Quantum Description of Impurities -Heavy Quarks and Quarkonia-

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Y.Akamatsu, A.Rothkopf, PRD85(2012),105011 (arXiv:1110.1203[hep-ph])
Y.Akamatsu, in preparation

Summary:
In studying quantum mechanical problem of heavy quark systems, like J/ψ and Υ suppression, we must incorporate physics of heavy quark diffusion (irreversible processes) in its description. Unified description of these physical processes is achieved by the closed-time path formalism of non-equilibrium QFT.

Recent progress in heavy quarkonium

- From complex to stochastic potential
- Schroedinger equation $i\frac{\partial}{\partial t}\Psi(X,t)=\left(2M-\frac{\nabla_1{}^2+\nabla_2{}^2}{2M}+V(r)\right)\Psi(X,t),\ X=(x_1,x_2)$
- Complex potential $V(r)=V_{\mathrm{Re}}(r)+iV_{\mathrm{Im}}(r)$

  pQCD (Laine, et al. '07)
  lattice (Rothkopf, et al. '11)
- Stochastic potential

$$\Psi(X,t)=U_{\Theta}^{(X)}(\Delta t\,|\,0)\Psi(X,0),\ U_{\Theta}^{(X)}(\Delta t\,|\,0)=\exp\left[-\frac{i}{\hbar}\Delta t\{H(X)+\Theta(X,t)\}\right],$$

$$\langle\Theta(X,t)\rangle=0,\ \langle\Theta(X,t)\Theta(X',t')\rangle=\hbar\Gamma(X,X')\delta_{tt'}/\Delta t,$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\Psi(X,t)=\left\{H(X)-\boxed{\frac{i}{2}\Gamma(X,X)}+\boxed{\Xi(X,t)}\right\}\Psi(X,t).$$

imaginary part　noise

$$\Xi(X,t)\equiv\Theta(X,t)-\frac{i\Delta t}{2\hbar}\left\{\Theta(X,t)^2-\langle\Theta(X,t)^2\rangle\right\},\ \langle\Xi(X,t)\rangle=0$$

Akamatsu, Rothkopf '12

Recent progress in heavy quarkonium

- Corresponding classical system
  - Stochastic Hamiltonian = Brownian motion w/o friction

⬇

  - Diffusion equation in momentum space

$$\left(\partial_t - D\nabla_p^2\right) f(p) = 0$$

⬇

c.f. Fokker-Planck equation $\left(\partial_t - \nabla_p\left(\Gamma p + D\nabla_p\right)\right) f(p) = 0$

  - Stationary solution = uniform in momentum space

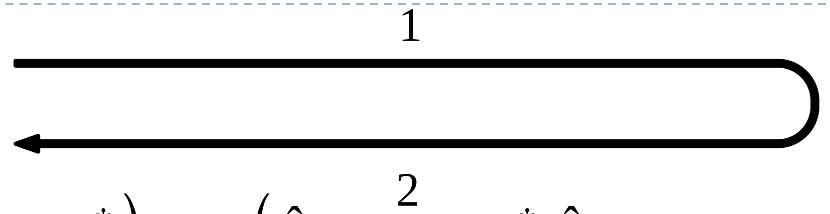
Without friction, energy rises forever …

(Quantum version: Ehrenfest relation)

Quantum description of noise = stochastic Hamiltonian
Quantum description of friction = ???

Path integral on CTP

- Closed-time path



$$Z[j_1, j_2] = \mathrm{Tr}\left(\hat{U}(\infty,0; j_1)\hat{\rho}\hat{U}(\infty,0; j_2)^\dagger\right) = \mathrm{Tr}\left(\hat{U}(\infty,0; j_2)^\dagger\hat{U}(\infty,0; j_1)\hat{\rho}\right)$$

$$\sim \int D\varphi_{1,2}\,\rho[\varphi_1^{\mathrm{ini}}, \varphi_2^{\mathrm{ini}}]\exp\left(iS(\varphi_1) - iS(\varphi_2) + i\int j_1\varphi_1 - i\int j_2\varphi_2\right)$$

$$\prod_i \frac{\delta}{\delta j_i(x_i)} \ln Z[j_1, j_2]\Big|_{j_{1,2}=0} \propto \left\langle \mathrm{T_C} \prod_i \hat{\varphi}_i(x_i)\right\rangle_{\mathrm{conn}}$$

$$\Rightarrow \begin{cases} \dfrac{\delta^2}{\delta j_1(x_1)\delta j_1(x_2)} \ln Z[j_1, j_2]\Big|_{j_{1,2}=0} \propto \left\langle \mathrm{T}\hat{\varphi}(x_1)\hat{\varphi}(x_2)\right\rangle_{\mathrm{conn}} = G^F(x_1, x_2) \\ \dfrac{\delta^2}{\delta j_1(x_1)\delta j_2(x_2)} \ln Z[j_1, j_2]\Big|_{j_{1,2}=0} \propto \left\langle \hat{\varphi}(x_2)\hat{\varphi}(x_1)\right\rangle_{\mathrm{conn}} = G^<(x_1, x_2) \\ \ldots \end{cases}$$

Path integral on CTP

- Application to QCD ($qA + \psi$)

$$Z[\eta_1,\eta_2] \sim \int D[\psi qA_{1,2}]\rho[\psi qA_1^{\text{ini}},\psi qA_2^{\text{ini}}]\exp\left(iS_1^{\psi} - iS_2^{\psi} + i\int \psi_1\eta_1 - i\int \psi_2\eta_2\right)$$

$$\times \exp\left(iS_1^{qA} - iS_2^{qA} + ig\int j_1 A_1 - ig\int j_2 A_2\right)$$

$$\rho = \rho_{\text{eq}}^{E} \otimes \rho^{S} \rightarrow \rho[\psi qA_1^{\text{ini}},\psi qA_2^{\text{ini}}] = \rho_{\text{eq}}^{E}[qA_1^{\text{ini}},qA_2^{\text{ini}}]\cdot \rho^{S}[\psi_1^{\text{ini}},\psi_2^{\text{ini}}]$$

E:Environment
S:System

* Ghost and FP term omitted for simplicity

Influence functional

$$= Z^{qA}[j_1, j_2]$$

$$= \exp\left(-1/2\int j_1 G^{F} j_1 + j_2 G^{\tilde{F}} j_2 - j_1 G^{>} j_2 - j_2 G^{<} j_1\right)$$

$$\times \exp\left(\int G_3\, jjj + G_4\, jjjj + \cdots\right)$$

$$\equiv \exp\left(iS^{FV}\right)$$

Weak coupling & Nonrelativistic limit

- Quantum dynamical equations in the leading order perturbation in g and in $(T/M)^{1/2}$.
  - Expansion in j up to 2nd order ($\sim g^2$)
  - Kinetic term and current in non-relativistic limit

$$\psi = (Q, Q_c^\dagger), \quad j^{a\mu} = \left(\rho^a, \vec{j}^{\,a}\right)$$

$$S^\psi = Q^\dagger\left(i\partial_0 - M + \nabla^2/2M\right)Q + Q_c\left(i\partial_0 + M - \nabla^2/2M\right)Q_c^\dagger + O\left(T^2/M\right) \sim M + T,$$

$$\rho^a = Q^\dagger t^a Q + Q_c t^a Q_c^\dagger \sim 1,$$

$$\left.\begin{array}{l} \vec{j}^{\,a} = Q^\dagger\left(\overleftrightarrow{\nabla}/2iM\right)t^a Q - Q_c\left(\overleftrightarrow{\nabla}/2iM\right)t^a Q_c^\dagger + O\left((T/M)^{3/2}\right) \\ \quad \cancel{+ Q^\dagger \vec{\sigma} t^a Q_c^\dagger + Q_c \vec{\sigma} t^a Q + O(T/M)} \end{array}\right] \sim O\left(\sqrt{T/M}\right),$$

Quenched approx.

$$\dot{\rho}^{0a} = -\vec{\nabla}\cdot\vec{j}^{\,a} + O\left(g^2\right) \sim O\left(\sqrt{T/M}\right), \quad \dot{\vec{j}}^{\,a} \sim O\left(g^2\right),$$

Weak coupling & Nonrelativistic limit

- Instantaneous approx.

Time scale for j is slow $\Leftrightarrow$ Time scale for G is fast

$$\tilde{G}(\vec{x}-\vec{y},\omega) \approx \tilde{G}(\vec{x}-\vec{y},0) + \omega \tilde{G}'(\vec{x}-\vec{y},0) \equiv \overline{G}(\vec{x}-\vec{y}) + \omega \overline{G}'(\vec{x}-\vec{y})$$

$$\Leftrightarrow \quad G(x-y) \approx \overline{G}(\vec{x}-\vec{y})\delta(x^0-y^0) + i\overline{G}'(\vec{x}-\vec{y})\frac{\partial}{\partial(x^0-y^0)}\delta(x^0-y^0)$$

$$\int_{xy} j(x)G(x-y)j(y) \approx \int_t \int_{\vec{x}\,\vec{y}} \left[\begin{array}{l} \overline{G}(\vec{x}-\vec{y})\, j(t,\vec{x})\, j(t,\vec{y}) \\ -\dfrac{i}{2}\overline{G}'(\vec{x}-\vec{y})\left\{\dot{j}(t,\vec{x})\, j(t,\vec{y}) - j(t,\vec{x})\,\dot{j}(t,\vec{y})\right\} \end{array} \right]$$

$\overline{G}_{0i}(\vec{x}-\vec{y}) = 0$ in Coulomb and covariant gauges.

➡ Only 00 components (charge density couplings) are relevant.

Weak coupling & Nonrelativistic limit

- Kinetic term $S_{NR}^{kin} = S_{1,NR}^{\psi} - S_{2,NR}^{\psi}$
- Influence functional

$$S_{LONR}^{FV}[j_1, j_2] = -1/2 \int_{t,\vec{x},\vec{y}} \left(V \rho_1^a \rho_1^a - V^* \rho_2^a \rho_2^a \right) + \int_{t,\vec{x},\vec{y}} \left(iD \rho_1^a \rho_2^a - 1/4T\, \vec{\nabla} D \cdot \left(\vec{j}_1^{\,a} \rho_2^a + \rho_1^a \vec{j}_2^{\,a} \right) \right)$$

$$\overline{G}_{ab,00}^{F}(\vec{x}-\vec{y}) = \left\{ -i\overline{G}_{00}^{R}(\vec{x}-\vec{y}) + \overline{G}_{00}^{<}(\vec{x}-\vec{y}) \right\} \delta_{ab} \Rightarrow V(\vec{x}-\vec{y}) \equiv -\left\{ \overline{G}_{00}^{R}(\vec{x}-\vec{y}) + i\overline{G}_{00}^{>}(\vec{x}-\vec{y}) \right\}$$

$$\overline{G}_{ab,00}^{\tilde{F}}(\vec{x}-\vec{y}) = \left\{ i\overline{G}_{00}^{R}(\vec{x}-\vec{y}) + \overline{G}_{00}^{>}(\vec{x}-\vec{y}) \right\} \delta_{ab} \qquad D(\vec{x}-\vec{y}) \equiv -\overline{G}_{00}^{>}(\vec{x}-\vec{y})$$

$$\overline{G}_{ab,00}^{R}(\vec{x}-\vec{y}) = g^2 \int_0^{\beta} d\tau \left\langle \hat{A}_{a0}(\vec{x},-i\tau) \hat{A}_{b0}(\vec{y},0) \right\rangle \propto \delta_{ab}$$

Complex potential

Dissipative coupling

$$\overline{G}_{ab,00}^{>}(\vec{x}-\vec{y}) = 2T\overline{G}'^{>}_{ab,00}(\vec{x}-\vec{y}) = \lim_{\omega\to 0} \frac{T}{\omega} \sigma_{ab,00}(\omega, \vec{x}-\vec{y}) \propto \delta_{ab}$$

$$\sigma_{ab,00}(\omega, \vec{x}-\vec{y}) = g^2 \int dt\, e^{i\omega t} \left\langle \left[\hat{A}_{a0}(\vec{x},t), \hat{A}_{b0}(\vec{y},0) \right] \right\rangle$$

Hamiltonian & Renormalization

- Action and Hamiltonian for what?
  - Initial value problem of a quantum system

$$Z[\eta_1,\eta_2] \sim \int D[\psi_{1,2}]\, \rho^S[\psi_1^{\text{ini}},\psi_2^{\text{ini}}] \text{= initial “wave function”}$$

$$\times \exp\left(iS_{NR}^{kin} + iS_{LONR}^{FV} + \cdots + i\int \psi_1\eta_1 - i\int \psi_2\eta_2 \right)$$

$$S_{LONR}^{FV}[j_1, j_2] = -1/2 \int_{t,\vec{x},\vec{y}} \left(V\rho_1^a\rho_1^a - V^*\rho_2^a\rho_2^a \right)$$

$$+ \int_{t,\vec{x},\vec{y}} \left(iD\rho_1^a\rho_2^a - 1/4T\, \vec{\nabla} D \cdot \left(\vec{j}_1^a\rho_2^a + \rho_1^a\vec{j}_2^a \right) \right) + \cdots$$

- Strategy
  - Obtain Hamiltonian H_{1+2} (~ladder approximation)
  - Functional Schroedinger equation for functional density matrix

$$\rho^S[\psi_1,\psi_2]$$

Hamiltonian & Renormalization

- Time arguments at *t*
- Which order in Hamiltonian?

$$\overline{\psi}\psi, \psi\overline{\psi}, N[\overline{\psi}\psi]??$$

- Fermion bilinears

Order of fermions → time arguments in path integral

For later purpose, define $\tilde{\psi}_2 \equiv \psi_2^\dagger, \tilde{\psi}_2^\dagger \equiv \psi_2, \tilde{\psi}_2^\dagger(t+\varepsilon)\cdots\tilde{\psi}_2(t-\varepsilon)$

➡ $\psi^*{}_1(t+\varepsilon)\cdots\psi_1(t-\varepsilon)$ and $\tilde{\psi}^*{}_2(t+\varepsilon)\cdots\tilde{\psi}_2(t-\varepsilon)$

- In instantaneous approximation

→ Symmetric in all possible order in time

Hamiltonian & Renormalization

- Hamiltonian
  - Inserting fermionic complete sets on a single time path
    → Fermions are time-ordered
  - Functional density matrix?

$$\left\langle \psi_1^{*\mathrm{fin}} \middle| \mathrm{Tr}_{\mathrm{E}}\left(\hat{U}(t,0)\hat{\rho}\hat{U}(t,0)^\dagger\right) \middle| \tilde{\psi}_2^{*\mathrm{fin}} \right\rangle$$

$$= \int d\psi_1^{*\mathrm{ini}} d\tilde{\psi}_2^{*\mathrm{ini}} \int_{\psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}}}^{\psi_1^{*\mathrm{fin}},\tilde{\psi}_2^{*\mathrm{fin}}} D[\psi_{1,2}]\, \rho^S[\psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}}] \exp\left(iS_{1,NR}^{\psi} - iS_{2,NR}^{\psi} + iS_{LONR}^{FV} + \cdots\right)$$

$$= \left\langle \psi_1^{*\mathrm{fin}},\tilde{\psi}_2^{*\mathrm{fin}} \middle| \exp\left[-i\int dt \hat{H}_{1+2}\right] \middle| \Psi_{\mathrm{ini}} \right\rangle, \quad \left\langle \psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}} \middle| \Psi_{\mathrm{ini}} \right\rangle = \rho^S[\psi_1^{*\mathrm{ini}},\tilde{\psi}_2^{*\mathrm{ini}}]$$

★Coherent state built on **empty Dirac sea** for HQ ≠ HQ **vacuum**

Hamiltonian & Renormalization

- Change basis to

$$\left\langle Q_1^*, Q_{1c}^* \right| \equiv \langle \Omega | \exp\left[-\int_x \hat{Q} Q_1^* + \hat{Q}_c Q_{1c}^* \right]$$

$$\left| \tilde{Q}_2^*, \tilde{Q}_{2c}^* \right\rangle \equiv \exp\left[-\int_x \tilde{Q}_2^* \hat{Q}^\dagger + \tilde{Q}_{2c}^* \hat{Q}_c^\dagger \right] |\Omega\rangle$$

⬇

$$\left\langle Q_{1(c)}^{*\,\mathrm{fin}} \right| \mathrm{Tr}_E \left(\hat{U}(t,0) \hat{\rho} \hat{U}(t,0)^\dagger \right) \left| \tilde{Q}_{2(c)}^{*\,\mathrm{fin}} \right\rangle$$

← This is what we want!

$$= \int dQ_{1(c)}^{*\mathrm{ini}} d\tilde{Q}_{2(c)}^{*\mathrm{ini}} \int_{Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}}}^{Q_{1(c)}^{*\mathrm{fin}}, \tilde{Q}_{2(c)}^{*\mathrm{fin}}} D[Q_{1(c),2(c)}^{(*)}] \, \rho^S[Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}}] \exp\left(iS_{1,NR}^{\psi} - iS_{2,NR}^{\psi} + iS_{LONR}^{FV} + \cdots + \right)$$

$$= \left\langle Q_{1(c)}^{*\,\mathrm{fin}}, \tilde{Q}_{2(c)}^{*\,\mathrm{fin}} \right| \exp\left[-i \int dt \hat{H}_{1+2} \right] \left| \Psi_{\mathrm{ini}} \right\rangle, \quad \left\langle Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}} \middle| \Psi_{\mathrm{ini}} \right\rangle = \rho^S[Q_{1(c)}^{*\mathrm{ini}}, \tilde{Q}_{2(c)}^{*\mathrm{ini}}]$$

Convenient to have H_{1+2} in **normal order** for Q_1 and $\tilde{Q}_2$ (not Q_2)

Hamiltonian & Renormalization

- Results

$$\boldsymbol{H}_{\rm kin}^{\rm NR} = \int d^3x Z_M M \left\{ \boldsymbol{Q}_1^\dagger \boldsymbol{Q}_1 + \boldsymbol{Q}_{1c}^\dagger \boldsymbol{Q}_{1c} + \tilde{\boldsymbol{Q}}_2^\dagger \tilde{\boldsymbol{Q}}_2 + \tilde{\boldsymbol{Q}}_{2c}^\dagger \tilde{\boldsymbol{Q}}_{2c} \right\}$$

$$+ \int d^3x Z_\psi \left\{ \begin{array}{c} \boldsymbol{Q}_1^\dagger \left(-\frac{\nabla^2}{2M}\right) \boldsymbol{Q}_1 + \boldsymbol{Q}_{1c}^\dagger \left(-\frac{\nabla^2}{2M}\right) \boldsymbol{Q}_{1c} \\ + \tilde{\boldsymbol{Q}}_2^\dagger \left(-\frac{\nabla^2}{2M}\right) \tilde{\boldsymbol{Q}}_2 + \tilde{\boldsymbol{Q}}_{2c}^\dagger \left(-\frac{\nabla^2}{2M}\right) \tilde{\boldsymbol{Q}}_{2c} \end{array} \right\}$$

$$\boldsymbol{H}_{\rm FV}^{\rm LONR} = \boldsymbol{H}_{(11)} + \boldsymbol{H}_{(22)} + \boldsymbol{H}_{(12)},$$

$$\boldsymbol{H}_{(11)} = \frac{Z_g^2}{2} \int d^3x d^3y V(\vec{x}-\vec{y}) \mathrm{N} \left\{ \boldsymbol{j}_1^{a0}(\vec{x}) \boldsymbol{j}_1^{a0}(\vec{y}) \right\}$$

$$+ \frac{Z_g^2 C_{\rm F}}{2} \int d^3x V(0) (\boldsymbol{Q}_1^\dagger \boldsymbol{Q}_1 + \boldsymbol{Q}_{1c}^\dagger \boldsymbol{Q}_{1c}) + \text{const.},$$

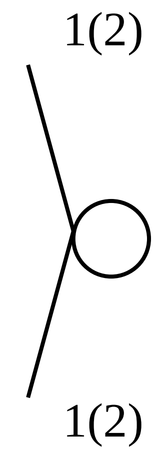
$$\boldsymbol{H}_{(22)} = -\frac{Z_g^2}{2} \int d^3x d^3y V^*(\vec{x}-\vec{y}) \mathrm{N} \left\{ \boldsymbol{j}_2^{a0}(\vec{x}) \boldsymbol{j}_2^{a0}(\vec{y}) \right\}$$

$$+ \frac{Z_g^2 C_{\rm F}}{2} \int d^3x V^*(0) (\tilde{\boldsymbol{Q}}_2^\dagger \tilde{\boldsymbol{Q}}_2 + \tilde{\boldsymbol{Q}}_{2c}^\dagger \tilde{\boldsymbol{Q}}_{2c}) + \text{const.},$$

(real) UV divergence + finite imaginary part

$$\boldsymbol{H}_{(12)} = -i Z_g^2 \int d^3x d^3y D(\vec{x}-\vec{y}) \mathrm{N} \left\{ \boldsymbol{j}_1^{a0}(\vec{x}) \boldsymbol{j}_2^{a0}(\vec{y}) \right\}$$

$$+ \frac{Z_g^2}{4T} \int d^3x d^3y \vec{\nabla}_x D(\vec{x}-\vec{y}) \cdot \mathrm{N} \left\{ \vec{\boldsymbol{j}}_{1,\rm NR}^a(\vec{x},t) \boldsymbol{j}_2^{a0}(\vec{y},t) + \boldsymbol{j}_1^{a0}(\vec{x},t) \vec{\boldsymbol{j}}_{2,\rm NR}^a(\vec{y},t) \right\}$$



Hamiltonian & Renormalization

- UV divergence and renormalization
- UV divergence

$V(0) = V_{\rm vac} + V_{\rm med}$

→ $V_{\rm vac}$ = UV divergent, $D = {\rm Im}[V(0)] = {\rm Im}[V_{\rm med}]$ = finite

- Rernormalization of vacuum contribution

- correct potential at r~$1/T$~(typical exchanged momentum)$^{-1}$

- correct kinetic term at k~$(MT)^{1/2}$~typical HQ momentum

➡
$$\begin{cases} Z_\psi(T) = 1, \quad Z_M(T) + \dfrac{V_{\rm vac} C_{\rm F}}{2M} Z_g(T)^2 = 1. \\ \text{Perturbatively, } Z_g(T) = g(T)/g_{\rm bare} \text{ or just } Z_g(T)=1,\ g_{\rm bare}=g(T) \end{cases}$$

Hamiltonian & Renormalization

- Renormalized Hamiltonian

Complex mass & potential

Dissipative coupling

$$\boldsymbol{H}_{1+2} = \int d^3x \left[aM \left(\boldsymbol{Q}_1^\dagger \boldsymbol{Q}_1 + \boldsymbol{Q}_{1c}^\dagger \boldsymbol{Q}_{1c} \right) + \left\{ \boldsymbol{Q}_1^\dagger \left(-\frac{\nabla^2}{2M} \right) \boldsymbol{Q}_1 + \boldsymbol{Q}_{1c}^\dagger \left(-\frac{\nabla^2}{2M} \right) \boldsymbol{Q}_{1c} \right\} \right]$$
$$+ \int d^3x \left[a^* M \left(\tilde{\boldsymbol{Q}}_2^\dagger \tilde{\boldsymbol{Q}}_2 + \tilde{\boldsymbol{Q}}_{2c}^\dagger \tilde{\boldsymbol{Q}}_{2c} \right) + \left\{ \tilde{\boldsymbol{Q}}_2^\dagger \left(-\frac{\nabla^2}{2M} \right) \tilde{\boldsymbol{Q}}_2 + \tilde{\boldsymbol{Q}}_{2c}^\dagger \left(-\frac{\nabla^2}{2M} \right) \tilde{\boldsymbol{Q}}_{2c} \right\} \right]$$
$$+ \frac{1}{2} \int d^3x d^3y \left[v(\vec{x}-\vec{y}) \mathrm{N} \left\{ \boldsymbol{j}_1^{a0}(\vec{x}) \boldsymbol{j}_1^{a0}(\vec{y}) \right\} - v^*(\vec{x}-\vec{y}) \mathrm{N} \left\{ \boldsymbol{j}_2^{a0}(\vec{x}) \boldsymbol{j}_2^{a0}(\vec{y}) \right\} - 2id(\vec{x}-\vec{y}) \mathrm{N} \left\{ \boldsymbol{j}_1^{a0}(\vec{x}) \boldsymbol{j}_2^{a0}(\vec{y}) \right\} + \frac{1}{2T} \vec{\nabla}_x d(\vec{x}-\vec{y}) \cdot \mathrm{N} \left\{ \vec{\boldsymbol{j}}_{1,\mathrm{NR}}^{a}(\vec{x},t) \boldsymbol{j}_2^{a0}(\vec{y},t) + \boldsymbol{j}_1^{a0}(\vec{x},t) \vec{\boldsymbol{j}}_{2,\mathrm{NR}}^{a}(\vec{y},t) \right\} \right]$$
$$+ \cdots, \quad (45)$$

$$a = 1 + Z_g(T)^2 C_{\mathrm{F}} V_{\mathrm{med}}/2M$$

$$v(\vec{r}) = Z_g(T)^2 V(\vec{r})$$

$$d(\vec{r}) = Z_g(T)^2 D(\vec{r})$$

Density matrix & master equation

- Functional density matrix

$$\rho_S\left[Q_1^*, Q_{1c}^*, \tilde{Q}_2^*, \tilde{Q}_{2c}^*, t\right] = \langle Q_1^*, Q_{1c}^* | \hat{\rho}_S(t) | \tilde{Q}_2^*, \tilde{Q}_{2c}^* \rangle = \langle Q_1^*, Q_{1c}^*, \tilde{Q}_2^*, \tilde{Q}_{2c}^* | \Psi(t) \rangle,$$

$$\langle Q_1^*, Q_{1c}^* | \equiv \langle \Omega | \exp\left[- \int d^3x \left\{ \hat{Q}(\vec{x}) Q_1^*(\vec{x}) + \hat{Q}_c(\vec{x}) Q_{1c}^*(\vec{x}) \right\} \right],$$

$$|\tilde{Q}_2^*, \tilde{Q}_{2c}^* \rangle \equiv \exp\left[- \int d^3x \left\{ \tilde{Q}_2^*(\vec{x}) \hat{Q}^\dagger(\vec{x}) + \tilde{Q}_{2c}^*(\vec{x}) \hat{Q}_c^\dagger(\vec{x}) \right\} \right] |\Omega\rangle.$$

- Time evolution

  In analogy to Schroedinger wave equation

$$i \frac{\partial}{\partial t} \rho_S \left[Q_{1(c)}^*, \tilde{Q}_{2(c)}^*, t \right]$$

$$= H_{1+2}\left[Q_{1(c)}^*, Q_{1(c)} = \frac{\delta}{\delta Q_{1(c)}^*}, \tilde{Q}_{2(c)}^*, \tilde{Q}_{2(c)} = -\frac{\delta}{\delta \tilde{Q}_{2(c)}^*}, t \right] \rho_S \left[Q_{1(c)}^*, \tilde{Q}_{2(c)}^*, t \right]$$

Density matrix & master equation

- Density matrix and master equation
- Single HQ

$$\rho_{\rm S}(\vec{x},\vec{y},t) \propto \langle\Omega|\hat{Q}(\vec{x})\hat{\rho}_{\rm S}(t)\hat{Q}^{\dagger}(\vec{y})|\Omega\rangle = -\frac{\delta}{\delta Q_1^*(\vec{x})}\frac{\delta}{\delta\tilde{Q}_2^*(\vec{y})}\rho_{\rm S}\left[Q_1^*, Q_{1c}^*, \tilde{Q}_2^*, \tilde{Q}_{2c}^*, t\right]\Big|_{Q_{1(c)}^*=\tilde{Q}_{2(c)}^*=0}$$

$$\rho_{\rm S}(\vec{x},\vec{y},t) \equiv \rho_{\rm S}^{ii}(\vec{x},\vec{y},t)$$

$$i\frac{\partial}{\partial t}\rho_{\rm S}(\vec{x},\vec{y},t) = \left\{(a-a^*)M + \left(-\frac{\nabla_x^2-\nabla_y^2}{2M}\right)\right\}\rho_{\rm S}(\vec{x},\vec{y},t) + C_{\rm F}\left\{-id(\vec{x}-\vec{y}) + \frac{\vec{\nabla}_x d(\vec{x}-\vec{y})}{4T}\cdot\frac{\vec{\nabla}_x-\vec{\nabla}_y}{iM}\right\}\rho_{\rm S}(\vec{x},\vec{y},t).$$

- Systems with arbitrary number of HQ can be treated similarly.

Density matrix & master equation

- HQ number conservation

$$\mathrm{Tr}\rho_{\mathrm{S}}(t) \equiv \int d^3x \rho_{\mathrm{S}}(\vec{x},\vec{x},t),$$

$$i\frac{d}{dt}\mathrm{Tr}\rho_{\mathrm{S}}(t) = \int d^3x d^3y \delta(\vec{x}-\vec{y}) \left(i\frac{\partial}{\partial t}\rho_{\mathrm{S}}(\vec{x},\vec{y},t) \right) = 0$$

- Ehrenfest relations
- Velocity
- Drag force
- Equilibration

$$\frac{d}{dt}\langle\vec{x}\rangle = \frac{\langle\vec{p}\rangle}{M},$$

$$\frac{d}{dt}\langle\vec{p}\rangle = -\frac{\gamma}{2MT}\langle\vec{p}\rangle,$$

$$\frac{d}{dt}\langle E\rangle = -\frac{\gamma}{MT}\left(\langle E\rangle - \frac{3T}{2}\right).$$

$$\gamma = \frac{C_F}{3}\nabla^2 d(x)\Big|_{x=0} = -\frac{g(T)^2 C_F}{9}\nabla^2 \tilde{G}^{>}_{00,aa}(\omega=0,x)\Big|_{x=0}$$

$$= \frac{g(T)^2 C_F}{9}\int \frac{d^3k}{(2\pi)^3} k^2 \tilde{G}^{>}_{00,aa}(\omega=0,k)$$

Consistent with classical Langevin drag force by Caron-Huot and Moore ’07,’08

Forward correlator and complex potential

- Forward correlation function

$$G^{>}_{Q\bar{Q}}(x,y,t) \propto \frac{\delta^2}{\delta Q_1^*(x)\delta Q_{1c}^*(y)} \rho_S\left[Q_{1(c)}^*, \tilde{Q}_{2(c)}^*, t\right]_{Q^*=0}$$

$$i\frac{\partial}{\partial t} G^{>}_{Q\bar{Q}}(x,y,t) = \left(-\frac{\nabla_x^2 + \nabla_y^2}{2M} + 2aM - v(x-y)\left(t^a\right)_x \left(t^a\right)_y \right) G^{>}_{Q\bar{Q}}(x,y,t)$$

Projection on singlet state → complex potential

$$V_{complex}(\vec{r}) = 2(a-1)M - C_F v(\vec{r})$$
$$= -\frac{g(T)^2 C_F}{4\pi}\left(\omega_D + \frac{e^{-\omega_D r}}{r} + iT\phi(\omega_D r) \right)$$

Consistent with potential from
Wilson loop spectral decomposition by Laine ’07

Conclusion

- Starting from non-equilibrium field theory on a closed-time path, we derive renormalized Hamiltonian H_{1+2} on a single time.
- Derivation in the leading order in g and in $(T/M)^{1/2}$.
- Master equation and forward correlator for any heavy quark system can be derived from the Hamiltonian H_{1+2}.
- HQ number conservation are confirmed.
- Heavy quark equilibration is derived through Ehrenfest relations.
- Imaginary part of the complex potential is obtained from thermal mass + complex potential.