q}] \beta_{0,Q} + 2\gamma_{gg}^{(0)} \beta_{1,Q} \right\} \ln^2 \left(\frac{\mathbf{m}^2}{\mu^2} \right) + \frac{1}{16} \left\{ 8\hat{\gamma}_{gg}^{(2)} - 8n_f a_{gg,Q}^{(2)} \hat{\gamma}_{gg}^{(0)} + 8\gamma_{gg}^{(0)} a_{Qg}^{(2)} \right. \\
&\quad \left. - 16a_{gg,Q}^{(2)} (2\beta_0 + 3\beta_{0,Q}) + 8\gamma_{gg}^{(0)} \beta_{1,Q}^{(1)} + \gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(0)} \zeta_2 \left(\gamma_{gg}^{(0)} - \gamma_{qq}^{(0)} + 6\beta_0 + 4n_f \beta_{0,Q} + 6\beta_{0,Q} \right) \right. \\
&\quad \left. + 4\beta_{0,Q} \zeta_2 \left(\gamma_{gg}^{(0)} + 2\beta_0 \right) \left(2\beta_0 + 3\beta_{0,Q} \right) \right\} \ln \left(\frac{\mathbf{m}^2}{\mu^2} \right) + 2(2\beta_0 + 3\beta_{0,Q}) \bar{a}_{gg,Q}^{(2)} + n_f \hat{\gamma}_{gg}^{(0)} \bar{a}_{gq,Q}^{(2)} - \gamma_{gg}^{(0)} \bar{a}_{Qg}^{(2)} \\
&\quad - \beta_{1,Q}^{(2)} \gamma_{gg}^{(0)} + \frac{\gamma_{gg}^{(0)} \hat{\gamma}_{gg}^{(0)} \zeta_3}{48} \left(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)} - 2[2n_f + 1] \beta_{0,Q} - 6\beta_0 \right) + \frac{\beta_{0,Q} \zeta_3}{12} \left([\beta_{0,Q} - 2\beta_0] \gamma_{gg}^{(0)} \right. \\
&\quad \left. + 2[\beta_0 + 6\beta_{0,Q}] \beta_{0,Q} - 4\beta_0^2 \right) - \frac{\hat{\gamma}_{gg}^{(0)} \zeta_2}{16} \left(\gamma_{gg}^{(1)} + \hat{\gamma}_{gg}^{(1)} \right) + \frac{\beta_{0,Q} \zeta_2}{8} \left(\hat{\gamma}_{gg}^{(1)} - 2\gamma_{gg}^{(1)} - 2\beta_1 - 2\beta_{1,Q} \right) \\
&\quad + \frac{\delta m_1^{(-1)}}{4} \left(8a_{gg,Q}^{(2)} + 24\delta m_1^{(0)} \beta_{0,Q} + 8\delta m_1^{(1)} \beta_{0,Q} + \zeta_2 \beta_{0,Q} \beta_0 + 9\zeta_2 \beta_{0,Q}^2 \right) + \delta m_1^{(0)} \left(\beta_{0,Q} \delta m_1^{(0)} + \hat{\gamma}_{gg}^{(1)} \right) \\
&\quad + \delta m_1^{(1)} \left(\hat{\gamma}_{gg}^{(0)} \gamma_{gg}^{(0)} + 2\beta_{0,Q} \gamma_{gg}^{(0)} + 4\beta_{0,Q} \beta_0 + 8\beta_{0,Q}^2 \right) - 2\delta m_2^{(0)} \beta_{0,Q} + a_{gg,Q}^{(3)}
\end{aligned}$$

The final renormalized contribution with the $\overline{\text{MS}}$ -mass \bar{m} :

$$\begin{aligned}
A_{gg,Q}^{(3),n_f T_F^2, \overline{\text{MS}}} &= n_f T_F^2 \left\{ \left(\mathbf{C}_F \frac{64(N^2 + N + 2)^2}{9(N-1)N^2(N+1)^2(N+2)} + \mathbf{C}_A \left[\frac{128(N^2 + N + 1)}{27(N-1)N(N+1)(N+2)} - \frac{64}{27} S_1 \right] \right) \ln^3 \left(\frac{\bar{m}^2}{\mu^2} \right) \right. \\
&- \mathbf{C}_F \frac{16}{3} \ln^2 \left(\frac{\bar{m}^2}{\mu^2} \right) + \left(\mathbf{C}_A \frac{1}{(N-1)(N+2)} \left[-\frac{4P_1}{81N^3(N+1)^3} - \frac{16P_2}{81N^2(N+1)^2} S_1 \right] \right. \\
&+ \mathbf{C}_F \frac{1}{(N-1)(N+2)} \left[\frac{16(N^2 + N + 2)^2}{N^2(N+1)^2} \left(S_1^2 - \frac{5}{3} S_2 \right) - \frac{4P_3}{9N^4(N+1)^4} - \frac{32P_4}{3N^3(N+1)^3} S_1 \right] \left. \right) \ln \left(\frac{\bar{m}^2}{\mu^2} \right) \\
&+ \mathbf{C}_A \frac{1}{(N-1)(N+2)} \left[-\frac{4P_5}{27N^2(N+1)^2} S_1^2 - \frac{8P_6}{729N^3(N+1)^3} S_1 + \frac{512}{27} (N-1)(N+2) \zeta_3 S_1 \right. \\
&- \left. \frac{2P_7}{729N^4(N+1)^4} - \frac{1024(N^2 + N + 1)}{27N(N+1)} \zeta_3 + \frac{4P_8}{27N^2(N+1)^2} S_2 \right] \\
&+ \mathbf{C}_F \frac{1}{(N-1)(N+2)} \left[\frac{64(N^2 + N + 2)^2}{9N^2(N+1)^2} \left(-\frac{1}{3} S_1^3 - 8\zeta_3 + \frac{4}{3} S_3 \right) + \frac{32P_9}{27N^3(N+1)^3} S_1^2 \right. \\
&- \left. \frac{64P_{10}}{81N^4(N+1)^4} S_1 - \frac{32P_{11}}{243N^5(N+1)^5} - \frac{32P_{12}}{3N^3(N+1)^3} S_2 \right] \left. \right\}
\end{aligned}$$

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gq,Q}$



The all- ε result constituting the color factor $T_F^2 n_f C_F$

$$\hat{A}_{gq, T_F^2 n_f}^{(3)} = -96 a_s^3 T_F^2 n_f C_F \left(\frac{m^2}{\mu^2} \right)^{\frac{3\varepsilon}{2}} S_\varepsilon^3 \frac{1 + (-1)^N}{2} e^{-\frac{3\varepsilon}{2}\gamma} \frac{(\varepsilon - 1)^2 (\varepsilon + 2) (\varepsilon + N^2 + N + 2)}{\varepsilon (\varepsilon + 1) (\varepsilon + 3)} \\ \times \Gamma(1 - \varepsilon)^2 \Gamma\left(-\frac{\varepsilon}{2} - 4\right) \Gamma\left(\frac{\varepsilon}{2} + 2\right) \frac{\Gamma\left(\frac{\varepsilon}{2} + 5\right) \Gamma\left(-\frac{3\varepsilon}{2}\right) \Gamma(N - 1)}{\Gamma(4 - 2\varepsilon) \Gamma\left(\frac{\varepsilon}{2} + N + 2\right)}$$

yields the renormalized contribution

$$A_{gq, Q}^{(3), n_f T_F^2, \overline{\text{MS}}} = n_f T_F^2 \frac{1 + (-1)^N}{2} \left\{ \mathbf{C}_F \frac{32 (N^2 + N + 2)}{9(N - 1)N(N + 1)} \ln^3 \left(\frac{\bar{m}^2}{\mu^2} \right) + \mathbf{C}_F \left[-\frac{16 (N^2 + N + 2)}{3(N - 1)N(N + 1)} (S_1^2 + S_2) \right. \right. \\ \left. \left. + \frac{32 (8N^3 + 13N^2 + 27N + 16)}{9(N - 1)N(N + 1)^2} S_1 + \frac{32 (19N^4 + 81N^3 + 86N^2 + 80N + 38)}{27(N - 1)N(N + 1)^3} \right] \ln \left(\frac{\bar{m}^2}{\mu^2} \right) \right. \\ \left. + \mathbf{C}_F \left[\frac{32 (N^2 + N + 2)}{27(N - 1)N(N + 1)} (S_1^3 + 3S_2 S_1 + 2 S_3 - 24\zeta_3) - \frac{32 (8N^3 + 13N^2 + 27N + 16)}{27(N - 1)N(N + 1)^2} (S_1^2 + S_2) \right. \right. \\ \left. \left. + \frac{64 (4N^4 + 4N^3 + 23N^2 + 25N + 8)}{27(N - 1)N(N + 1)^3} S_1 + \frac{64 (197N^5 + 824N^4 + 1540N^3 + 1961N^2 + 1388N + 394)}{243(N - 1)N(N + 1)^4} \right] \right\}$$

Here we **confirm** the n_f contribution to the anomalous dimension:

[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$\hat{\gamma}_{gq}^{(2),n_f} = n_f T_F^2 C_F \left(\frac{64(N^2 + N + 2)}{3(N-1)N(N+1)} - (S_1^2 + S_2) + \frac{128(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^2} S_1 - \frac{128(4N^4 + 4N^3 + 23N^2 + 25N + 8)}{9(N-1)N(N+1)^3} \right)$$

in an independent calculation.

Furthermore we are able to **check** a result for the combination

$$\tilde{\gamma}_{gg}^{(2)} + \frac{\tilde{\gamma}_{gq}^{(2)} \gamma_{qg}^{(0)}}{\tilde{\gamma}_{gg}^{(0)} n_f}$$

of 3-loop anomalous dimensions, derived from the **large n_f expansion** in QCD

by [Bennett, Gracey 1997]; where we denote with $\tilde{\gamma}_{ij}^{(k)}$ the leading n_f coefficient of $\gamma_{ij}^{(k)}$.

Graphs with m_c and m_b

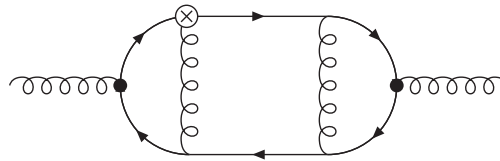
$$\begin{aligned}
a_{Qg}^{(3)}(N=6) = & T_F^2 C_A \left\{ \frac{69882273800453}{367569090000} - \frac{395296}{19845} \zeta_3 + \frac{1316809}{39690} \zeta_2 + \frac{832369820129}{14586075000} x + \frac{1511074426112}{624023544375} x^2 - \frac{84840004938801319}{690973782403905000} x^3 \right. \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{11771644229}{194481000} + \frac{78496}{2205} \zeta_2 - \frac{1406143531}{69457500} x - \frac{105157957}{180093375} x^2 + \frac{2287164970759}{7669816654500} x^3 \right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{324148}{19845} + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{156992}{6615} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{128234}{3969} - \frac{112669}{330750} x + \frac{98746}{51975} x^2 + \frac{31340489}{17027010} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{68332}{6615} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{83755534727}{583443000} + \frac{78496}{2205} \zeta_2 + \frac{1406143531}{69457500} x + \frac{105157957}{180093375} x^2 - \frac{2287164970759}{7669816654500} x^3 \right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{2668087}{79380} + \frac{112669}{661500} x - \frac{49373}{51975} x^2 - \frac{31340489}{34054020} x^3 \right] + \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{412808}{19845} \left. \right\} \\
& + T_F^2 C_F \left\{ -\frac{3161811182177}{71471767500} + \frac{447392}{19845} \zeta_3 + \frac{9568018}{4862025} \zeta_2 - \frac{64855635472}{2552563125} x + \frac{1048702178522}{97070329125} x^2 + \frac{1980566069882672}{2467763508585375} x^3 \right. \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{1786067629}{204205050} - \frac{111848}{15435} \zeta_2 - \frac{128543024}{24310125} x - \frac{22957168}{3361743} x^2 - \frac{2511536080}{2191376187} x^3 \right] \\
& + \ln^2\left(\frac{m_2^2}{\mu^2}\right) \left[\frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_2^2}{\mu^2}\right) \frac{111848}{19845} - \ln^2\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
& + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{22238456}{4862025} - \frac{1504864}{231525} x - \frac{355888}{40425} x^2 - \frac{255717856}{42567525} x^3 \right] + \ln\left(\frac{m_2^2}{\mu^2}\right) \ln^2\left(\frac{m_1^2}{\mu^2}\right) \frac{223696}{46305} \\
& + \ln\left(\frac{m_1^2}{\mu^2}\right) \left[-\frac{24797875607}{1021025250} - \frac{111848}{15435} \zeta_2 + \frac{128543024}{24310125} x + \frac{22957168}{3361743} x^2 + \frac{2511536080}{2191376187} x^3 \right] \\
& + \ln^2\left(\frac{m_1^2}{\mu^2}\right) \left[\frac{3232799}{4862025} + \frac{752432}{231525} x + \frac{177944}{40425} x^2 + \frac{127858928}{42567525} x^3 \right] - \ln^3\left(\frac{m_1^2}{\mu^2}\right) \frac{1230328}{138915} \left. \right\} + O(x^4 \ln^3(x))
\end{aligned}$$

These moments have been calculated referring [qexp](#) by Steinhauser et al. [with operator insertions](#). Despite being [universal](#), these contribution do not belong to the charm or bottom PDF. [This is then the end of the VFNF](#).

Calculation of Convergent Massive 3-Loop Graphs

- Aim:
 - Compute fixed Mellin moments of convergent 3-loop diagrams
 - Find general N representations for all convergent 3-loop topologies
- Here we work in the α -representation to calculate the integrals.
- The corresponding graph polynomials of a graph G are given by
 - $U = \sum_T \prod_{l \notin T} \alpha_l$, where T denotes the spanning trees of G
 - $V = \sum_{l \in massive} \alpha_l$
 - different Dodgson polynomials, which can be derived from the corresponding tadpole diagram, for the operator insertions

Calculation of Convergent Massive 3-Loop Graphs



$$\begin{aligned}
 I_4(N) &= \int \cdots \int d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 d\alpha_7 d\alpha_8 \frac{\sum_{j=0}^N T_{4\alpha}^{N-j} T_{4b}^j}{U^2 V^2} \\
 T_{4\alpha} &= \alpha_5 \alpha_7 \alpha_4 + \alpha_2 \alpha_3 \alpha_5 + \alpha_2 \alpha_5 \alpha_4 + \alpha_3 \alpha_5 \alpha_7 + \alpha_2 \alpha_5 \alpha_8 + \alpha_8 \alpha_5 \alpha_4 + \alpha_5 \alpha_7 \alpha_8 + \alpha_2 \alpha_3 \alpha_8 \\
 &\quad + \alpha_7 \alpha_2 \alpha_8 + \alpha_6 \alpha_2 \alpha_8 + \alpha_3 \alpha_7 \alpha_2 + \alpha_2 \alpha_3 \alpha_6 + \alpha_4 \alpha_2 \alpha_8 + \alpha_2 \alpha_6 \alpha_4 + \alpha_4 \alpha_7 \alpha_2 \\
 T_{4b} &= +\alpha_2 \alpha_5 \alpha_4 + \alpha_4 \alpha_2 \alpha_8 + \alpha_4 \alpha_7 \alpha_2 + \alpha_2 \alpha_5 \alpha_8 + \alpha_2 \alpha_3 \alpha_5 + \alpha_7 \alpha_2 \alpha_8 + \alpha_3 \alpha_7 \alpha_2 + \alpha_8 \alpha_5 \alpha_4 \\
 &\quad + \alpha_5 \alpha_7 \alpha_4 + \alpha_4 \alpha_1 \alpha_8 + \alpha_1 \alpha_7 \alpha_4 + \alpha_3 \alpha_5 \alpha_7 + \alpha_5 \alpha_7 \alpha_8 + \alpha_8 \alpha_1 \alpha_7 + \alpha_1 \alpha_3 \alpha_7 \\
 U &= \alpha_2 \alpha_5 \alpha_4 + \alpha_2 \alpha_3 \alpha_5 + \alpha_1 \alpha_3 \alpha_5 + \alpha_5 \alpha_7 \alpha_4 + \alpha_1 \alpha_6 \alpha_4 + \alpha_1 \alpha_3 \alpha_6 + \alpha_2 \alpha_3 \alpha_6 + \alpha_2 \alpha_6 \alpha_4 \\
 &\quad + \alpha_5 \alpha_6 \alpha_4 + \alpha_1 \alpha_5 \alpha_4 + \alpha_3 \alpha_5 \alpha_7 + \alpha_1 \alpha_3 \alpha_7 + \alpha_1 \alpha_7 \alpha_4 + \alpha_3 \alpha_7 \alpha_2 + \alpha_4 \alpha_7 \alpha_2 + \alpha_3 \alpha_5 \alpha_6 \\
 &\quad + \alpha_2 \alpha_3 \alpha_8 + \alpha_2 \alpha_5 \alpha_8 + \alpha_5 \alpha_7 \alpha_8 + \alpha_8 \alpha_5 \alpha_4 + \alpha_8 \alpha_5 \alpha_6 + \alpha_5 \alpha_3 \alpha_8 + \alpha_1 \alpha_8 \alpha_5 + \alpha_1 \alpha_8 \alpha_6 \\
 &\quad + \alpha_6 \alpha_2 \alpha_8 + \alpha_1 \alpha_8 \alpha_3 + \alpha_4 \alpha_1 \alpha_8 + \alpha_4 \alpha_2 \alpha_8 + \alpha_7 \alpha_2 \alpha_8 + \alpha_8 \alpha_1 \alpha_7 \\
 V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
 \end{aligned}$$

- The integral above is a projective integral, one α -parameter may be set 1
- The operators sit on on-shell diagrams which obey specific symmetries. These are generally not obeyed by the operator insertion.
- For the above example : after applying symmetry transformations $\alpha_1 \rightarrow x_1 - \alpha_2$, $\alpha_3 \rightarrow x_2 - \alpha_4$, $\alpha_5 \rightarrow x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.

Calculation of Convergent Massive 3-Loop Graphs

- Feynman parameter integrals are performed in terms of **Hyperlogarithms**,

[Brown 2008 Comm. Math. Phys.]

$L(\vec{w}, z) : \mathbb{C} \setminus \Sigma \rightarrow \mathbb{C}$, where

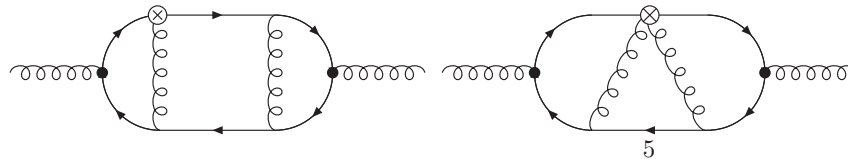
- $\Sigma = \{\sigma_0, \sigma_1, \dots, \sigma_N\}$ are distinct points in \mathbb{C} which may contain variables
 - \vec{w} is a word over the alphabet $\mathfrak{A} = \{a_0, a_1, \dots, a_N\}$ where each letter a_i corresponds to a point σ_i
- $L(\vec{w}, z)$ is uniquely defined by the following properties
 1. $L(\{\}, z) = 1$, and $L(0^n, z) = \frac{1}{n!} \log^n(z)$ for $n \geq 1$
 2. $\frac{\partial}{\partial z} L(\{a_i \vec{w}\}, z) = \frac{1}{z - \sigma_i} L(\vec{w}, z)$ for $z \in \mathbb{C} \setminus \Sigma$
 3. If \vec{w} is not of the form $w = (0, 0, \dots, 0)$, then $\lim_{z \rightarrow 0} L(\vec{w}, z) = 0$.
 - e.g. $L(a_i, z) = \log(z - \sigma_i) - \log(\sigma_i)$
 - The weight of $L(\vec{w}, z)$ is given by the number of letters in \vec{w}

- The hyperlogarithms satisfy shuffle relations $L(\vec{w}_1, z) L(\vec{w}_2, z) = L(\vec{w}_1 \sqcup \vec{w}_2, z)$, e.g.:
 $L(\{a_1, a_2\}, z) L(\{a_3\}, z) = L(\{a_3, a_1, a_2\}, z) + L(\{a_1, a_3, a_2\}, z) + L(\{a_1, a_2, a_3\}, z)$
- The indices a_i contain further **integration variables**.
- Using these properties after partial fractioning and integration by parts, one can express any primitive for expressions consisting of rational and hyperlogarithmic functions in terms of different hyperlogarithmic functions
- These primitives have to be evaluated at the respective integration limits
 - The limit at $z \rightarrow 0$ is trivially obtained by computing the regularized Taylor series for the hyperlogarithmic functions
 - The limit at $z \rightarrow \infty$ is more sophisticated. General idea:
 1. Choose the integration order (**Fubini**).
 2. Compute the derivative with respect to the next integration variable x , (this lowers the weight by one).
 3. Perform the series expansion of the derivative.
 4. Perform the indefinite integration with respect to x .
 5. Determine the respective integration constant.

- For example: $\lim_{y \rightarrow \infty} L(\{-x-1, 0\}, y)$
- Compute $\frac{\partial}{\partial x} L(\{-x-1, 0\}, y)$:
 - $\frac{\partial}{\partial x} \frac{\partial}{\partial y} L(\{-x-1, 0\}, y) = -\frac{L(\{0\}, y)}{(y+1+x)^2}$
 - $$\begin{aligned} \frac{\partial}{\partial x} L(\{-x-1, 0\}, y) &= \int dy \frac{\partial}{\partial x} \frac{\partial}{\partial y} L(\{-x-1, 0\}, y) + Const \\ &= \left(\frac{1}{y+x+1} - \frac{1}{x+1} \right) L(\{0\}, y) + \frac{L(\{-x-1\}, y)}{x+1} + Const \end{aligned}$$
 - Fix $Const$ such that $\lim_{y \rightarrow 0} \frac{\partial}{\partial x} L(\{-x-1, 0\}, y) = 0$, in this case $Const = 0$
 - The asymptotic limit of the derivative is then given by $-\frac{L(\{-1\}, x)}{x+1}$
 - $\int dx \lim_{y \rightarrow \infty} \frac{\partial}{\partial x} L(\{-x-1, 0\}, y) = L(\{-1, -1\}, x)$
 - We have to add the respective integration constant, which is given by $\lim_{y \rightarrow \infty} L(\{-1, 0\}, y) = L(\{0, 0\}, y) - \zeta_2$
- Thus $\lim_{y \rightarrow \infty} L(\{-x-1, 0\}, y) = L(\{-1, -1\}, x) + L(\{0, 0\}, y) - \zeta_2$.

Fixed Mellin Moments

- Using this method we have computed a number of fixed Mellin-Moments from $N = 0..19$
e.g.:



N	Diag 4	Diag 5 _a	Diag 5 _b
0	$2 - 2\zeta_3$	$2\zeta_3$	$2\zeta_3$
1	$-2 + 2\zeta_3$	$-\frac{5}{2} - \zeta_3$	$-2 - 2\zeta_3$
2			
3			
4			
...
19	$-\frac{5825158236879253094413489658569181}{2503562235895708381108915200000}$ $-\frac{104899807174743864253}{54192375991353600} \zeta_3$	$-\frac{128090266890628029062643215783549}{133523319247771113659142144000}$ $+\frac{238388793949217497}{301068755507520} \zeta_3$	$-\frac{254116903575797385411050257769}{25288507433289983647564800000}$ $-\frac{1968329}{635040} \zeta_3$

General Values of N

- Due to the operator-insertions leading to power-type functions, the integrals do not fit directly into the framework of the algorithm for general values of N .
- In order to use the algorithm also on integrals with general values of N , a generating function is constructed e.g. by the mapping

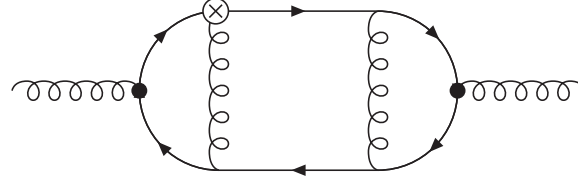
$$p(\alpha_1, \dots, \alpha_n)^N \rightarrow \frac{1}{1 - x p(\alpha_1, \dots, \alpha_n)} .$$

- Performing the Feynman-parameter integrations then leads to an expression which contains hyperlogarithms $L_w(x)$ in the variable x .
- Finally the N th coefficient of this expression in x has to be extracted **analytically**. This has been done with the package `HarmonicSums` by J.Ablinger.
- Generalized harmonic sums occur:

$$S_{n_1, \dots, n_k}(a_1, \dots, a_k)(N), \quad n_i \in \mathbb{N}, a_i \in \mathbb{Q}$$

.

Six Massive Lines & Vertex Insertion



$$\begin{aligned}
\hat{I}_4 = & \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2} \zeta_3 + \frac{(-1)^N (65 + 101N + 56N^2 + 13N^3 + N^4)}{2(1+N)^2(2+N)^2(3+N)^2} S_{-3} \\
& + \frac{(-24 - 5N + 2N^2)}{12(2+N)^2(3+N)^2} S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)} S_2^2 + \frac{1}{(2+N)(3+N)} S_1^2 S_2 \\
& + \frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2} S_1^2 - \frac{3}{2} S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2} S_3 - 2S_{-2,-3} - 2\zeta_3 S_{-2} - S_{-2,1} S_{-2} \\
& + \frac{(-1)^N (65 + 101N + 56N^2 + 13N^3 + N^4)}{(1+N)^2(2+N)^2(3+N)^2} S_{-2,1} + \frac{(59 + 42N + 6N^2)}{2(1+N)(2+N)(3+N)} S_4 + \frac{(5+N)}{(1+N)(3+N)} \zeta_3 S_1 \quad (2) \\
& - \frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2} S_2 - \zeta_3 S_2 - \frac{3}{2} S_3 S_2 - 2S_{2,1} S_2 + \frac{(99 + 225N + 190N^2 + 65N^3 + 7N^4)}{2(1+N)^2(2+N)^2(3+N)} S_{2,1} \\
& + \frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4} S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)} \zeta_3 S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2} S_2 S_1 - S_{2,3} \\
& + \frac{(53+29N)}{2(1+N)(2+N)(3+N)} S_3 S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)} S_1 S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)} S_{3,1} - 3S_{4,1} \\
& + S_{-2,1,-2} + \frac{2^{N+1}(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2} S_{1,2} \left(\frac{1}{2}, 1 \right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)} S_{2,1,1} \\
& + 5S_{2,2,1} + 6S_{3,1,1} + \frac{2^N(-28-25N-4N^2+N^3)}{(1+N)^2(2+N)(3+N)^2} S_{1,1,1} \left(\frac{1}{2}, 1, 1 \right) \\
& - \frac{(5+N)}{(1+N)(3+N)} S_{1,1,2} \left(2, \frac{1}{2}, 1 \right) - \frac{(5+N)}{2(1+N)(3+N)} S_{1,1,1,1} \left(2, \frac{1}{2}, 1, 1 \right)
\end{aligned}$$

We encounter several **limits of S-Sums**:

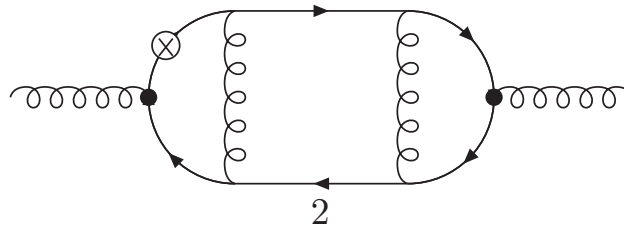
$$\begin{aligned}
S_1\left(\frac{1}{2}; \infty\right) &= \ln(2) \\
S_2\left(\frac{1}{2}; \infty\right) &= \frac{1}{2}(\zeta_2 - \ln^2(2)) \\
S_{1,1}\left(\frac{1}{2}, 1; \infty\right) &= \frac{\zeta_2}{2} \\
S_3\left(\frac{1}{2}; \infty\right) &= \frac{\ln^3(2)}{6} - \frac{1}{2}\zeta_2 \ln(2) \\
S_{1,2}\left(\frac{1}{2}, 1; \infty\right) &= \frac{5\zeta_3}{8} \\
S_{2,1}\left(1, \frac{1}{2}; \infty\right) &= \frac{1}{12}(6\zeta_2 \ln(2) + 3\zeta_3 + 2\ln^3(2)) \\
S_{2,1}\left(\frac{1}{2}, 1; \infty\right) &= \zeta_3 - \frac{1}{2}\zeta_2 \ln(2) \\
S_{2,1}\left(\frac{1}{2}, 2; \infty\right) &= \frac{3}{8}(7\zeta_3 - 4\zeta_2 \ln(2)) \\
S_{1,2}\left(\frac{1}{2}, 2; \infty\right) &= \frac{3}{2}\zeta_2 \ln(2) \\
S_{1,1,1}\left(\frac{1}{2}, 1, 1; \infty\right) &= \frac{3\zeta_3}{4}
\end{aligned}$$

The 2^N factors cancel in the large N limit:

$$\begin{aligned}
\hat{I}_4 \approx & \zeta_2^2 \left[\frac{1115231}{20N^{10}} - \frac{74121}{4N^9} + \frac{122951}{20N^8} - \frac{40677}{20N^7} + \frac{13391}{20N^6} - \frac{873}{4N^5} + \frac{1391}{20N^4} - \frac{417}{20N^3} + \frac{101}{20N^2} \right] \\
& + \zeta_3 \left[\left(-\frac{95855}{2N^{10}} + \frac{31525}{2N^9} - \frac{10295}{2N^8} + \frac{3325}{2N^7} - \frac{1055}{2N^6} + \frac{325}{2N^5} - \frac{95}{2N^4} + \frac{25}{2N^3} - \frac{5}{2N^2} \right) \ln(N) \right. \\
& \left. - \frac{23280115}{2016N^{10}} + \frac{2093041}{1008N^9} - \frac{177251}{1008N^8} - \frac{25843}{336N^7} + \frac{2569}{48N^6} - \frac{155}{8N^5} + \frac{91}{24N^4} + \frac{2}{3N^3} - \frac{11}{12N^2} \right] \\
& + \zeta_2 \left[\left(\frac{19171}{N^{10}} - \frac{6305}{N^9} + \frac{2059}{N^8} - \frac{665}{N^7} + \frac{211}{N^6} - \frac{65}{N^5} + \frac{19}{N^4} - \frac{5}{N^3} + \frac{1}{N^2} \right) \ln^2(N) \right. \\
& \left. + \left(\frac{103016863}{2520N^{10}} - \frac{3091261}{315N^9} + \frac{2571839}{1260N^8} - \frac{6215}{21N^7} - \frac{293}{20N^6} + \frac{2071}{60N^5} - \frac{103}{6N^4} + \frac{67}{12N^3} - \frac{1}{N^2} \right) \ln(N) \right. \\
& \left. + \frac{292993001621}{302400N^{10}} - \frac{4402272031}{30240N^9} + \frac{22261739}{840N^8} - \frac{78507473}{14112N^7} + \frac{180961}{144N^6} - \frac{111807}{400N^5} + \frac{629}{12N^4} - \frac{319}{72N^3} - \frac{7}{4N^2} \right] \\
& + \left(\frac{249223}{6N^{10}} - \frac{145015}{12N^9} + \frac{10295}{3N^8} - \frac{11305}{12N^7} + \frac{1477}{6N^6} - \frac{715}{12N^5} + \frac{38}{3N^4} - \frac{25}{12N^3} + \frac{1}{6N^2} \right) \ln^3(N) \\
& + \left(\frac{193493767}{10080N^{10}} + \frac{210658237}{10080N^9} - \frac{21541697}{2520N^8} + \frac{243269}{96N^7} - \frac{30539}{48N^6} + \frac{2123}{16N^5} - \frac{59}{3N^4} + \frac{5}{8N^3} + \frac{1}{2N^2} \right) \ln^2(N) \\
& + \left(-\frac{2207364771673}{4233600N^{10}} + \frac{1390655509}{352800N^9} + \frac{285594061}{22050N^8} - \frac{67234111}{14400N^7} + \frac{8617073}{7200N^6} - \frac{35209}{144N^5} + \frac{116}{3N^4} - \frac{119}{24N^3} + \frac{1}{N^2} \right) \ln(N) \\
& + \frac{1344226725047831}{889056000N^{10}} - \frac{165849841805771}{889056000N^9} + \frac{808151260279}{27783000N^8} - \frac{708430537}{120960N^7} + \frac{304474703}{216000N^6} \\
& - \frac{606811}{1728N^5} + \frac{1867}{24N^4} - \frac{1813}{144N^3} + \frac{1}{N^2} + O(N^{-11})
\end{aligned}$$

General Values of N

The method has been applied also to other graphs, e.g.:



$$\begin{aligned}
 I_{2a} = & \frac{1}{(N+1)(N+2)(N+3)} \left\{ \frac{8(3+2N)}{(1+N)(2+N)} - 2 \left(-3 + (-1)^N + 2^{3+N} \right) \zeta_3 \right. \\
 & - (-1)^N S_{-3} + \frac{(16+12N+N^2) S_1^2}{2(1+N)(2+N)} + \frac{1}{6} S_1^3 + \frac{(56+40N+3N^2) S_2}{2(1+N)(2+N)} \\
 & + \frac{4(3+2N)}{(1+N)(2+N)} S_1 - \frac{1}{2} S_1 S_2 - (17+3N) S_3 - 2(-1)^N S_{-2,1} \\
 & \left. - (N+3) S_{2,1} + 2^{4+N} S_{1,2} \left(\frac{1}{2} \right) + 2^{3+N} S_{1,1,1} \left(\frac{1}{2} \right) \right\}
 \end{aligned}$$

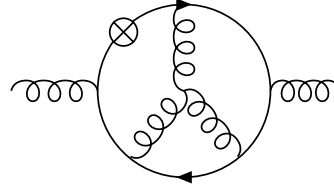
Calculated also with summation \implies **Sigma**.

Characteristics of associated Recursions

Diagram	rational			ζ_3		
	# Moments	Degree	Order	# Moments	Degree	Order
I_{1a}	203	26	8	15	3	2
I_{1b}	269	36	9	15	3	2
I_{2a}	215	31	8	19	3	3
I_{2b}	269	42	9	35	6	3
I_4	623	90	13	131	24	6

Diagram	ε^{-2}			ε^{-1}			ε^0 rat.			$\varepsilon^0 \zeta_2$		
	#	Deg.	Ord.	#	Deg.	Ord.	#	Deg.	Ord.	#	Deg.	Ord.
I_{6a}	15	3	2	55	11	3	142	25	5	15	3	2
I_{6b}	15	3	2	55	12	3	142	27	5	15	3	2
I_{8a}	19	4	2	69	14	3	164	30	5	19	4	2
I_{8b}	19	4	2	79	16	3	175	34	5	19	4	2
I_9	142	26	9	463	83	10	1199	210	16	142	26	5
I_{10a}	47	6	4	341	57	10	949	156	16	109	17	6
I_{10b}	109	17	6	323	53	10	911	152	16	47	6	4

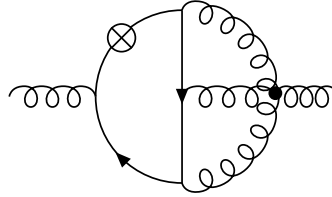
General Values of N : Higher Topologies



$$\begin{aligned}
 I(x) = & \frac{1}{(1+N)(2+N)x} \left\{ \zeta_3 \left[2L(\{-1\}, x) - 2(-1+2x)L(\{1\}, x) - 4L(\{1, 1\}, x) \right] - 3L(\{-1, 0, 0, 1\}, x) \right. \\
 & + 2L(\{-1, 0, 1, 1\}, x) - 2xL(\{0, 0, 1, 1\}, x) + 3xL(\{0, 1, 0, 1\}, x) - xL(\{0, 1, 1, 1\}, x) \\
 & + (-3+2x)L(\{1, 0, 0, 1\}, x) + 2xL(\{1, 0, 1, 1\}, x) - (-1+5x)L(\{1, 1, 0, 1\}, x) + xL(\{1, 1, 1, 1\}, x) \\
 & - 2L(\{1, 0, 0, 1, 1\}, x) + 3L(\{1, 0, 1, 0, 1\}, x) - L(\{1, 0, 1, 1, 1\}, x) + 2L(\{1, 1, 0, 0, 1\}, x) \\
 & \left. + 2L(\{1, 1, 0, 1, 1\}, x) - 5L(\{1, 1, 1, 0, 1\}, x) + L(\{1, 1, 1, 1, 1\}, x) \right\}
 \end{aligned}$$

$$\begin{aligned}
 I(N) = & \frac{1}{(N+1)(N+2)(N+3)} \left\{ \frac{648 + 1512N + 1458N^2 + 744N^3 + 212N^4 + 32N^5 + 2N^6}{(1+N)^3(2+N)^3(3+N)^3} \right. \\
 & - \frac{2 \left(-1 + (-1)^N + N + (-1)^N N \right)}{(1+N)} \zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)} S_1^3 + \frac{1}{24} S_1^4 \\
 & - \frac{(7 + 22N + 10N^2)}{2(1+N)^2(2+N)} S_2 - \frac{19}{8} S_2^2 - \frac{1 + 4N + 2N^2}{2(1+N)^2(2+N)} S_1^2 + \frac{9}{4} S_2 - \frac{(-9 + 4N)}{3(1+N)} S_3 \\
 & - \frac{1}{4} S_4 - 2(-1)^N S_{-2,1} + \frac{(-1 + 6N)}{(1+N)} S_{2,1} + \frac{54 + 207N + 246N^2 + 130N^3 + 32N^4 + 3N^5}{(1+N)^3(2+N)^2(3+N)^2} S_1 \\
 & \left. + 4\zeta_3 S_1 - \frac{(-2 + 7N)}{2(1+N)} S_2 S_1 + \frac{13}{3} S_3 S_1 - 7S_{2,1} S_1 - 7S_{3,1} + 10S_{2,1,1} \right\}
 \end{aligned}$$

General Values of N : Higher Topologies



$$\begin{aligned}
 I(N) = & \frac{1}{(N+1)(N+2)} \left\{ \frac{2(1-13(-1)^N + (-1)^N 2^{3+N} + N - 7(-1)^N N + 3(-1)^N 2^{1+N} N)}{(1+N)(2+N)} \zeta_3 \right. \\
 & + \frac{1}{(2+N)} S_3 + \frac{(-1)^N}{2(2+N)} S_1^3 - \frac{(-1)^N (3+2N)}{2(1+N)^2(2+N)} S_2 + \frac{5(-1)^N}{2} S_2^2 \\
 & + \frac{(-1)^N (3+2N)}{2(1+N)^2(2+N)} S_1^2 - \frac{(-1)^N}{2} S_2 S_1^2 + \frac{3(-1)^N (4+3N)}{(1+N)(2+N)} S_3 + 3(-1)^N S_4 + \frac{2}{(2+N)} S_{-2,1} \\
 & + 2(-1)^N \zeta_3 S_{1,(2)} + \frac{2(-1)^N (3+N)}{(1+N)(2+N)} S_{2,1} - 12(-1)^N S_1 \zeta_3 \\
 & + \frac{(-1)^N (5+7N)}{2(1+N)(2+N)} S_1 S_2 + 3(-1)^N S_1 S_3 + 4(-1)^N S_{2,1} S_1 - 4(-1)^N S_{3,1} \\
 & - \frac{4((-1)^N 2^{2+N} - 3(-2)^N N + 3(-1)^N 2^{1+N} N)}{(1+N)(2+N)} S_{1,2} \left(\frac{1}{2}, 1 \right) - 5(-1)^N S_{2,1,1} \\
 & + \frac{2(-(-1)^N 2^{2+N} - 13(-2)^N N + 5(-1)^N 2^{1+N} N)}{(1+N)(2+N)} S_{1,1,1} \left(\frac{1}{2}, 1, 1 \right) \\
 & \left. - 2(-1)^N S_{1,1,2} \left(2, \frac{1}{2}, 1 \right) - (-1)^N S_{1,1,1,1} \left(2, \frac{1}{2}, 1, 1 \right) \right\}
 \end{aligned}$$

Conclusions

- A series of moments for the transition matrix elements A_{ij} at 3-loop order were given in [Bierenbaum, Blümlein, Klein 2009 Nucl. Phys. B].
- The corresponding quarkonic 3-loop contributions of $O(n_f T_F^2 C_{A,F})$ to A_{qq} and A_{qg} were calculated in [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl. Phys. B]. Now also $A_{gg,Q}$ and $A_{gq,Q}$ have been obtained for these color coefficients at general N .
- A series of OMEs were fully calculated A_{qq} and A_{qg} in $O(T_F^2 C_{A,F})$
- Ladder topologies, including poles, are currently calculated using **Sigma** and the method of hyperlogarithms.
- The moments $N = 2,4,6$ have been calculated for graphs depending on both m_c and m_b ; general N results in the NS and PS case have been obtained already. Starting with 3-loops, graphs exist which conflict with the ideology of the VFNS.
This was expected, since the heavy quarks are produced in the final state!
- With the help of hyperlogarithms non-divergent 3-loop graphs can be calculated, if moments are considered. For general values of N first analytic results have been obtained, including Benz-topologies, performing the calculation automatically.
- 3-loop moments of **polarized** massive OMEs up to the constant terms have been calculated.