Instantons and Sphalerons in a Magnetic Field

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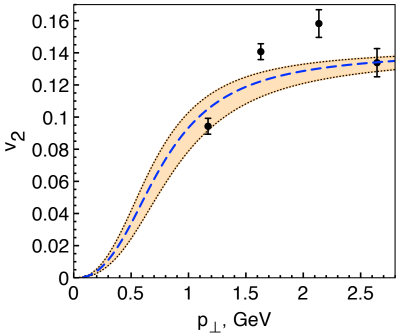
08/17/2012

Quark Matter 2012, Washington D.C.

GB, G.Dunne & D. Kharzeev , arXiv:1112.0532, PRD **85** 045026
GB, D. Kharzeev, arXiv:1202.2161, PRD **85** 086012

Magnetic field generated in heavy ion collisions $\sim m_\pi^2$ combined with:

- Axial Anomaly $\Rightarrow$ C.M.E. (charge separation)
  C.M.W. (charge dependent v_2)
- Conformal Anomaly $\Rightarrow$ photon v_2
  (Başar, Kharzeev, Skokov, arXiv:1206.1334)



(talk by V. Skokov at xQCD, 08/22)

Part I

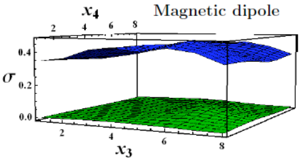
Instanton in a Magnetic Field

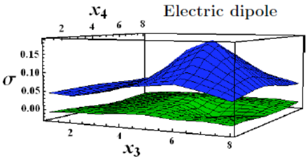
Motivation & some lattice results

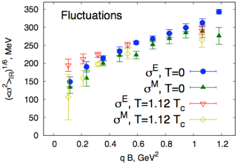
Interplay between topology & magnetic field

- Chiral magnetic effect $\vec{J} \propto \mu_5 \vec{B}$
- What sources μ_5? sphalerons, η domains, etc..
- Instanton + magnetic field
- Lattice results
  - ITEP group (electric & dipole moments)
  - T. Blum et al. (zero modes $\propto$ B)
  - A. Yamamoto (C.M. conductivity)

(Polikarpov et al. '09)







Notation & conventions

work in: $\mathbb{R}^4$

chiral basis: $$\gamma_\mu = \begin{pmatrix} 0 & \alpha_\mu \\ \bar{\alpha}_\mu & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\alpha_\mu = (\mathbb{1}, -i\vec{\sigma}) \quad , \quad \bar{\alpha}_\mu = (\mathbb{1}, i\vec{\sigma}) = \alpha_\mu^\dagger$$

Dirac operator: $$\not{\mathcal{D}} = \begin{pmatrix} 0 & \alpha_\mu \mathcal{D}_\mu \\ \bar{\alpha}_\mu \mathcal{D}_\mu & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & D \\ -D^\dagger & 0 \end{pmatrix}$$

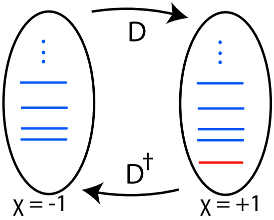
gauge field: $$\mathcal{A}_\mu = A_\mu + a_\mu$$

Notation & conventions

diagonal form: $(i\not{D})^2\,\psi_\lambda = \begin{pmatrix} DD^\dagger & 0 \\ 0 & D^\dagger D \end{pmatrix}\psi_\lambda = \lambda^2\psi_\lambda$

$$\chi = +1: \qquad DD^\dagger = -\mathcal{D}_\mu^2 - \mathcal{F}_{\mu\nu}\bar{\sigma}_{\mu\nu}$$
$$\chi = -1: \qquad D^\dagger D = -\mathcal{D}_\mu^2 - \mathcal{F}_{\mu\nu}\sigma_{\mu\nu}$$

"supersymmetry:" for $\lambda \neq 0$, $DD^\dagger$ and $D^\dagger D$ has identical spectra



Instanton & magnetic field

$$DD^\dagger = -\mathcal{D}_\mu^2 - B\sigma_3, \quad , \quad D^\dagger D = -\mathcal{D}_\mu^2 - F_{\mu\nu}\sigma_{\mu\nu} - B\sigma_3$$

Zero modes: Both spins, both chiralities

Index thm: $\mathrm{tr}\left(\mathcal{F}_{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu}\right) = \mathrm{tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right)$

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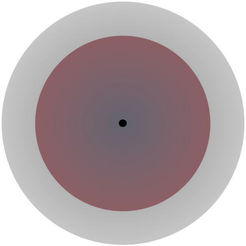
Competition between instanton and magnetic field

$\downarrow$ try to align chiralities (instanton)

$\downarrow$ align spins (magnetic field)

Instanton zero mode: $|\psi_0|^2 = \frac{64\,\rho^2}{(x^2+\rho^2)^3}$

Topological charge: $q_5(x) = \frac{192\,\rho^4}{(x^2+\rho^2)^4}$



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B field zero mode: $|\psi_0|^2 \propto f(x_1 + ix_2)e^{-B|x_1+ix_2|^2}$

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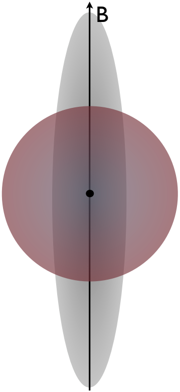

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suppose: $\frac{1}{\sqrt{B}} << \rho$

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after appropriate gauge rotation & Lorentz transformation:

$$\mathcal{A}_\mu = -\frac{F}{2}(-x_2, x_1, -x_4, x_3)\tau^3 + \frac{B}{2}(-x_2, x_1, 0, 0)\mathbb{1}_{2\times 2}$$

quasi-abelian, covariantly constant $\rightarrow$ soluble!

Large instanton limit

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$$\mathcal{F}_{12} = \begin{pmatrix} B-F & 0 \\ 0 & B+F \end{pmatrix}$$

$$\mathcal{F}_{34} = \begin{pmatrix} -F & 0 \\ 0 & F \end{pmatrix}$$

Landau problem with field strengths $\mathcal{F}_{12}$ & $\mathcal{F}_{34}$

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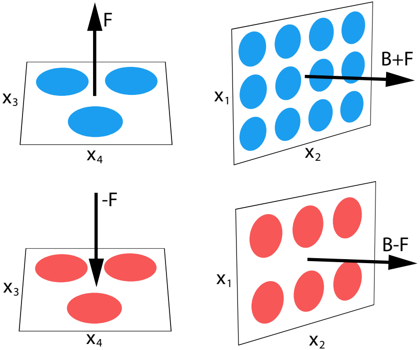
Zero modes

$$\tau = -1\,,\ \chi = -1\,,\ spin \uparrow,\ n_- = \frac{(B+F)}{2\pi}\ \frac{F}{2\pi}$$

$$\tau = +1\,,\ \chi = +1\,,\ spin \uparrow,\ n_+ = \frac{(B-F)}{2\pi}\ \frac{F}{2\pi}$$

$$n_+ + n_- = B\frac{F}{2\pi^2} \quad , \quad n_+ - n_- = -\frac{F^2}{2\pi^2}$$



Dipole moments

$$\sigma_3^M = \frac{1}{2}\langle\bar{\psi}\Sigma_{12}\psi\rangle \quad , \quad \sigma_3^E = \langle\bar{\psi}\Sigma_{34}\psi\rangle$$

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$$\begin{aligned} m\langle\bar{\psi}\Sigma_{12}\psi\rangle &= \mathrm{tr}_{2\times 2}\left(\sigma_3 \frac{m^2}{m^2 + DD^\dagger}\right) + \mathrm{tr}_{2\times 2}\left(\sigma_3 \frac{m^2}{m^2 + D^\dagger D}\right) \\ m\langle\bar{\psi}\Sigma_{34}\psi\rangle &= -\mathrm{tr}_{2\times 2}\left(\sigma_3 \frac{m^2}{m^2 + DD^\dagger}\right) + \mathrm{tr}_{2\times 2}\left(\sigma_3 \frac{m^2}{m^2 + D^\dagger D}\right) \end{aligned}$$

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$$\begin{aligned} m\langle\bar{\psi}\Sigma_{12}\psi\rangle &\approx \frac{BF}{2\pi^2} \\ m\langle\bar{\psi}\Sigma_{34}\psi\rangle &\approx \frac{F^2}{2\pi^2} \\ \langle\bar{\psi}\Sigma_{34}\psi\,\bar{\psi}\Sigma_{34}\psi\rangle &\approx \left(\frac{F}{2\pi^2 m^2 L^4}\right) B \end{aligned}$$

- $\sigma_3^M > \sigma_3^E$

Part II

Sphaleron Rate in a Magnetic Field

Sphaleron rate (basics)

$$\Gamma_{CS} = \frac{(\Delta Q_5)^2}{Vt} = \int d^4x \left\langle \frac{g^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a(x) \frac{g^2}{32\pi^2} F^a_{\alpha\beta}\tilde{F}^{\alpha\beta}_a(0) \right\rangle$$

Diffusion of topological charge: $\frac{dN_5}{dt} = -c\, N_5\, \frac{\Gamma_{CS}}{T^3}$

- CP odd effects in QCD (CME)
- Baryon number (B+L) violation in E.W.

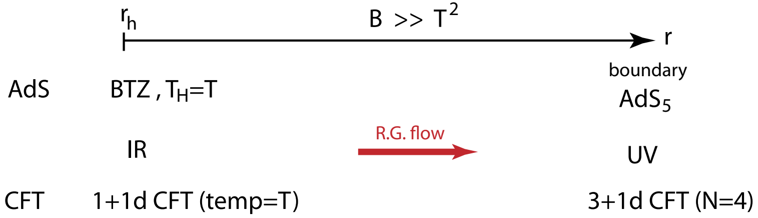
Weak coupling: $\Gamma_{CS} = \kappa\, g^4 T \log(1/g)\,(g^2T)^3$ (Bödeker '98)

Strong coupling: $\Gamma_{CS} = \frac{(g^2N)^2}{256\pi^3} T^4$ (Son, Starinets '02)

Sphaleron rate with B field (holography)

Gauge theory in magnetic field $\Leftrightarrow$ Einstein-Maxwell theory

Dynamics with magnetic field $\Leftrightarrow$ Self-consistent solutions



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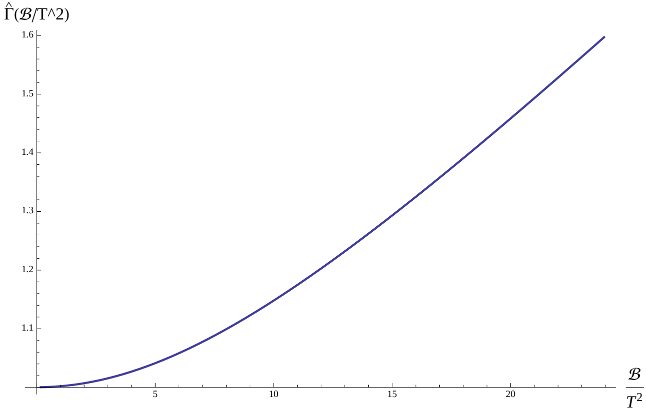

$$\Gamma_{CS} = \begin{cases} \frac{(g^2N)^2}{256\pi^3}\left(T^4 + \frac{1}{6\pi^4}B^2 + \mathcal{O}(\frac{B^4}{T^2})\right) & , \quad B << T^2 \\ \frac{(g^2N)^2}{384\sqrt{3}\pi^5}\left(B\,T^2 + 15.9\,T^4 + \mathcal{O}(\frac{T^6}{B})\right) & , \quad B >> T^2 \end{cases}$$

Sphaleron rate with B field (holography)

$$B \sim T^2$$

- for $B = T^2$ the effect is $\sim \%0.2$
- it is safe to ignore the effects of B field on Γ_{CS} for CME estimates

Sphaleron rate with B field (holography)

$$B >> T^2$$

$$\Gamma_{CS} = \frac{(g^2 N)^2}{384\sqrt{3}\,\pi^4}\,\frac{B}{\pi}\;\times\;T^2$$

Sphaleron rate with B field (holography)

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$$\Gamma_{CS} = \frac{(g^2 N)^2}{384\sqrt{3}\,\pi^4} \underset{\text{Landau level density}}{\overset{\uparrow}{\frac{B}{\pi}}} \times T^2$$

Sphaleron rate with B field (holography)

$$B >> T^2$$

Landau level density

$$\Gamma_{CS} = \frac{(g^2 N)^2}{384\sqrt{3}\,\pi^4}\,\overset{\substack{\text{Landau level density}\\ \uparrow}}{\frac{B}{\pi}} \times \underset{\substack{\downarrow \\ \text{diffusion scale in 1+1d}}}{T^2}$$

diffusion scale in 1+1d

Conclusions

- Anomalies + magnetic field have a rich structure
- Electric and magnetic dipole moments
- Zero modes play a crucial role
- 1^{st} order derivative expansion captures some lattice results
- Confinement ? (instantons with nonzero holonomy), CSB ?
- At strong coupling:
  - Magnetic field always increases the sphaleron rate
  - Back-reaction of magnetic field into non-abelian sector
  - Strong magnetic field leads to dimensional reduction
- Weak coupling?