Hydrodynamics and Transport

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Fluid dynamics = Conservation of energy and momentum for long wavelength modes

If the system is strongly interacting, i.e., has a short mean free path compared to the scales of interest, hydrodynamics should work

It was a surprise at RHIC that hydrodynamics worked so well (so well that we are still using it a lot)

I will try to give an overview of some of the important facts about relativistic hydrodynamics for heavy-ion collisions and explain different concepts that most speakers at QM2012 will assume to be known
Non-relativistic hydrodynamics

Equations of hydrodynamics can be obtained from a simple argument:

Variation of mass in the volume $V$ is due to in- and out-flow through the surface $\partial V$:

$$\frac{\partial}{\partial t} \int \rho dV = - \int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} dS$$

Gauss’ theorem:

$$\frac{\partial}{\partial t} \int \rho dV = - \int_V \nabla \cdot (\rho \mathbf{u}) dV$$

Conservation of mass: Continuity Equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

with mass density $\rho$ and fluid velocity $\mathbf{u}$.

Conservation of momentum: Euler Equation

$$\partial_t \mathbf{u} + \mathbf{u} (\nabla \cdot \mathbf{u}) = -\frac{1}{\rho} \nabla p$$
Relativistic hydrodynamics

Relativistic system: mass density is not a good degree of freedom: Does not account for kinetic energy (large for motions close to $c$).

- Replace $\rho$ by the total energy density $\varepsilon$.
- Replace $u$ by Lorentz four-vector $u^\mu$.

Ideal energy momentum tensor is built from pressure $p$, energy density $\varepsilon$, flow velocity $u^\mu$, and the metric $g^{\mu\nu}$.

Properties: symmetric, transforms like a Lorentz-tensor.
So the most general form is

$$T^{\mu\nu} = \varepsilon (c_0 g^{\mu\nu} + c_1 u^\mu u^\nu) + p (c_2 g^{\mu\nu} + c_3 u^\mu u^\nu)$$

Constraints:

$T^{00} = \varepsilon$ and $T^{0i} = 0$ and $T^{ij} = \delta^{ij} p$ in the local rest frame.

It follows:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - p (g^{\mu\nu} - u^\mu u^\nu)$$
Relativistic hydrodynamics

Conservation of energy and momentum:

\[ \partial_{\mu} T^{\mu\nu} = 0 \]

together with

\[ T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - p (g^{\mu\nu} - u^{\mu} u^{\nu}) \]

is ideal fluid dynamics.

In the non-relativistic limit \((u^2/c^2 \ll 1\) and \(p \ll mc^2)\):

\[ \partial_{\mu} T^{\mu 0} = 0 \rightarrow \text{Continuity equation} \]

\[ \partial_{\mu} T^{\mu i} = 0 \rightarrow \text{Euler equation} \]
Relativistic viscous hydrodynamics

Generally:

\[ T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu} + \pi^{\mu\nu}. \]

First order Navier Stokes theory (shear only):

\[ \pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} = \eta(\nabla^{\mu}u^{\nu} + \nabla^{\nu}u^{\mu} - \frac{2}{3}\Delta^{\mu\nu}\nabla_{\alpha}u^{\alpha}). \]

\[ \Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu} \]

Relativistic Navier Stokes is unstable (short wavelength modes become superluminal)

Second order theory:

\[ \pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} + \text{second derivatives}. \]

Israel-Stewart theory for a conformal fluid:

\[ \pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} - \tau_{\pi} \left( \frac{4}{3}\pi^{\mu\nu}\partial_{\alpha}u^{\alpha} + \Delta^{\mu}_{\alpha}\Delta^{\nu}_{\beta}u^{\sigma}\partial_{\sigma}\pi^{\alpha\beta} \right) \]

in flat space and neglecting vorticity and all terms that seem numerically unimportant


\[ \eta = \text{shear viscosity} \]

\[ \tau_{\pi} = \text{shear relaxation time} \]
Typical set of equations for heavy-ion physics

Using the set of equations

$$\partial_\mu T^{\mu\nu} = 0$$

and

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} - \tau_\pi \left( \frac{4}{3} \pi^{\mu\nu} \partial_\alpha u^\alpha + \Delta_\alpha^\mu \Delta_\beta^\nu u^\sigma \partial_\sigma \pi^{\alpha\beta} \right)$$

is now standard.

When **bulk viscosity** is included (non-conformal fluid)

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - Pg^{\mu\nu} + \pi^{\mu\nu} - \Pi \Delta^{\mu\nu}$$


Also heat flow and vorticity are sometimes included.
A heavy-ion collision

0 fm/c
before collision

pre-equilibrium

∼ 0.5 fm/c
quark-gluon-plasma

∼ 3 – 5 fm/c
hadronization

hadr. rescattering

∼ 10 fm/c
freeze-out

detection

initial state
(e.g. color glass condensate)

thermalization (glasma state)

Hydrodynamics, Jet quenching, ...

Hydrodynamics
Hadronic transport

compare theory
to experiment
Describing heavy-ion collisions with hydro

Hydrodynamics works for all systems with short mean free path. (comparing to size scales of interest)

How do we incorporate the physics of heavy-ion collisions?

1. Equation of state $p(\varepsilon, \rho_B)$
2. Initial conditions
3. Freeze-out and conversion of energy densities into particles
4. Values of transport coefficients (e.g. shear viscosity)
Is hydro useful for HICs?

**Within hydro:**

- Equation of state: unknown
- Initial conditions: unknown
- Freeze-out: unknown
- Transport coefficients: unknown

⇒ Predictive power?
Is hydro useful for HICs?

Within hydro:

- Equation of state: want to study
- Initial conditions: want to study
- Freeze-out: unknown
- Transport coefficients: want to study

⇒ Predictive power?

⇒ Need more constraints!

Hydrodynamics can provide the link from different models for the initial state, equation of state, etc. to experimental data.
Method

1. Use **another model** to fix unknowns:
   - e.g. take initial conditions from color glass condensate
   - Input equation of state from lattice QCD and hadron gas models

2. Use experimental data to **fix parameters**:
   - use one set of data to fix parameters:
     e.g. \( \frac{dN}{dy p_T dp_T} \bigg|_{b=0 \text{ fm}} \) and \( \frac{dN}{dy} (b) \)

Example parameters at RHIC:
\( \varepsilon_{0,\text{max}} \approx 30 \text{ GeV/fm}^3 \), \( \tau_0 \approx 0.6 \text{ fm/c} \), \( T_{fo} \approx 130 \text{ MeV} \)
- predict another set of data:
  Flow, photons and dileptons, HBT, ...
By the way: Initial energy density

The initial maximal energy densities needed to reproduce the experimental data are $\sim 30 \text{ GeV/} \text{fm}^3$. How much is that?

Critical energy density to create quark-gluon-plasma: $1 \text{ GeV/} \text{fm}^3$ (lattice QCD).
By the way: Initial energy density

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Landau and Bjorken hydrodynamics

- **Landau hydrodynamics**
  - Initial fireball at rest: \( u^\mu = (1, 0, 0, 0) \) everywhere
  - Start with a slab of radius \( r_{\text{nucleus}} \) and thickness \( 2r/\gamma \)
    \((\gamma \text{ is the } \gamma\text{-factor of the colliding nuclei})\)
  - Assumption of \( v_z = 0 \) seems unrealistic

- **Bjorken hydrodynamics**
  - At large energies \( \gamma \to \infty \), Landau thickness \( \to 0 \)
  - No longitudinal scale \( \to \) scaling flow
    \[
    v = \frac{z}{t}
    \]
    Because all particles are assumed to have been produced at 
\((t, z) = (0 \text{ fm}/c, 0 \text{ fm})\)
  a particle at point \((z, t)\) must have had average \( v = z/t \)
Practical coords. for scaling flow expansion

- Longitudinal proper time $\tau$:
  \[ \tau = \sqrt{t^2 - z^2} \]

- Space-time rapidity $\eta_s$:
  \[ \eta_s = \frac{1}{2} \ln \frac{t + z}{t - z} \]

Inversely: \( t = \tau \cosh \eta_s \) and \( z = \tau \sinh \eta_s \)

Boost-invariance: Results are independent of $\eta_s$.
This is assumed when you see 2+1D hydro calculations.
Good assumption when studying mid-rapidity at highest RHIC and LHC energies.
Initial conditions - all including fluctuations

You will see different initial conditions being used:

- **MC-Glauber**: geometric model determining wounded nucleons based on the inelastic cross section (different implementations)
- **MC-KLN**: Color-Glass-Condensate (CGC) based model using $k_T$-factorization
  
  Same fluctuations in the wounded nucleon positions as MC-Glauber
- **MCrcBK**: Similar to MC-KLN but with improved energy/rapidity dependence following from solutions to the running coupling Balitsky Kovchegov equation
- **IP-Glasma**: Recent CGC based model using classical Yang-Mills evolution of early-time gluon fields, including additional fluctuations in the particle production
- **Also hadronic cascades**: UrQMD or NEXUS and partonic cascades (e.g. BAMPS) can provide initial conditions
Initial energy densities

IP-Glasma

$\tau = 0.2\ \text{fm}$

MC-Glauber geometry

MC-KLN

uses $k_T$-factorization

MC-KLN: Drescher, Nara, nucl-th/0611017

mckln-3.52 from http://physics.baruch.cuny.edu/files/CGC/CGC_IC.html with defaults, energy density scaling
More choices

- Initial time $\tau_0$: thermalization time - should be of order $1 \text{ fm}/c$
- Initial transverse flow: often set to zero (cascade models provide initial flow, so does IP-Glasma)
- Assume boost-invariance: 2+1D hydrodynamics
  The viscous 3+1D hydrodynamic simulations are
- Initial $\pi^{\mu\nu}$: zero or Navier-Stokes value
Equation of State - QCD enters here

Need an equation of state \( p(\varepsilon) \) to close the set of hydro equations

- Early days: 1st order phase transition EoS from MIT bag model
- Today: EoS from lattice QCD + hadron resonance gas model


also S. Borsanyi et al, JHEP 1011:077 (2010)
The end of hydro...

Well - the end of the hydrodynamic evolution.

- Particles are observed. Not a fluid.
- How to convert fluid into particles?
- So how far is hydro valid - when to switch to a particle description?

![Graphical representation of hydrodynamic evolution](image)

Kinetic equilibrium requires scattering rate $\gg$ expansion rate

- scattering rate $\tau_{sc}^{-1} \sim \sigma n \sim \sigma T^3$
- expansion rate $\theta = \partial_\mu u^\mu$
  $\Rightarrow \tau^{-1}$ in 1+1D

Fluid description breaks down when $\tau_{sc}^{-1} \approx \theta$

$\Rightarrow$ momentum distributions freeze out

$\tau_{sc}^{-1} \propto T^3 \Rightarrow$ rapid transition to free streaming
Cooper-Frye freeze-out

Approximation: Decoupling takes place on constant temperature hypersurface $\Sigma$ at $T = T_{fo}$

- Number of particles emitted = number of particles crossing $\Sigma$:
  \[ N = \int_{\Sigma} d\Sigma_\mu N^\mu \]

- We can compute the particle current:
  \[ N^\mu = \int \frac{d^3p}{E} p^\mu f(x, \partial_\mu u^\mu) \]
  \[ N = \int \frac{d^3p}{E} \int_{\Sigma} d\Sigma_\mu p^\mu f(x, \partial_\mu u^\mu) \]

So we get the invariant inclusive momentum spectrum (Cooper-Frye formula):

\[ E \frac{dN}{d^3p} = \int_{\Sigma} d\Sigma_\mu p^\mu f(x, \partial_\mu u^\mu) \]

Freeze-out in the viscous case

Viscous correction to the equilibrium distribution functions:

\[ f \rightarrow f + \delta f \]

with

\[ \delta f = f_0 (1 \pm f_0) p^\alpha p^\beta \pi_{\alpha\beta} \frac{1}{2(\epsilon + P)T^2} \]

The choice \( \delta f \sim p^2 \) is not unique

depends on microscopic interactions

Ambiguity in \( \delta f \) leads to uncertainty

When you hear “afterburner”: Late hadronic gas stage

Combining hydrodynamic evolution with microscopic hadronic transport models.

The alternative being to just take the thermal spectra and compute resonance decays.

Use of a hadron cascade like UrQMD in hadron gas:
large dissipation and freeze-out naturally included
Less extreme transition than going from hydro right to free streaming
Different strategies

average, then evolve

initial energy density
faster
more approximate

You will hear words like
single shot hydrodynamics

evolve, then average

initial energy density
more precise
more costly

event-by-event hydrodynamics
Successes of hydro: Describes anisotropic flow

Non-central collision

beam direction

impact parameter $b$
Successes of hydro: Describes anisotropic flow

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beam direction

larger pressure gradient in this direction...

...than in this direction
Successes of hydro: Describes anisotropic flow

Non-central collision

beam direction

...than in this direction

larger pressure gradient in this direction...

Similar behavior in very different system:

“A cigar-shaped cloud of fermionic $^6$Li atoms is confined and rapidly cooled to degeneracy in a CO$_2$ laser trap [...] Upon abruptly turning off the trap, the gas exhibits a spectacular anisotropic expansion.”

Successes of hydro: Describes anisotropic flow

Non-central collision

beam direction

larger pressure gradient in this direction...

...than in this direction

Particle distribution in **momentum space** will be anisotropic. Quantify using a Fourier decomposition:

\[
\frac{dN}{d\phi} = \frac{N}{2\pi} \left( 1 + \sum_n (2v_n \cos(n\phi)) \right) \Rightarrow v_2 \text{ characterizes elliptic flow}
\]
Free streaming vs. hydro (1P-Glasma initial condition)

2+1D CYM

$t = 0.1 \text{ fm/c}$

Hydro

after $\tau = 0.2 \text{ fm/c}$ (CYM before)
Free streaming vs. hydro (IP-Glasma initial condition)

$2+1$D CYM

$t = 0.1 \text{ fm/c}$

Hydro

after $\tau = 0.2 \text{ fm/c}$ (CYM before)
Success of hydro: charged hadron $v_2(p_T)$ at RHIC

Ideal hydro, first order phase transition (EOS Q), avg init cond

Au+Au, $\sqrt{s} = 130$ A GeV, minimum bias


LDL: low density limit - not hydro
identified particle $v_2(p_T)$ at RHIC

Ideal hydro, average Glauber initial conditions

Au+Au, minimum bias

No perfect agreement but EoS with plasma phase favored

P. Huovinen (2001)
ideal hydro, Au+Au at $\sqrt{s} = 200$ A GeV
chemical equilibrium

s95p: $T_{FO} = 140$ MeV
EoS Q: first order phase transition at $T_c = 170$ MeV, $T_{FO} = 125$ MeV
Chemical freeze-out (when you see “PCE”)

Hadronic phase: ideal gas of massive hadrons and resonances assumed to be in chemical equilibrium.

- Thermal model fits to particle ratios indicate a chemical freeze-out temperature of $T_{ch} \approx 160$ MeV.
- We evolve down to $T_{FO} \approx 120$ MeV
  $\Rightarrow$ particle ratios will come out wrong in hydro
- Solution:
  include (partial) chemical non-equilibrium (PCE)
  fixing particles ratios at $T_{ch} > T_{FO}$.
  Number of pions, Kaons, etc. are conserved quantities below $T_{ch}$.
- This modifies the EoS, in particular $T(\varepsilon, n_b)$. 
Extracting transport properties of the QGP

Viscous hydrodynamics differs from ideal hydro, especially for $v_n$: Early work (smooth initial conditions):

Higher harmonic flow

\[
\frac{dN}{d\phi} = \frac{N}{2\pi} \left(1 + \sum_n (2v_n \cos[n(\phi - \psi_n)])\right)
\]

When including fluctuations, all moments appear:

\(n = 2\) \quad \(n = 3\) \quad \(n = 4\) \quad \(n = 5\) \quad \(n = 6\)

also \(v_1\) and \(n > 6\)

Compute \(v_n = \langle \cos[n(\phi - \psi_n)] \rangle\)

with the event-plane angle \(\psi_n = \frac{1}{n} \arctan \frac{\langle \sin(n\phi) \rangle}{\langle \cos(n\phi) \rangle}\)
Fluctuations and viscosity

We can see the effect of viscosity on higher harmonics in event-by-event simulations!

after 6 fm/c

initial

\[ \tau = 0.4 \text{ fm/c} \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{initial.png}
\caption{Energy density in the transverse plane after 0.4 fm/c.}
\end{figure}

ideal

\[ \tau = 6.0 \text{ fm/c, ideal} \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{ideal.png}
\caption{Energy density in the transverse plane after 6.0 fm/c, ideal.
\end{figure}

viscous

\[ \tau = 6.0 \text{ fm/c, } \eta/s = 0.16 \]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{viscous.png}
\caption{Energy density in the transverse plane after 6.0 fm/c, viscous.
\end{figure}

energy density in the transverse plane

Viscosity in a single event

B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

ideal

$\eta/s = 0.16$

t = 0.7 fm/c

energy density (scale adjusted with time)
Viscosity in a single event

B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

ideal

\[ \frac{\eta}{s} = 0.16 \]

t = 0.7 fm/c

energy density (scale adjusted with time)
More quantitatively: Sensitivity to $\eta/s$

Higher Fourier coefficients are suppressed more by viscosity.


$vn(\eta/s=0.08)/vn(ideal)$
$vn(\eta/s=0.16)/vn(ideal)$

20-30%
Using higher harmonics to determine $\eta/s$

B. Schenke, S. Jeon, C. Gale, arXiv:1109.6289

Data is from event-plane method. Calculations are $\sqrt{\langle v_n^2 \rangle}$. 

This is promising.

Need systematic study of all $v_n$ as function of initial conditions, granularity, $\eta/s$, ...

Experimental data: PHENIX, arXiv:1105.3928
Beyond constant $\eta/s$

Determine dependence of $v_2$ on modeled $\eta/s(T)$.
L=“low”, H=“high”
H=hadronic phase, Q=QGP


Weak dependence on QGP $\eta/s(T)$ at RHIC. Dependent on minimum.
Different at LHC energies (longer QGP phase, smaller gradients in the hadronic phase)
Do we really need local thermal equilibrium?

Rapid expansion → large momentum space anisotropies 
→ shear ≥ isotropic pressure.
Breakdown of expansion in terms of shear corrections

Try this: Hydrodynamics for highly anisotropic systems

Hydrodynamic expansion around anisotropic mom. dist. function 
→ New evolution equations

- by requiring energy-momentum conservation and using an ansatz for an entropy source
- by taking moments of the Boltzmann equation

One formalism for anisotropic early time dynamics and late time near-equilibrium dynamics (right limits)
Other recent developments and different approaches

- **Thermal fluct. in sound and shear waves contribute to viscosity**
  Important when microscopic $\eta/s = 0.08$ (breakdown of 2$^{nd}$ order visc. hydro)
  negligible when 0.16

- **Hydrodynamic fluctuations**
  in addition to initial state fluctuations, should be there and can be important

- **Shear viscosity from parton cascade with 2 ↔ 3 processes:**
  $\eta/s = 0.13 - 0.16$, also elliptic flow computation

- **Flow in multiphase transport model (AMPT)**

- **Flux-tube model with split into hydrodynamic and high-momentum part, EPOS - describes wide momentum range.**
Conclusions

- Hydrodynamics is a powerful tool to describe the low momentum, bulk properties of heavy-ion collisions
- QCD enters through the equation of state (EoS) and (sometimes) through the initial state
- Many 2+1D and 3+1D simulations are on the market. 2 of them viscous 3+1D event-by-event
- Some simulations are coupled to hadronic afterburners (like UrQMD) to describe hadronic stage better
- Different models for the initial state exist. Some are mainly geometric (Glauber), some based on QCD (MC-KLN, IP-Glasma), some are extracted from cascade or string models
- Particle spectra and anisotropic flow are well described by e-b-e viscous hydrodynamics
- Using hydrodynamics we can learn about transport properties of the QGP and test models for the initial state
Open questions

- Thermalization. How does the system become isotropic or even thermal in the first $\sim 0.5 \text{ fm}/c$?
- Does the system actually become isotropic and thermal in the first $\sim 0.5 \text{ fm}/c$?
- What exactly happens in the first $\sim 0.5 \text{ fm}/c$? Is there flow built up?
- Freeze-out. Is Cooper-Frye at constant $T$ good enough? What about different species?
- Viscous corrections. Do they become too large? At least for photons they do...
- Need more studies on bulk viscosity