Derivation of transient relativistic fluid dynamics from the Boltzmann equation for a multi-component system

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Abstract

We derive the non-equilibrium single-particle momentum distribution function of a hadron resonance gas. We then study the effects that this newly derived expression can have in the freeze-out description of fluid-dynamical models of heavy ion collisions and compare it with the method traditionally employed, the 14-moment approximation.

1. Introduction

Fluid-dynamical models have been able to successfully describe the transverse momentum spectra and azimuthal transverse momentum anisotropies of particles observed in ultrarelativistic heavy ion collisions. One of the main ingredients of fluid-dynamical models is the so-called freeze-out procedure in which the fluid degrees of freedom are matched to kinetic or particle degrees of freedom. Since experiments at RHIC and LHC measure particles and not fluid elements, the freeze-out is an essential step towards the comparison with experimental data.

The Cooper-Frye formalism, usually applied to describe this matching process, demands the knowledge of microscopic information: the (single-particle) momentum distribution function of hadrons on the hypersurface in which freeze-out is performed (usually a constant temperature hypersurface). While this is well known for ideal fluids (Bose-Einstein or Fermi-Dirac distributions in the local rest frame of the fluid), it has remained an open problem for viscous fluids. We remark that, so far, Israel and Stewart's (IS) simple ansatz for *single* component systems is still being employed in most fluid-dynamical calculations.

In this case, the non-equilibrium correction to the momentum distribution of the i–th hadronic species , $\delta f_{\mathbf{k}}^{(i)}$, is assumed to be [1]

$$\delta f_{\mathbf{k}}^{(i)} = \varepsilon_{\mu\nu} k_i^{\mu} k_i^{\nu}, \tag{1}$$

where k_i^{μ} is the four-momentum of the corresponding hadron and $\varepsilon_{\mu\nu}$ is an expansion coefficient that should be matched to the fluid-dynamical variables.

The above expression has two distinct approximations: 1) the momentum dependence was assumed to be quadratic and 2) the coefficient $\varepsilon_{\mu\nu}$ was assumed to be the same for all hadronic species, being matched to the shear stress tensor, $\pi^{\mu\nu}$, as $\varepsilon_{\mu\nu} = \pi^{\mu\nu} / \left[2 \left(\varepsilon + P \right) T^2 \right]$. Here, ε, P, and T are the energy density, thermodynamic pressure, and temperature of the fluid. In principle, both these assumptions are incorrect. The limitations of the first assumption were already investigated in some works [2]. On the other hand, the second assumption and its domain of validity were just investigated in Ref. [3]. The purpose of this work is to further elaborate the studies regarding how $\delta f_{\mathbf{k}}^{(i)}$ actually depends on the different particle species.

2. Method of moments

We use the method of moments, as developed in Ref. [4], to compute the $\delta f_{\mathbf{k}}^{(i)}$ of a multi-component system without making any *a priori* assumption regarding its momentum dependence and the particle dependence of the expansion coefficients. First we factorize $\delta f_{\mathbf{k}}^{(i)}$ in the following way $\delta f_{\mathbf{k}}^{(i)} = f_{0\mathbf{k}}^{(i)} \tilde{f}_{0\mathbf{k}}^{(i)} \phi_{\mathbf{k}}^{(i)}$, where $f_{0\mathbf{k}}^{(i)}$ is the local equilibrium distribution function, $\tilde{f}_{0\mathbf{k}}^{(i)} = 1 + a \tilde{f}_{0\mathbf{k}}^{(i)}$ ($a = -1/1$ for fermions/bosons), and $\phi_{\mathbf{k}}^{(i)}$ is an out-of-equilibrium contribution. Next, we expand $\phi_{\mathbf{k}}^{(i)}$ in terms of its moments using a complete and orthogonal basis constructed from particle four-momentum, k_i^{μ}, and fluid four-velocity, u^{μ}. As done in Ref. [4], we use an expansion basis with two basic ingredients: 1) the irreducible tensors $1, k_i^{\langle\mu\rangle}, k_i^{\langle\mu} k_i^{\nu\rangle}, k_i^{\langle\mu} k_i^{\nu} k_i^{\lambda\rangle}, \cdots$, analogous to the well-known set of spherical harmonics and constructed by the symmetrized traceless projection of $k_i^{\mu_1} \cdots k_i^{\mu_m}$, i.e., $k_i^{\langle\mu_1} \cdots k_i^{\mu_m\rangle} \equiv \Delta^{\mu_1 \cdots \mu_m}_{\nu_1 \cdots \nu_m} k_i^{\nu_1} \cdots k_i^{\nu_m}$, and 2) the orthonormal polynomials $P_{i\mathbf{k}}^{(n\ell)} = \sum_{r=0}^{n} a_{nr}^{(\ell)i} \left(u_{\mu} k_i^{\mu}\right)^r$, which are equivalent to the associated Laguerre polynomials in the limit of massless, classical particles [4].

Then, the distribution function becomes,

$$f_{\mathbf{k}}^{(i)} = f_{0\mathbf{k}}^{(i)} + f_{0\mathbf{k}}^{(i)} \tilde{f}_{0\mathbf{k}}^{(i)} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{H}_{i\mathbf{k}}^{(n\ell)} \rho_{i,n}^{\mu_1 \cdots \mu_\ell} k_{i,\mu_1} \cdots k_{i,\mu_\ell}, \tag{2}$$

where we introduced the energy-dependent coefficients, $\mathcal{H}_{i\mathbf{p}}^{(n\ell)} \equiv \left[N_i^{(\ell)}/\ell!\right] \sum_{m=n}^{\infty} a_{mn}^{(\ell)i} P_{i\mathbf{k}}^{(m\ell)} \left(u_{\mu} k_i^{\mu}\right)$. The fields $\rho_{i,n}^{\mu_1 \cdots \mu_\ell}$ can be determined exactly using the orthogonality relations satisfied by the expansion basis and can be shown to correspond to the irreducible moments of $\delta f_{\mathbf{k}}^{(i)}$,

$$\rho_{i,r}^{\mu_1 \ldots \mu_\ell} \equiv \left\langle E_{i\mathbf{k}}^r k_i^{\langle\mu_1} \ldots k_i^{\mu_\ell\rangle} \right\rangle_{\delta}, \quad \langle \ldots \rangle_{\delta} = \int dK_i \ (\ldots) \, \delta f_{\mathbf{k}}^{(i)}, \tag{3}$$

where g_i is the degeneracy factor of the i–th hadron species and $dK_i = g_i d^3\mathbf{k}/\left[(2\pi)^3 k_i^0\right]$. As long as this basis is complete, the above expansion fully describes $f_{\mathbf{k}}^{(i)}$, no matter how far from equilibrium the system is.

Here, we are interested only on the effects arising from the shear-stress tensor. For this case, it is enough to fix $\ell = 2$ (take only irreducible second-rank tensors) in the expansion above, i.e., neglect all scalar terms, e.g. bulk viscous pressure, irreducible first-rank tensors, e.g. heat flow, and tensors with rank higher than two (that never appear in fluid dynamics). The next approximation is the truncation of the expansion in momentum space, keeping only the term with $n = 0$ (with $\ell = 2$ already fixed). Then, we obtain (for classical particles)

$$f_{\mathbf{k}}^{(i)} = f_{0\mathbf{k}}^{(i)} + \frac{f_{0\mathbf{k}}^{(i)}}{2\left(\varepsilon_i + P_i\right) T^2} \pi_i^{\mu\nu} k_{i,\mu} k_{i,\nu} \ . \tag{4}$$

Above, $\pi_i^{\mu\nu} = \rho_{i,0}^{\mu\nu}$, ε_i, and P_i are the shear-stress tensor, the energy density, and the thermodynamic pressure *of the i–th particle species*, respectively. Note that by keeping only the term with

$n = 0$ (for $\ell = 2$) we have the same momentum dependence obtained in the IS formula. However, our expansion coefficients are not independent of the particle type. In this formalism, this did not have to be assumed, but appeared naturally as a consequence of the orthogonality relations satisfied by the basis.

In order to apply this expression to describe freeze-out, further approximations are required. This happens because in fluid dynamics we only evolve the total shear-stress tensor of the system ($\pi^{\mu\nu} = \sum_i \pi_i^{\mu\nu}$) and do not know, from fluid dynamics itself, how it divides into the individual shear-stress tensors of each hadron species ($\pi_i^{\mu\nu}$). In order to extract this information, it is mandatory to know how transient relativistic fluid dynamics emerges from the underlying microscopic theory.

Basically, transient relativistic fluid dynamics is derived as the long, but not asymptotically long, time limit of the Boltzmann equation. In this limit, it is possible to relate the shear-stress tensor of individual particle species with the total shear stress tensor, $\pi^{\mu\nu}$, and the shear tensor, $\sigma^{\mu\nu} = \partial^{\langle\mu} u^{\nu\rangle}$, of the system. This was done in Ref. [4], for single component systems, and can be easily extended to multi-component systems. Here, we just write down the solution and leave the detailed derivation of this relation to a future work. The solution is,

$$\pi_i^{\mu\nu} = \frac{\Omega^{i0}}{\sum_j \Omega^{j0}} \pi^{\mu\nu} + 2 \left[\eta_i - \eta \frac{\Omega^{i0}}{\sum_j \Omega^{j0}} \right] \sigma^{\mu\nu} \equiv \alpha_i \pi^{\mu\nu} + \beta_i \sigma^{\mu\nu}, \tag{5}$$

where η_i is the shear viscosity of the i–th species, while $\eta = \sum_i \eta_i$ is the total shear viscosity. The matrices Ω are defined in such a way as to diagonalize the collision integral $\mathcal{M}$, $\Omega^{-1}\mathcal{M}\Omega = \mathrm{diag}(\chi_1, \cdots, \chi_N)$, with χ_i being the eigenvalues of $\mathcal{M}$. With the truncation in momentum assumed above and considering only elastic 2-to-2 collisions between *classical* particles, $\mathcal{M}$ has the following simple form

$$\mathcal{M}^{ij} = \mathcal{A}^i \delta^{ij} + C^{ij}, \tag{6}$$

$$\mathcal{A}^i = \frac{1}{5} \sum_{j=1}^{N_{\mathrm{hadr.}}} \int dK_i dK'_j dP_i dP'_j \gamma_{ij} W^{ij}_{\mathbf{pp'}-\mathbf{kk'}} f^{(0)}_{i\mathbf{k}} f^{(0)}_{j\mathbf{k'}} \left(u_\lambda k_i^\lambda\right)^{-1} k_i^{\langle\mu} k_i^{\nu\rangle} \left[p_{i\langle\mu} p_{i\,\nu\rangle} - k_{i\langle\mu} k_{i\,\nu\rangle} \right], \tag{7}$$

$$C^{ij} = \frac{1}{5} \int dK_i dK'_j dP_i dP'_j \gamma_{ij} W^{ij}_{\mathbf{pp'}-\mathbf{kk'}} f^{(0)}_{i\mathbf{k}} f^{(0)}_{j\mathbf{k'}} \left(u_\lambda k_i^\lambda\right)^{-1} k_i^{\langle\mu} k_i^{\nu\rangle} \left[p'_{j\langle\mu} p'_{j\,\nu\rangle} - k'_{j\langle\mu} k'_{j\,\nu\rangle} \right], \tag{8}$$

where $\gamma^{ij} = 1 - (1/2)\,\delta^{ij}$ and $W^{ij}_{\mathbf{pp'}-\mathbf{kk'}}$ is the transition rate.

3. Simple model of hadrons

In order to compute the coefficients α_i and β_i appearing in Eq. 5 we must provide a set of hadronic cross sections. In this work, we estimate these coefficients for the first time using a simple hadronic model, in which all hadrons have the same constant cross section, $\sigma_{ij} = 30$ mb. We consider only elastic collisions between the hadrons and include all hadrons up to a mass of 1.8 GeV. In this case, α_i and β_i are actually independent of the value chosen for the cross section and all the difference between the various hadrons species originate solely from their different masses. In this simple scheme, we computed α_i and β_i for all hadron species up to 1.8 GeV and included all the decays of unstable resonances.

In order to probe the effect this more sophisticated $\delta f^{(i)}_{\mathbf{k}}$ can have on heavy-ion observables, we computed the elliptic flow of pions, kaons, and protons using the fluid-dynamical model applied in Ref. [5], with exactly the same parameters (GLmix initialization, HH-HQ shear viscosity parametrization), but considering several choices of $\delta f_{\mathbf{k}}^{(i)}$. The resutls are shown in Figs.1. The different lines correspond to the following cases: 1) solid line uses Israel-Stewart's ansatz, 2) dotted line uses the $\delta f_{\mathbf{k}}^{(i)}$ derived in this paper, and 3) dashed line uses the $\delta f_{\mathbf{k}}^{(i)}$ derived in this paper in the Navier-Stokes limit ($\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$).

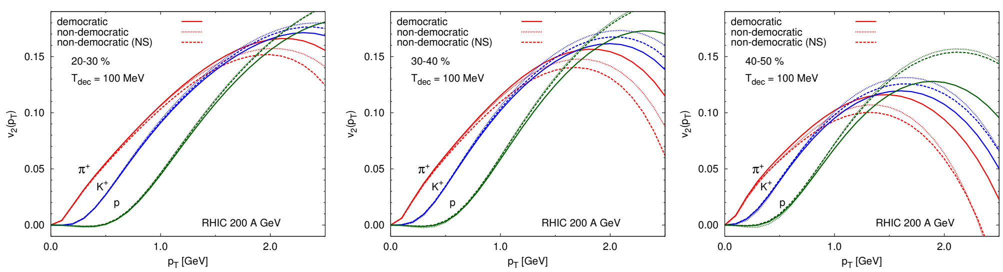

Figure 1: Differential elliptic flow for pions, kaons, and protons for three different centrality classes, (a) $20-30$ %, (b) $30-40$ %, (c) $40-50$ %, and various choices of $\delta f_{\mathbf{k}}^{(i)}$.

We see that the particle dependence of the coefficients do not have a strong effect on the differential elliptic flow, which looks very similar to the one computed using Israel-Stewart's ansatz. At very non-central collisions, the difference is larger, but even then, cannot be considered as a crucial effect. Note that one of the reasons that makes this difference small is that we are not in the Navier-Stokes limit at freeze-out: the elliptic flow computed assuming the Navier-Stokes limit actually deviates more strongly from the one computed using Israel-Stewart's ansatz. Furthermore, note that there is a qualitative particle dependence in our result, the elliptic flow of pions is below the one computed with Israel-Stewart's ansatz while the elliptic flow of kaons and protons are always above.

We remark that these results arise from an over simplified hadronic description and this insensitivity to the particle species might be coming just from the simple choice of hadronic cross sections assumed. In future calculations, we will come back to this discussion using more realistic cross sections for the hadrons and also including inelastic collisions between the hadrons as well.

The work of H.N. was supported by the Academy of Finland, Project No. 133005. G.S.D thanks D. Molnar for helpful discussions.

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