Moments of Generalized Parton Distributions from lattice QCD

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Outline

Introduction

QCD-Gauge theory of the strong interaction
Lagrangian: formulated in terms of quarks and gluons

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{f=u,d,s,c,b,t} \bar{\psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \psi_f$$

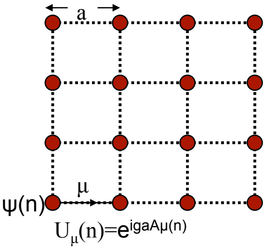
$$D_\mu = \partial_\mu - ig \frac{\lambda^a}{2} A^a_\mu$$

This “simple” Lagrangian produces the amazingly rich structure of strongly interacting matter in our universe.

Numerical simulation of QCD provides essential input for a wide class of complex strong interaction phenomena

- QCD phase diagram relevant for Quark-Gluon Plasma studied in heavy ion collisions at RHIC and LHC affecting the early evolution of the Universe
- Nuclear forces that affect the large scale structure of the Universe
- Hadron structure studied in experimental programs at CERN, JLab, Mainz, DESY
  - Momentum distribution of quarks and gluons in the nucleon
  - Hadron form factors e.g. the nucleon axial charge g_A

QCD on the lattice



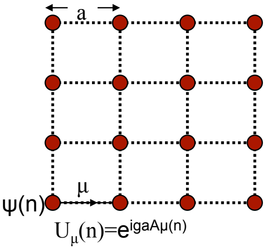
- Discretization of space-time in 4 Euclidean dimensions
  $\rightarrow$ Rotation into imaginary time $x_0 \rightarrow ix_4$ is the most drastic modification
  It forms the basis of LQCD $\rightarrow$ by relating quantum field theory to statistical mechanics it allows for a purely numerical treatment by means of Monte Carlo techniques.
- The quantities usually studied are matrix elements of local operators $\langle h'(p')|\mathcal{O}|h(p)\rangle$ between hadronic states h.
- Most LQCD results for matrix elements have to be matched to typically the MS scheme at a certain scale μ. This requires a matching of renormalization effects. Some exceptions exist like hadron masses.

Like continuum QCD lattice QCD has as unknown input parameters the coupling constant α_s and the masses of the up, down, strange, charm and bottom quarks.
$\Longrightarrow$ Lattice QCD provides a well-defined approach to calculate observables non-perturbative starting directly from the QCD Langragian.

Consider simplest isotropic hypercubic grid: $a = a_S = a_T$ and size $N_S \times N_S \times N_S \times N_T$, $N_T > N_S$.

- **Finite Volume:**
  1. Finite volume effects need to be studied $\rightarrow$ Take box sizes such that $L_S m_\pi \gtrsim 3.5$.
  2. Only discrete values of momentum in units of $2\pi/L_S$ are allowed.
- **Finite lattice spacing:** Need at least three values of the lattice spacing in order to extrapolate to the continuum limit.
- **q^2-values:** Fourier transform of lattice results in coordinate space taken numerically $\rightarrow$ for large values of momentum transfer results are too noisy $\Longrightarrow$ Limited to $Q^2 = -q^2 \sim 2$ GeV^2.

QCD on the lattice



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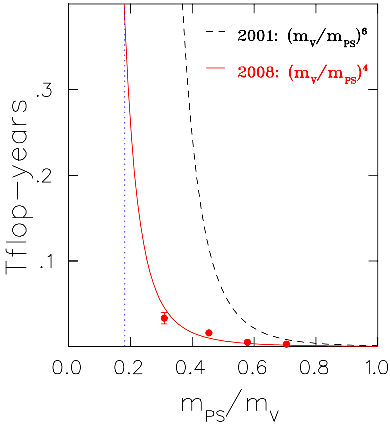
Computational cost

Simulation cost: $C_{\rm sim} \propto \left(\frac{300\rm MeV}{m_\pi}\right)^{c_m} \left(\frac{L}{2\rm fm}\right)^{c_L} \left(\frac{0.1\rm fm}{a}\right)^{c_a}$



Coefficients c_m, c_L and c_a depend on the discretized action used for the fermions.
State-of-the-art simulations use improved algorithms:

- Mass preconditioner, M. Hasenbusch, Phys. Lett. B519 (2001) 177
- Multiple time scales in the molecular dynamics updates

$\Longrightarrow$ for twisted mass fermions: $c_m \sim 4$, $c_L \sim 5$ and $c_a \sim 6$.

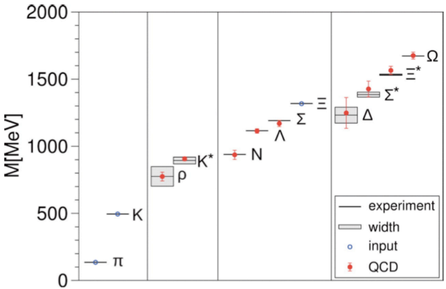
- Results at physical quark masses require $\mathcal{O}(1)$ Pflop.Years.
- After post-diction of well measured quantities the goal is to predict quantities that are difficult or impossible to measure experimentally.

L=2.1 fm, a=0.089 fm, K. Jansen and C. Urbach, arXiv:0905.3331

We are currently at the Petaflop scale: 10^{15} Flops (arithmetic operations per sec)
Exaflops machines are already being planned: 10^{18} Flops

Mass of low-lying hadrons

$N_F = 2+1$ smeared Clover fermions, BMW Collaboration, S. Dürr et al. Science 322 (2008)
$N_F = 2$ twisted mass fermions, ETM Collaboration, C. Alexandrou et al. PRD (2008)



- BMW with $N_F = 2+1$:
  - 3 lattice spacings: $a \sim 0.125, 0.085, 0.065$ fm set by m_Ξ
  - Pion masses: $m_\pi \gtrsim 190$ MeV
  - Volumes:$m_\pi^{\min} L \gtrsim 4$
- ETMC with $N_F = 2$:
  - 3 lattice spacings: $a = 0.089, 0.070, a = 0.056$ fm, set by m_N
  - $m_\pi \gtrsim 260$ MeV
  - Volumes:$m_\pi^{\min} L \gtrsim 3.3$

Good agreement between different discretization schemes $\Longrightarrow$ Significant progress in understanding the masses of low-lying mesons and baryons

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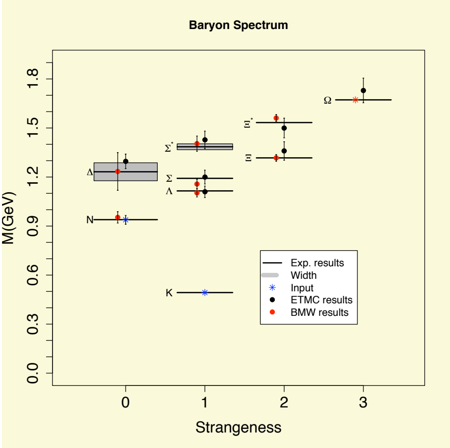

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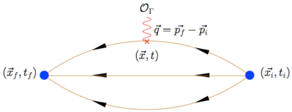
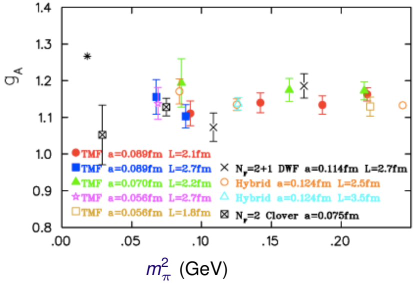
Nucleon axial charge

- Many lattice studies down to lowest pion mass of $m_\pi \sim 300$ MeV
  $\Longrightarrow$ Lattice data in general agreement
- Axial-vector FFs: $A^a_\mu = \bar\psi\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi(x)$
  $\Longrightarrow \frac{1}{2}\left[\gamma_\mu\gamma_5 G_A(q^2) + \frac{q^\mu\gamma_5}{2m}G_p(q^2)\right]$



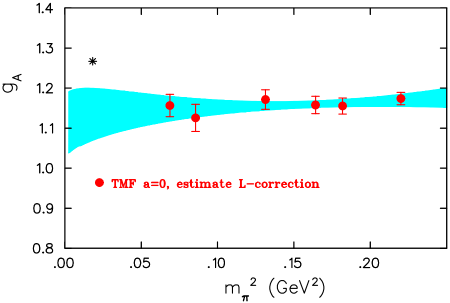
Axial charge is well known experimentally, straight forward to compute in lattice QCD



- Agreement among recent lattice results - all use non-perturbative Z_A
- Weak light quark mass dependence
- What can we say about the physical value of g_A?

- TMF: C. A. *et al.* (ETMC), PRD 83 (2011) 045010
- DWF: T. Yamazaki *et al.*, (RBC-UKQCD), PRD 79 (2009) 14505; S. Ohta, arXiv:1011.1388
- Hybrid:J. D. Bratt *et al.* (LHPC),PRD 82 (2010) 094502
- Clover:D. Pleiter *et al.* (QCDSF), arXiv:1101.2326

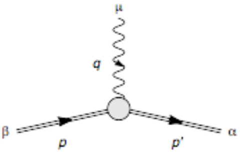
Physical results on g_A



- What can we say about the physical value of g_A?
- Use results obtained with twisted mass fermions C. A. *et al.* (ETMC), Phys. Rev. D83 (2011) 045010
- Take continuum limit and estimate volume corrections, A. Ali Khan, *et al.*, PRD **74**, 094508 (2006)
- Use one-loop chiral perturbation theory in the small scale expansion (SSE), T. R. Hemmert, M. Procura and W. Weise, PRD **68**, 075009 (2003).
- 3 fit parameters, $g_A^0 = 1.10(8)$, $g_{\Delta\Delta}$=2.1(1.3), $C^{SSE}(1\ GeV) = -0.7(1.7)$, axial N$\Delta$ coupling fixed to 1.5: $\Rightarrow g_A = 1.14(6)$
- Fitting lattice results directly leads to $g_A = 1.12(7)$

Lattice determination of the axial charges of other baryons can provide input for χPT, H.- W. Lin and K. Orginos, PRD 79, 034507 (2009); M. Gockeler *et l.*, arXiv:1102.3407

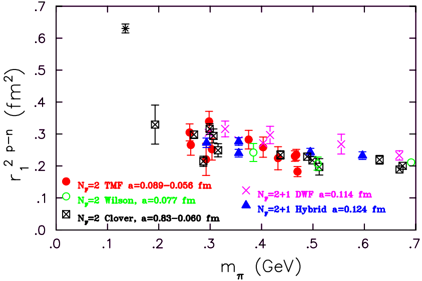
Dirac and Pauli isovector radii of the nucleon

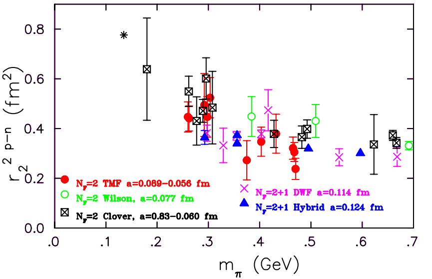


$$\langle N(p',s')|j^\mu(0)|N(p,s)\rangle = \bar{u}_N(p',s')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u_N(p,s)$$

Dirac and Pauli radii: $r^2_{1,2} = -\frac{6}{F_{1,2}(0)}\frac{dF_{1,2}}{dq^2}|_{q^2\to 0}$

Use a dipole Ansatz to fit the q^2-dependence of F_1 and F_2.





- TMF: C. A. *et al.* (ETMC), PRD83 (2011) 094502
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- DWF: S. N. Syritsyn *et al.* (LHPC), PRD 81, 034507 (2010); T. Yamazaki *et al.* (RBC-UKQCD), PRD 79, 114505 (2009)
- Hybrid:J. D. Bratt *et al.* (LHPC), Phys. Rev. D82, 094502 (2010)

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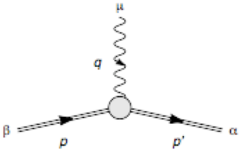
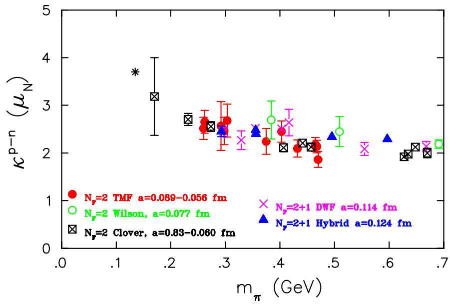


$$\langle N(p',s')|j^\mu(0)|N(p,s)\rangle = \bar{u}_N(p',s')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)\right] u_N(p,s)$$

Anomalous magnetic moment: $F_2(0)\,\frac{m_N^{\rm phys}}{m_N^{\rm lat}}$



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Δ electromagnetic form factors

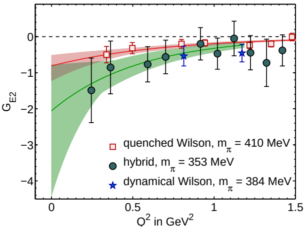
$$\langle \Delta(p', s') | j^\mu(0) | \Delta(p, s) \rangle = -\bar{u}_\alpha(p', s') \left\{ \left[F_1^*(Q^2) g^{\alpha\beta} + F_3^*(Q^2) \frac{q^\alpha q^\beta}{(2M_\Delta)^2} \right] \gamma^\mu + \left[F_2^*(Q^2) g^{\alpha\beta} + F_4^*(Q^2) \frac{q^\alpha q^\beta}{(2M_\Delta)^2} \right] \frac{i\sigma^{\mu\nu} q_\nu}{2M_\Delta} \right\} u_\beta(p, s)$$

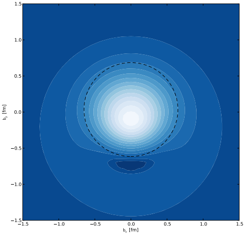
with e.g. the quadrupole form factor given by: $G_{E2} = \left(F_1^* - \tau F_2^*\right) - \frac{1}{2}(1+\tau)\left(F_3^* - \tau F_4^*\right)$, where $\tau \equiv Q^2/(4M_\Delta^2)$

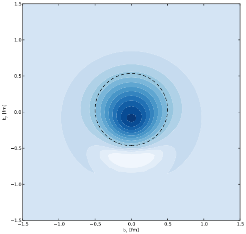
Construct an optimized source to isolate $G_{E2} \rightarrow$ additional sequential propagators needed.
Neglect disconnected contributions in this evaluation.

Transverse charge density of a Δ polarized along the x-axis can be defined in the infinite momentum frame $\rightarrow$ $\rho^{\Delta}_{T\,\frac{3}{2}}(\vec{b})$ and $\rho^{\Delta}_{T\,\frac{1}{2}}(\vec{b})$.

Using G_{E2} we can predict 'shape' of Δ.







Δ with spin 3/2 projection elongated along spin axis compared to the Ω^-

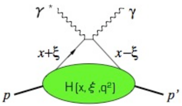
C. A., T. Korzec, G. Koutsou, C. Lorcé, J. W. Negele, V. Pascalutsa, A. Tsapalis, M. Vanderhaeghen, NPA825 ,115 (2009).

Definition of Generalized Form Factors

High energy scattering: Formulate in terms of light-cone correlation functions, M. Diehl, Phys. Rep. 388 (2003)
Consider one-particle states p' and $p \rightarrow$ Generalized Parton Distributions (GPDs), X. Ji, J. Phys. G24(1998)1181

$$F_{\Gamma}(x,\xi,q^2) = \frac{1}{2}\int \frac{d\lambda}{2\pi} e^{ix\lambda} \langle p'|\bar{\psi}(-\lambda n/2)\Gamma\mathcal{P}e^{ig\int_{-\lambda/2}^{\lambda/2} d\alpha\, n\cdot A(n\alpha)}\psi(\lambda n/2)|p\rangle\ ,$$

"Handbag" diagram



where $q = p' - p$, $\bar{P} = (p'+p)/2$, n is a light-cone vector and $\bar{P}.n = 1$

$$\begin{aligned}
\Gamma &= \not{n} :\rightarrow \frac{1}{2}\bar{u}_N(p')\left[\not{n}H(x,\xi,q^2) + i\frac{n_\mu q_\nu \sigma^{\mu\nu}}{2m_N}E(x,\xi,q^2)\right]u_N(p) \\
\Gamma &= \not{n}\gamma_5 :\rightarrow \frac{1}{2}\bar{u}_N(p')\left[\not{n}\gamma_5\tilde{H}(x,\xi,q^2) + \frac{n.q\gamma_5}{2m_N}\tilde{E}(x,\xi,q^2)\right]u_N(p) \\
\Gamma &= n_\mu\sigma^{\mu\nu} :\rightarrow \text{tensor GPDs} \quad .
\end{aligned}$$

Forward matrix elements $F_{\Gamma}(x,0,0)$, measured in DIS, connected to the parton distributions $q(x)$, $\Delta q(x)$, $\delta q(x)$

Expansion of the light cone operator leads to a tower of local twist-2 operators $\mathcal{O}^{\mu\mu_1\cdots\mu_n}$

- Diagonal matrix element $\langle P|\mathcal{O}(x)|P\rangle$ (DIS) $\rightarrow$ moments of parton distributions:

$$\mathcal{O}_q^{\mu\mu_1\cdots\mu_n} = \bar{\psi}\gamma^{\{\mu}iD^{\mu_1}\ldots iD^{\mu_n\}}\psi \quad \xrightarrow{unpolarized} \quad \langle x^n\rangle_q = \int_0^1 dx\, x^n\left[q(x) - (-1)^n\bar{q}(x)\right]$$

$$\tilde{\mathcal{O}}_{\Delta q}^{\mu\mu_1\cdots\mu_n} = \bar{\psi}\gamma_5\gamma^{\{\mu}iD^{\mu_1}\ldots iD^{\mu_n\}}\psi \quad \xrightarrow{helicity} \quad \langle x^n\rangle_{\Delta q} = \int_0^1 dx\, x^n\left[\Delta q(x) + (-1)^n\Delta\bar{q}(x)\right]$$

$$\mathcal{O}_{\delta q}^{\rho\mu\mu_1\cdots\mu_n} = \bar{\psi}\sigma^{\rho\{\mu}iD^{\mu_1}\ldots iD^{\mu_n\}}\psi \quad \xrightarrow{transversity} \quad \langle x^n\rangle_{\delta q} = \int_0^1 dx\, x^n\left[\delta q(x) - (-1)^n\delta\bar{q}(x)\right]$$

where $q = q_\downarrow + q_\uparrow$, $\Delta q = q_\downarrow - q_\uparrow$, $\delta q = q_T + q_\perp$

- Off-diagonal matrix elements (DVCS) $\rightarrow$ generalized form factors

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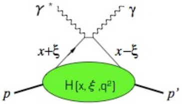

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$$\Gamma = \not{n}\gamma_5 :\to \frac{1}{2}\bar{u}_N(p')\left[\not{n}\gamma_5 \tilde{H}(x,\xi,q^2) + \frac{n.q\gamma_5}{2m_N}\tilde{E}(x,\xi,q^2)\right]u_N(p)$$

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- Off-diagonal matrix elements (DVCS) $\to$ generalized form factors

Nucleon generalized form factors

Decomposition of matrix elements into generalized form factors:

$$\langle N(p',s')|\mathcal{O}_q^{\mu\mu_1\cdots\mu_n}|N(p,s)\rangle = \bar{u}_N(p',s')\Big[\sum_{i=0,2,\ldots}^{n}\Big(A_{n+1,i}(q^2)\gamma^{\{\mu} + B_{n+1,i}(q^2)\frac{i\sigma^{\{\mu\alpha}q_\alpha}{2m}\Big)q^{\mu_1}\ldots q^{\mu_i}\overline{P}^{\mu_{i+1}}\ldots\overline{P}^{\mu_n\}}$$
$$+\mathrm{mod}(n,2)C_{n+1,0}(q^2)\frac{1}{m}q^{\{\mu}q^{\mu_1}\ldots q^{\mu_n\}}\Big]u_N(p,s)$$

Similarly for $\mathcal{O}_{\Delta q}^{\mu\mu_1\cdots\mu_n}$ (in terms of $\tilde{A}_{ni}(q^2)$, $\tilde{B}_{ni}(q^2)$) and $\mathcal{O}_{\delta q}^{\mu\mu_1\cdots\mu_n}$ (in terms of $A^T_{ni},\ B^T_{ni},\ C^T_{ni},\ D^T_{ni}$).

Special cases:

- $n=1$: ordinary nucleon form factors

$$A_{10}(q^2) = F_1(q^2) = \int_{-1}^{1} dx H(x,\xi,q^2), \quad B_{10}(q^2) = F_2(q^2) = \int_{-1}^{1} dx E(x,\xi,q^2)$$
$$\tilde{A}_{10}(q^2) = G_A(q^2) = \int_{-1}^{1} dx \tilde{H}(x,\xi,q^2), \quad \tilde{B}_{10}(q^2) = G_p(q^2) = \int_{-1}^{1} dx \tilde{E}(x,\xi,q^2)$$

  where

  - $j_\mu = \bar{\psi}\gamma_\mu\psi \Longrightarrow \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2)$
    The Dirac F_1 and Pauli F_2 are related to the electric and magnetic Sachs form factors:
    $G_E(q^2) = F_1(q^2) - \frac{q^2}{(2m)^2}F_2(q^2), \qquad G_M(q^2) = F_1(q^2) + F_2(q^2)$
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- $A_{n0}(0)$, $\tilde{A}_{n0}(0)$, $A^T_{n0}(0)$ are moments of parton distributions, e.g. $\langle x\rangle_q = A_{20}(0)$ and $\langle x\rangle_{\Delta q} = \tilde{A}_{20}(0)$ are the spin independent and helicity distributions
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Nucleon generalized form factors

Decomposition of matrix elements into generalized form factors:

$$\langle N(p',s')|\mathcal{O}_q^{\mu\mu_1\cdots\mu_n}|N(p,s)\rangle = \bar{u}_N(p',s')\Big[\sum_{i=0,2,\ldots}^{n}\Big(A_{n+1,i}(q^2)\gamma^{\{\mu} + B_{n+1,i}(q^2)\frac{i\sigma^{\{\mu\alpha}q_\alpha}{2m}\Big)q^{\mu_1}\ldots q^{\mu_i}\overline{P}^{\mu_{i+1}}\ldots\overline{P}^{\mu_n\}}$$

$$+\mathrm{mod}(n,2)C_{n+1,0}(q^2)\frac{1}{m}q^{\{\mu}q^{\mu_1}\ldots q^{\mu_n\}}\Big]u_N(p,s)$$

Similarly for $\mathcal{O}_{\Delta q}^{\mu\mu_1\cdots\mu_n}$ (in terms of $\tilde{A}_{ni}(q^2)$, $\tilde{B}_{ni}(q^2)$) and $\mathcal{O}_{\delta q}^{\mu\mu_1\cdots\mu_n}$ (in terms of $A^T_{ni},\ B^T_{ni},\ C^T_{ni},\ D^T_{ni}$).

Special cases:

- $n=1$: ordinary nucleon form factors

$$A_{10}(q^2) = F_1(q^2) = \int_{-1}^{1} dx H(x,\xi,q^2), \quad B_{10}(q^2) = F_2(q^2) = \int_{-1}^{1} dx E(x,\xi,q^2)$$
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  where

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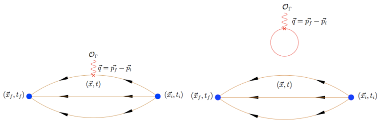
Lattice QCD evaluation

Evaluation of two-point and three-point functions

$$G(\vec{q},t) = \sum_{\vec{x}_f} e^{-i\vec{x}_f\cdot\vec{q}}\,\Gamma^4_{\beta\alpha}\,\langle J_\alpha(\vec{x}_f,t_f)\bar{J}_\beta(0)\rangle$$

$$G^{\mu\nu}(\Gamma,\vec{q},t) = \sum_{\vec{x}_f,\vec{x}} e^{i\vec{x}\cdot\vec{q}}\,\Gamma_{\beta\alpha}\,\langle J_\alpha(\vec{x}_f,t_f)\mathcal{O}^{\mu\nu}(\vec{x},t)\bar{J}_\beta(0)\rangle$$



Sequential inversion "through the sink" $\rightarrow$ fix sink-source separation $t_f - t_i$, final momentum $\vec{p}_f = 0$, Γ

Apply smearing techniques to improve ground state dominance in three-point correlators

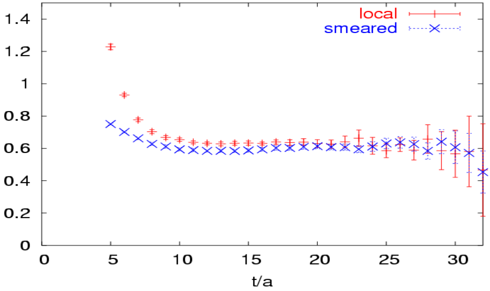
Take ratios: Leading time dependence cancels like in determination of hadron masses $\rightarrow$ Talk by M. Peardon

$$aE_{\rm eff}(\vec{q},t) = \ln\left[G(\vec{q},t)/G(\vec{q},t+a)\right]$$

$$\rightarrow aE(\vec{q}) \stackrel{\vec{q}=0}{\rightarrow} am$$

$$R^{\mu\nu}(\Gamma,\vec{q},t) = \frac{G^{\mu\nu}(\Gamma,\vec{q},t)}{G(\vec{0},t_f)}\sqrt{\frac{G(\vec{p}_i,t_f-t)G(\vec{0},t)G(\vec{0},t_f)}{G(\vec{0},t_f-t)G(\vec{p}_i,t)G(\vec{p}_i,t_f)}}$$

$$\rightarrow \Pi^{\mu\nu}(\vec{q},\Gamma)$$



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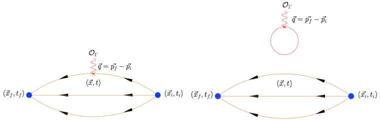

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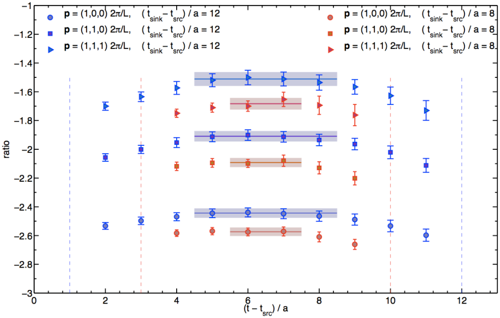


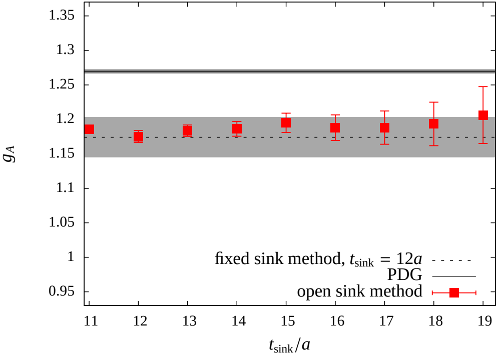
For nucleon form factors: $t_f - t_i > 1$ fm

However, this might be operator dependent

Study of excited state contributions

$N_F = 2 + 1 + 1$ with $m_\pi \sim 380$ MeV and $a = 0.08$ fm

Vary source- sink separation:



S. Dinter, C.A., M. Constantinou, V. Drach, K. Jansen and D. Renner, arXiv: 1108.1076

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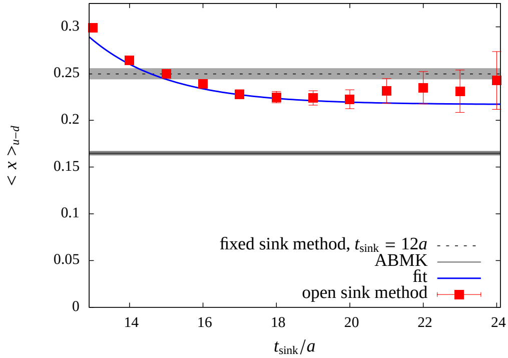


$\Longrightarrow$ Excited contributions are operator dependent

g_A unaffected, $\langle x \rangle_{u-d}$ 10% lower

S. Dinter, C.A., M. Constantinou, V. Drach, K. Jansen and D. Renner, arXiv: 1108.1076

Non-perturbative renormalization

Connect lattice results to measurements: $\mathcal{O}_{\overline{\rm MS}}(\mu) = Z(\mu, a)\mathcal{O}_{\rm latt}(a)$

Most collaborations evaluate $Z(\mu, a)$ non-perturbatively

ETMC: RI'-MOM renormalization scheme as in e.g. M. Göckeler *et al.*, Nucl. Phys. B544,699

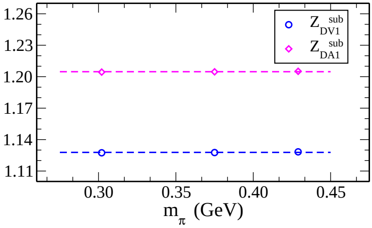
- Fix to Landau gauge and compute:

  $S^u(p) = \frac{a^8}{V} \sum_{x,y} e^{-ip(x-y)} \langle u(x)\bar{u}(y)\rangle$

  $G(p) = \frac{a^{12}}{V} \sum_{x,y,z,z'} e^{-ip(x-y)} \langle u(x)\bar{u}(z)\mathcal{J}(z,z')d(z')\bar{d}(y)\rangle$

  $\rightarrow$ Amputated vertex functions: $\Gamma(p) = (S^u(p))^{-1}\, G(p)\, (S^d(p))^{-1}$
- Renormalization functions: Z_q and $Z_{\mathcal{O}}$:

  $Z_q = \frac{1}{12}{\rm Tr}\left[(S^L(p))^{-1} S^{(0)}(p)\right]|_{p^2=\mu^2}\,, \qquad Z_q^{-1} Z_{\mathcal{O}}\, \frac{1}{12}{\rm Tr}\left[\Gamma^L(p)(\Gamma^{(0)}(p))^{-1}\right]|_{p^2=\mu^2}$
- Mass independent renormalization scheme $\rightarrow$ need chiral extrapolations



C.A., M. Constantinou, T. Korzec, H. Panagopoulos, Stylianou, arXiv:1201.5025, PRD83 (2011) 014503

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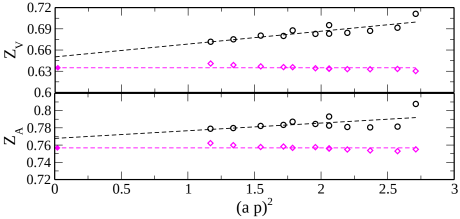
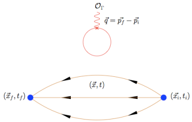
- Mass independent renormalization scheme $\rightarrow$ need chiral extrapolations
- Subtract $\mathcal{O}(a^2)$ perturbatively.

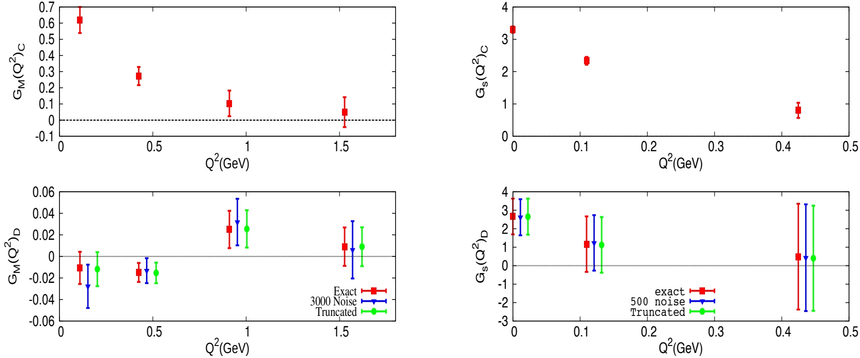


C.A., M. Constantinou, T. Korzec, H. Panagopoulos, Stylianou, arXiv:1201.5025, PRD83 (2011) 014503

Disconnected contributions

- Approximate using stochastic techniques
- Loops with a scalar inversion are much easier to compute
- Disconnected loops contributing to nucleon form factors show slow convergence
- The truncated solver method is best suited, G. Bali, S. Collins, A. Schafer Comput.Phys.Commun. 181 (2010) 1570



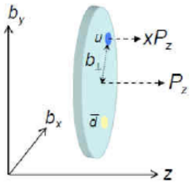


C.A., K. Hadjiyiannakou, G. Koutsou, A. 'O Cais, A. Strelchenko, arXiv:1108.2473: Comparison of stochastic methods to the exact evaluation enabled using GPUs; $N_f = 2$ Wilson fermions (SESAM Collaboration)

Results on nucleon parton distributions

Transverse quark distributions:

M. Burkardt, PRD62 (2000)



$$
\begin{aligned}
q(x, \mathbf{b}_\perp) &= \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\mathbf{b}_\perp \cdot \Delta_\perp} H(x, \xi = 0, -\Delta_\perp^2) \\
\int_{-1}^{1} dx x^{n-1} q(x, \mathbf{b}_\perp) &= \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{-i\mathbf{b}_\perp \cdot \Delta_\perp} A^q_{n0}(-\Delta_\perp^2) \\
q(x \to 1, \mathbf{b}_\perp) &\propto \delta^2(\mathbf{b}_\perp)
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$\rightarrow$ the slope of A_{n0} decrease as n increases

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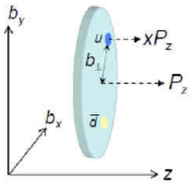

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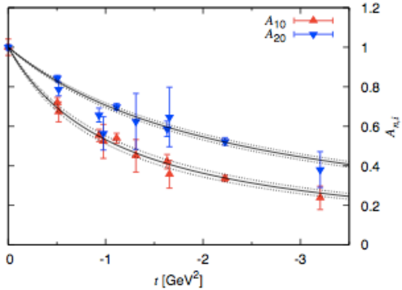
Pion: D. Brömmel *et al.* (QCDSF), hep-lat/0509133, PRL 101 (2008) 1229001



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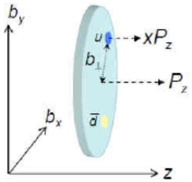

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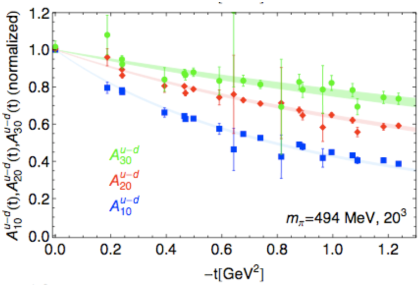
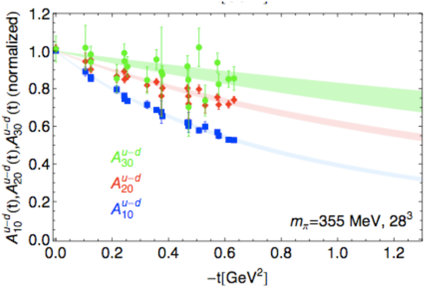
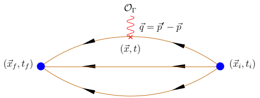
Nucleon: J. D. Bratt *et al.* (LHPC), arXiv:1001:3620

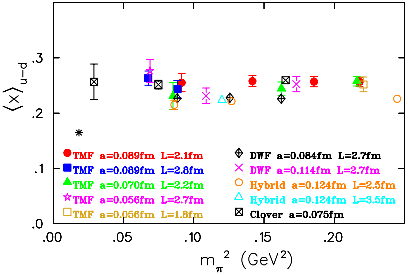


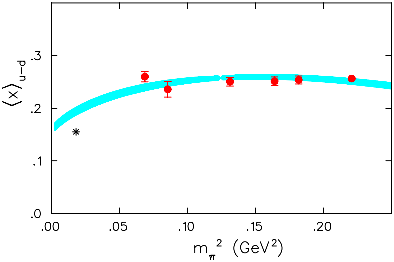


Nucleon momentum fraction

Momentum fraction $\langle x \rangle_{u-d} = A_{20}^{\text{isovector}}$







Physical point: $\langle x \rangle_{u-d}$ from S. Alekhin *et al.* arXiv:0908.2766

HBχPT for $\langle x \rangle_{u-d}$ and $\langle x \rangle_{\Delta u - \Delta d}$, D. Arndt, M. Savage, NPA 697, 429 (2002); W. Detmold, W Melnitchouk, A. Thomas, PRD 66, 054501 (2002)
Fit ETMC results with $\lambda^2 = 1$ GeV^2

$$\langle x \rangle_{u-d} = C \left[1 - \frac{3g_A^2 + 1}{(4\pi f_\pi)^2} m_\pi^2 \ln \frac{m_\pi^2}{\lambda^2} \right] + \frac{c_8(\lambda^2) m_\pi^2}{(4\pi f_\pi)^2} \qquad \langle x \rangle_{\Delta u - \Delta d} = \tilde{C} \left[1 - \frac{2g_A^2 + 1}{(4\pi f_\pi)^2} m_\pi^2 \ln \frac{m_\pi^2}{\lambda^2} \right] + \frac{\tilde{c}_8(\lambda^2) m_\pi^2}{(4\pi f_\pi)^2}$$

Nucleon spin

Spin sum: $\frac{1}{2} = \frac{1}{2}\Delta\Sigma + L_q + J_G$

Non-relativistic quark model:

If $\Delta\Sigma_{u,d} = 1 \Rightarrow L_q = 0$ and $J_G = 0$, as well as $\Delta s = 0$, where Δq contains both the spin of q and $\bar{q}$.

- Integrate over the range of data:
  - COMPASS data for $x \geq 0.004$, M. G. Alekseev *et al.* NPL B 693, 227 (2010)
  - HERMES data $x \geq 0.02$, A. Airapetian *et al.* PRD 75, 012007 (2007)

  $\Longrightarrow \Delta s \sim 0$.
- Global analyses give $\Delta s \sim -0.12$, i.e. a large negative $\Delta s(x)$ at very small x, E. Leader, A. V. Sidorov and D. B. Stamenov, PRD 82, 114018 (2010); J. Rojo et al. (NNPDF), PoS DIS 2010, 244 (2010); D. de Florian, R. Sassot, M. Stratmann and W. Vogelsang, PRD 80, 034030 (2009).
- Gluon helicity distribution from both COMPASS and STAR experiments is found to be close to zero, M.Stolarski(COMPASS),Nucl.Phys.Proc.Suppl.207-208,53(2010) ;P.Djawotho (STAR), J. Phys. Conf. Ser. 295, 012061 (2011)

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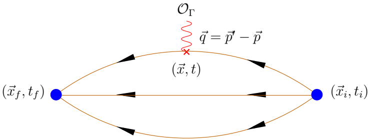
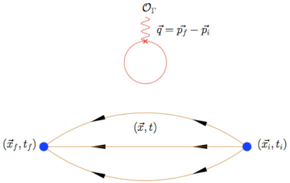
- Integrate over the range of data:
  - COMPASS data for $x \geq 0.004$, M. G. Alekseev *et al.* NPL B 693, 227 (2010)
  - HERMES data $x \geq 0.02$, A. Airapetian *et al.* PRD 75, 012007 (2007)

  $\Longrightarrow \Delta s \sim 0$.
- Global analyses give $\Delta s \sim -0.12$, i.e. a large negative $\Delta s(x)$ at very small x, E. Leader, A. V. Sidorov and D. B. Stamenov, PRD 82, 114018 (2010); J. Rojo et al. (NNPDF), PoS DIS 2010, 244 (2010); D. de Florian, R. Sassot, M. Stratmann and W. Vogelsang, PRD 80, 034030 (2009).
- Gluon helicity distribution from both COMPASS and STAR experiments is found to be close to zero, M.Stolarski(COMPASS),Nucl.Phys.Proc.Suppl.207-208,53(2010) ;P.Djawotho (STAR), J. Phys. Conf. Ser. 295, 012061 (2011)

Lattice QCD: Need both connected and disconnected contributions to evaluate contributions to spin

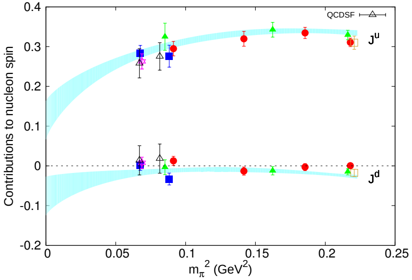
Bali *et al.*, arXiv:1112.3354: $\Delta u + \Delta d + \Delta s$ = 0.45 (4)(9) with $\Delta s = -0.020(10)(4)$ at $\mu = \sqrt{7.4}$ GeV
$\Longrightarrow$ Small strangeness (disconnected) contribution to the nucleon spin





Lattice results on the nucleon spin

$J_q = \frac{1}{2}[A_{20}(0) + B_{20}(0)] = \frac{1}{2}\Delta\Sigma_q + L_q$
$\Delta\Sigma_q = \tilde{A}_{10}$
Only connected contribution
Results using $N_F = 2$ TMF for 270 MeV $< m_\pi < 500$ MeV, C. Alexandrou *et al.* (ETMC), arXiv:1104.1600
In agreement with A. Sternbeck *et al.* (QCDSF) arXiv:1203.6579
In qualitative agreement with J. D. Bratt *et al.* (LHPC), PRD82 (2010) 094502



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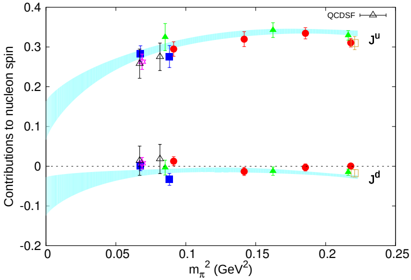

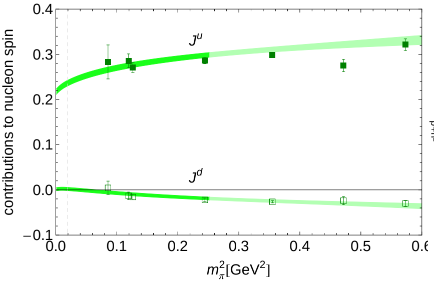

$\Longrightarrow$ Total spin for u-quarks $J^u \sim 0.25$ and for d-quark $J^d \sim 0$

Lattice results on the nucleon spin

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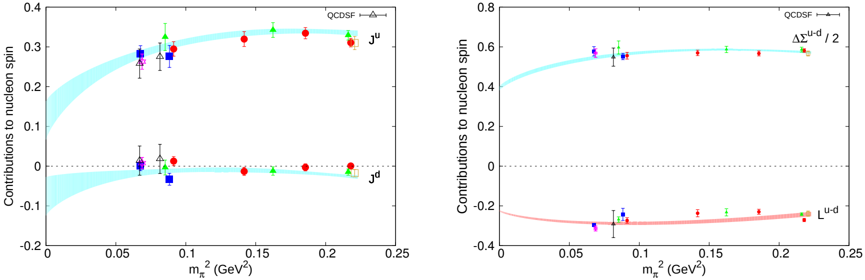
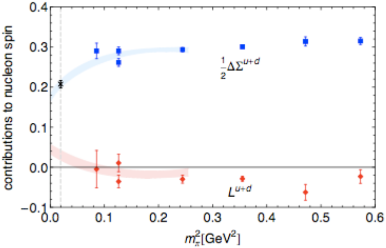
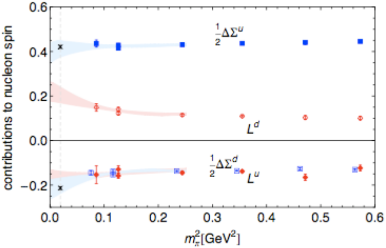


$\Longrightarrow$ Good agreement also for $\Delta\Sigma^{u-d}$ and ΔL^{u-d}

Spin of the Nucleon

- $N_F = 2 + 1$ domain wall valence quarks on staggered sea J. D. Bratt *et al.* (LHPC), PRD82 (2010) 094502





Physical points from HERMES 2007 analysis

Conclusions

- Large scale simulations using the underlying theory of the Strong Interactions have made spectacular progress
  $\Longrightarrow$ we now have simulations of the full theory at near physical parameters
- The low-lying hadron spectrum is reproduced
- Nucleon form factors are being computed by a number of collaborations aiming at reproducing the experimental values
- For resonances and such as the Δ lattice QCD provides a prediction for the form factors
- Moments of GPDs are being computed using a number of discretization schemes $\rightarrow$ provide insight into the structure of nucleon

Simulations at physical pion mass are becoming available $\Longrightarrow$ we expect many physical results on key observables

Thank you for your attention