Introduction to Lattice QCD -- Monte Carlo Method and Lattice Field Theories --

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２０１２, Oct. 11. Wuhan

Plan

- Monte Carlo Method and Lattice QCD
- Gauge Transformation on the Lattice
- Hadrons on Lattice
- Lattice Tool Kit in Fortran 90 (LTKf90)

This is a Talk for Students, and non-lattice experts such as Experimentalists.

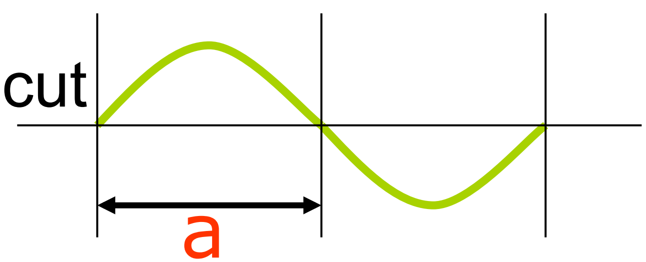
Lattice QCD

- Path Integral in Euclid space (Imaginary time)

$$Z = \int DU D\overline{\psi} D\psi \, e^{-(S_G + \overline{\psi}\Delta\psi)} = \int DU \det\Delta \, e^{-S_G}$$

  - U:Gluon Fields、ψ:Quark Fields

- Quantam fluctuation of Gauge field（Gluon field）
  ⇨ Monte Carlo
- Fermion (Quark) propagators
  ⇨ Linear algebra（Inverse of DeterminantΔ）
- Lattice
  ⇨ Ultra-violet cut



$$p_{\max} = \frac{\pi}{a}$$

Link variables U

$$Z = \int DUD\overline{\psi}D\psi\, e^{-(S_G + \overline{\psi}\Delta\psi)} = \int DU \det\Delta\, e^{-S_G}$$

$$U_\mu(x) = e^{iA_\mu(x)}$$

μ=x,y,z,t or 1,2,3,4

$$x = (x, y, z, t)$$

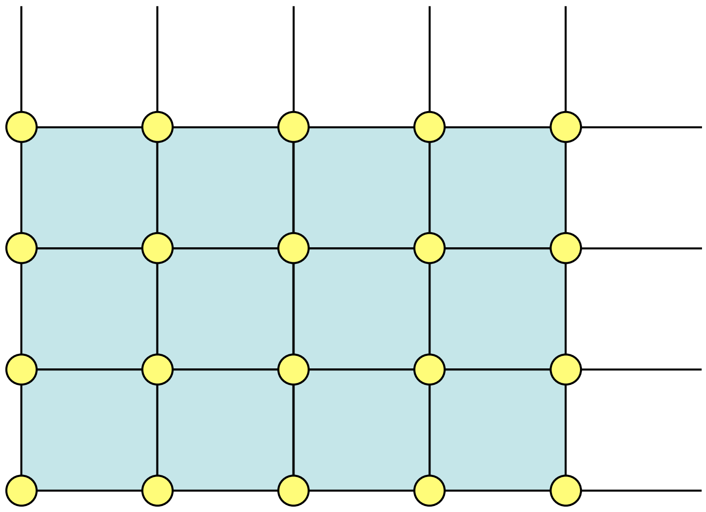
$$= (x_1, x_2, x_3, x_4)$$

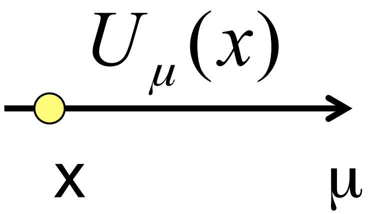
$$x_1 = 1, 2, \cdots, N_x$$

$$x_2 = 1, 2, \cdots, N_y$$

$$x_3 = 1, 2, \cdots, N_z$$

$$x_4 = 1, 2, \cdots, N_t$$



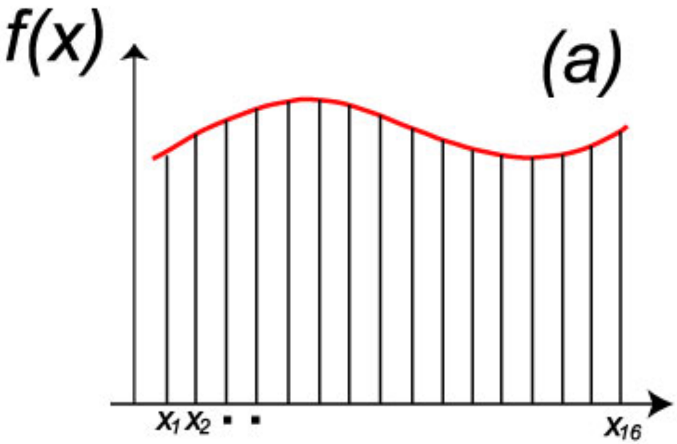


$$\int DU = \int \prod_{\mu=1,2,3,4} \prod_{x_1=1}^{N_x} \prod_{x_2=1}^{N_y} \prod_{x_3=1}^{N_z} \prod_{x_4=1}^{N_t} dU_\mu(x_1, x_2, x_3, x_4)$$

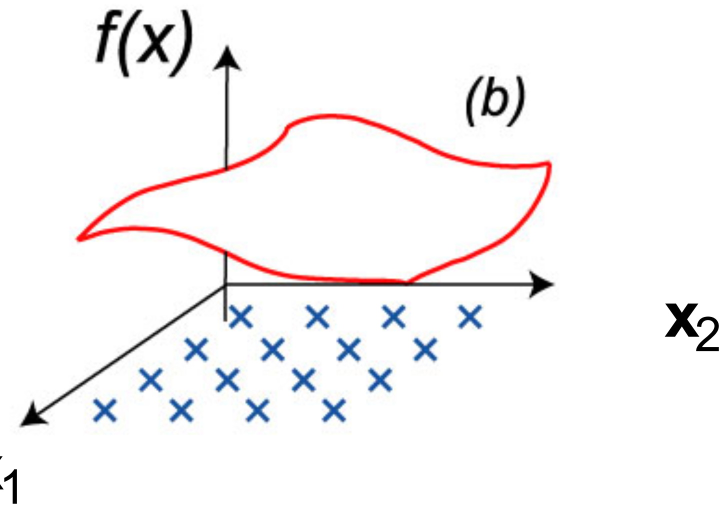
Integral in many high dimensions

Integral in high dimension and Monte Carlo method

$$I = \int f(x) dx_1 dx_2 dx_3 \cdots dx_n$$



1-dimension



2-dimension

Errors in Numerical Integration

$$\text{Error} \propto \frac{1}{\text{The number of points along a direction}} = \frac{1}{N^{1/n}}$$

N: The number of total points

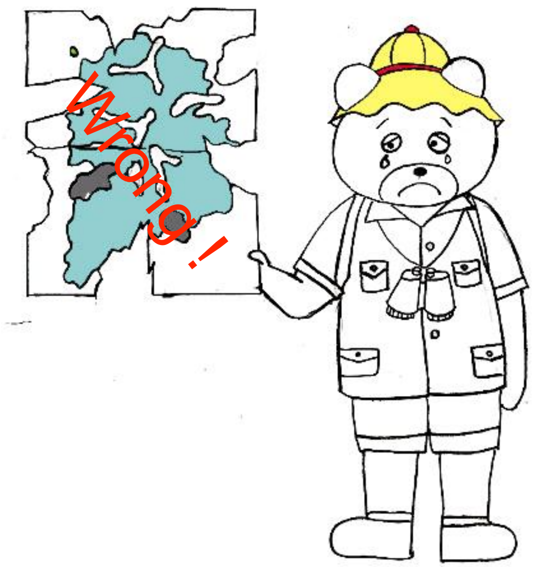
CPU Time is proportional to N

N=1000, n=10 $N^{1/n} = 1.99526$

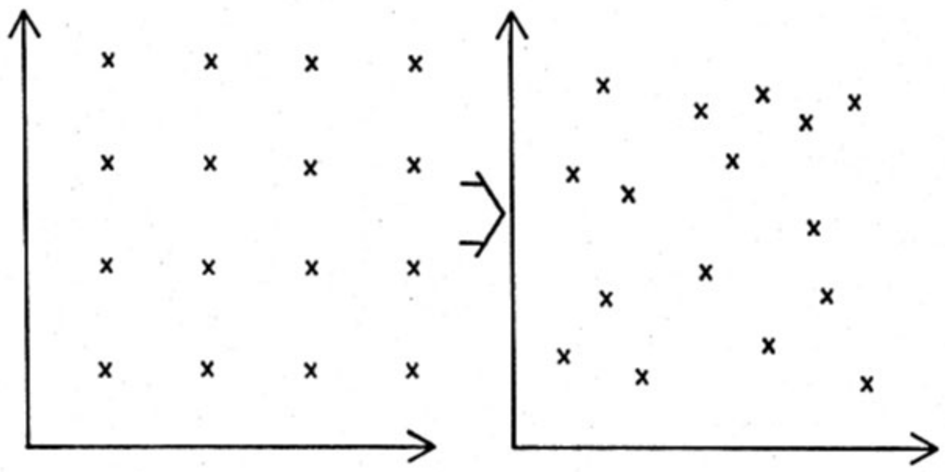
In case of Lattice QCD

$$n = 4N_x N_y N_z N_t \times 8$$

Standard numerical integral method does not work.



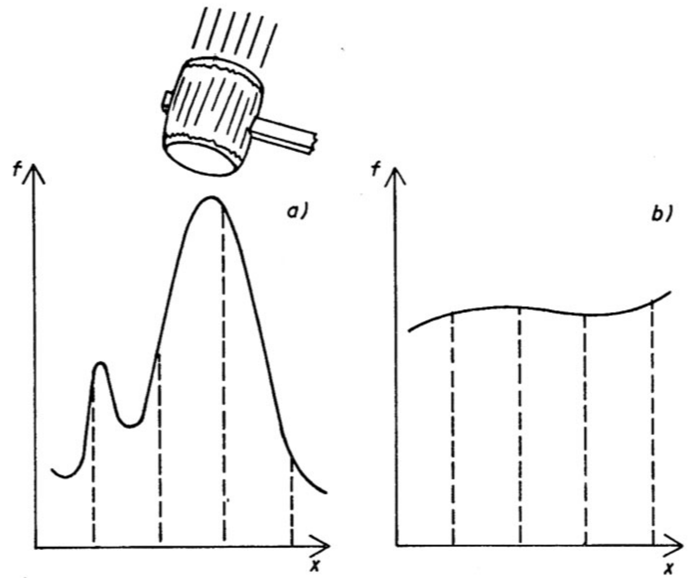
Error in Monte Carlo method



$$\text{Error} \sim \frac{1}{\sqrt{N}}$$

Independent of n !

Importance Sampling



If an Integrand is flat, it it easy to integrate in numerically.

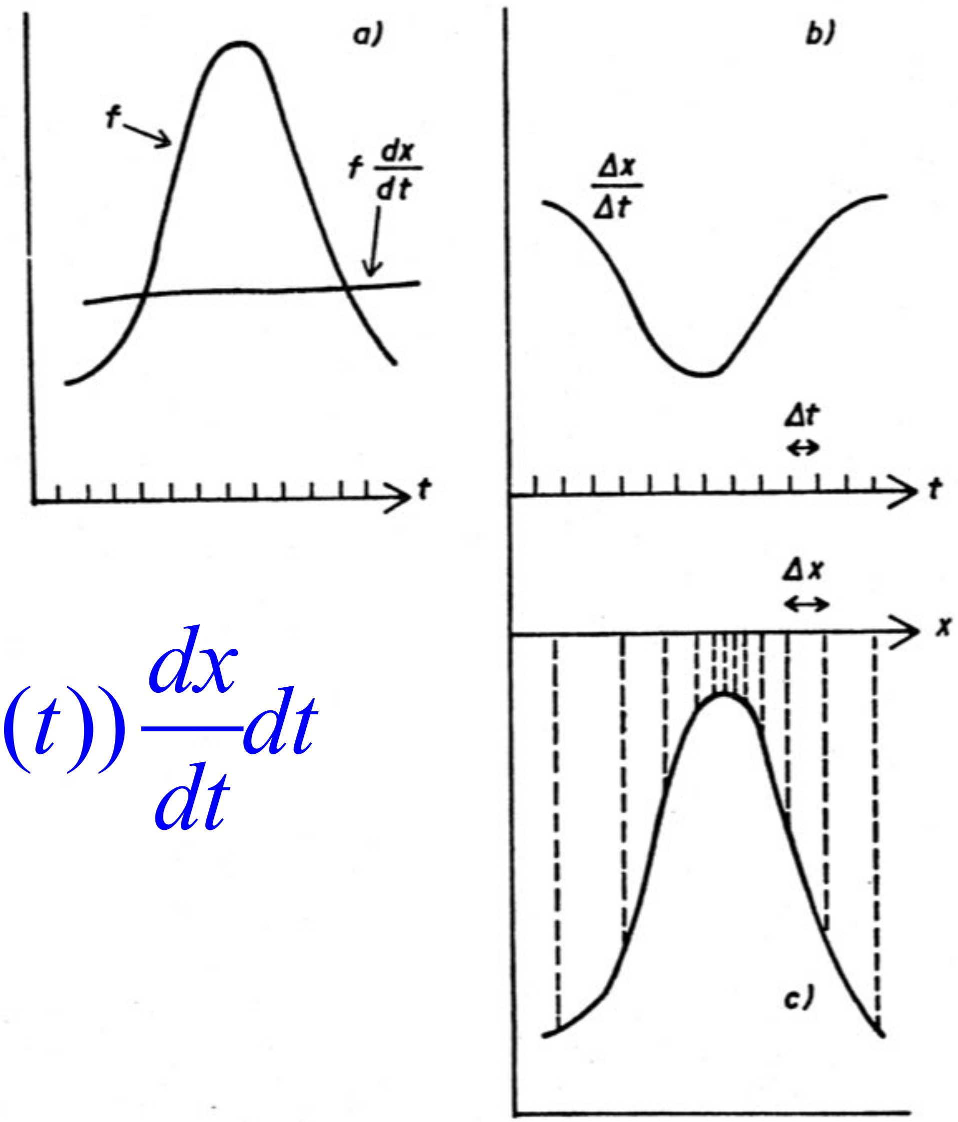
Change variable x ⇒ t

so that $$\frac{dx}{dt} : \frac{1}{f}$$

$$I = \int f(x)dx = \int f(x(t))\frac{dx}{dt}dt$$

Almost flat

Importance Sampling (2)

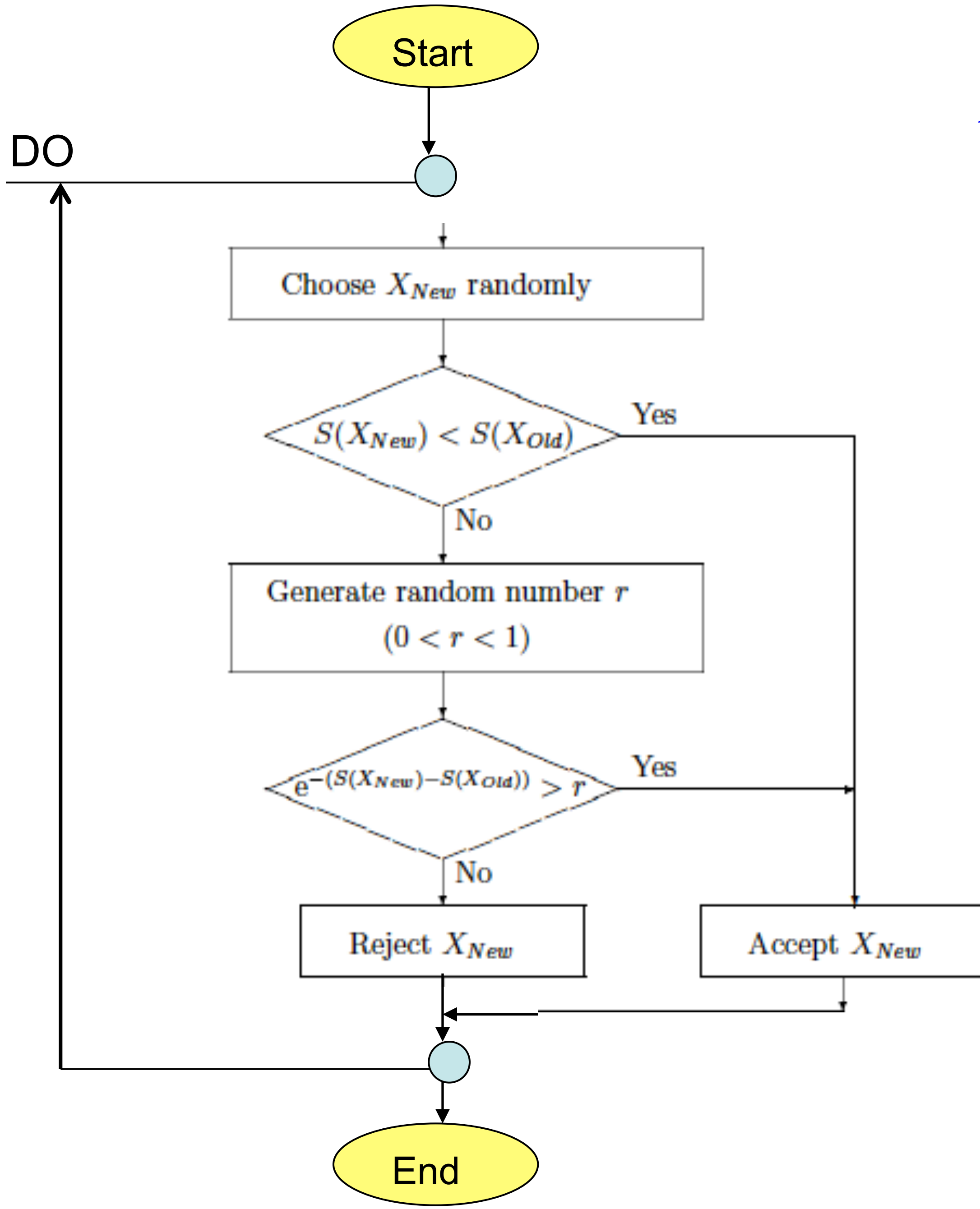


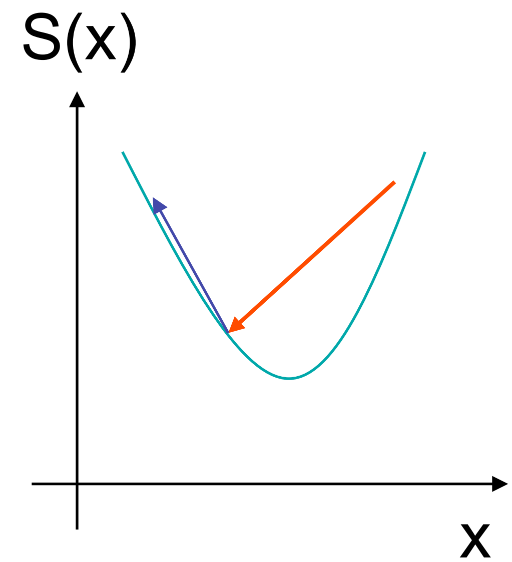
$$I = \int f(x(t)) \frac{dx}{dt} dt$$

Metropolis Algorithm

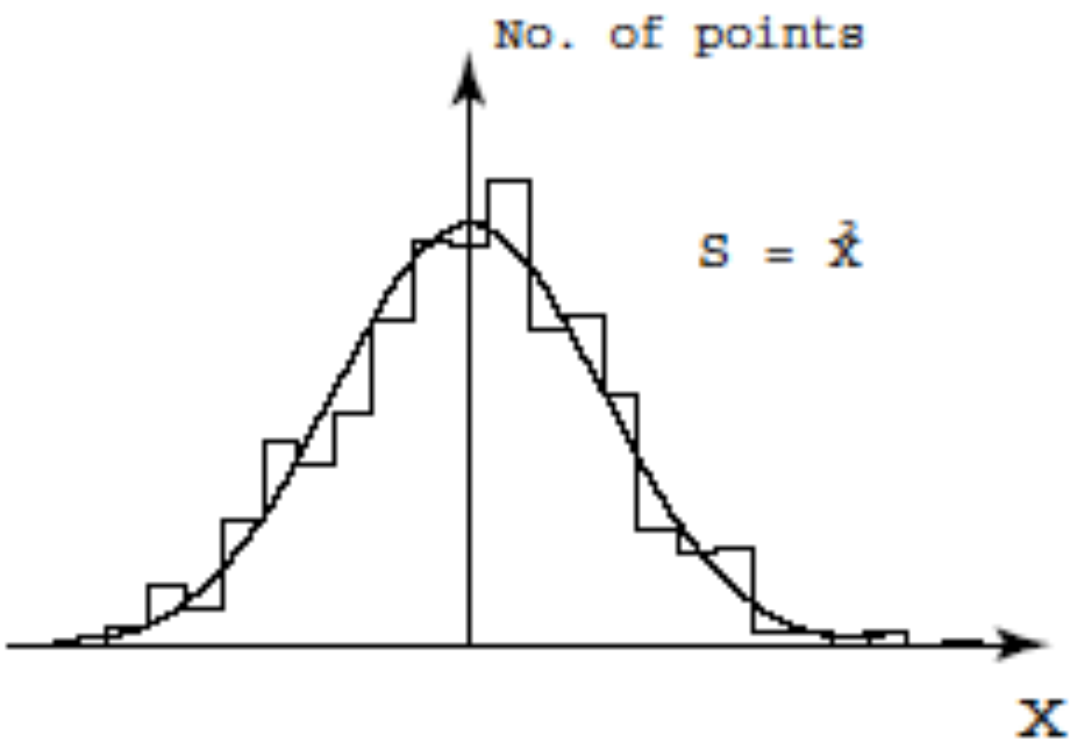
- Importance Sampling + Random Sampling
  ⇒
  Monte Carlo method in many-dimension
- Is it possible ?
  Yes !
  - N.Metropolis et al.
    J. Chem. Phys. 21, 1087 (1953)
    Very readable paper for physicists.

$$I = \int e^{-S(x)} dx$$


Start
DO
Choose X_{New} randomly
$S(X_{New}) < S(X_{Old})$
Yes
No
Generate random number r
$(0 < r < 1)$
$e^{-(S(X_{New})-S(X_{Old}))} > r$
Yes
No
Reject X_{New}
Accept X_{New}
End


S(x)
x

$$I = \int e^{-S(x)} dx = \int e^{-x^2} dx$$



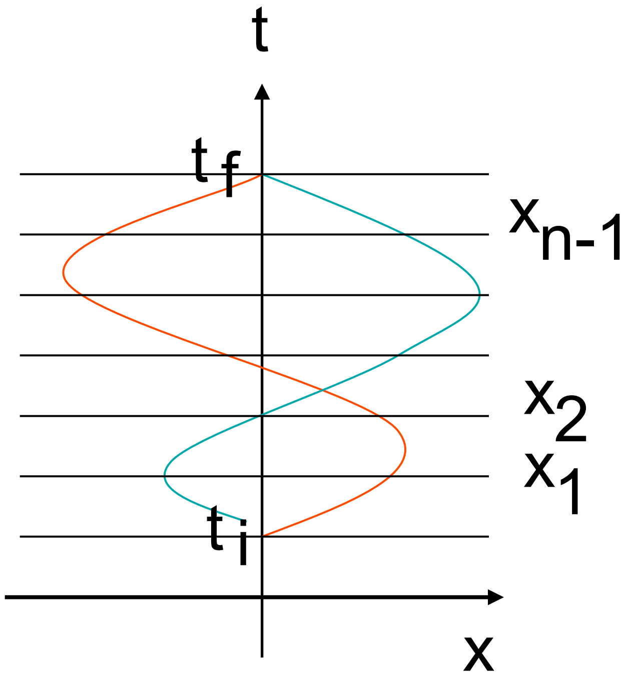
Quantum Mechanics in 1-Dimension

(Creutz & Freedman, Ann. Phys. 32 427,1981)

$$Z = \int Dx e^{\frac{i}{\hbar}\int dt L},$$

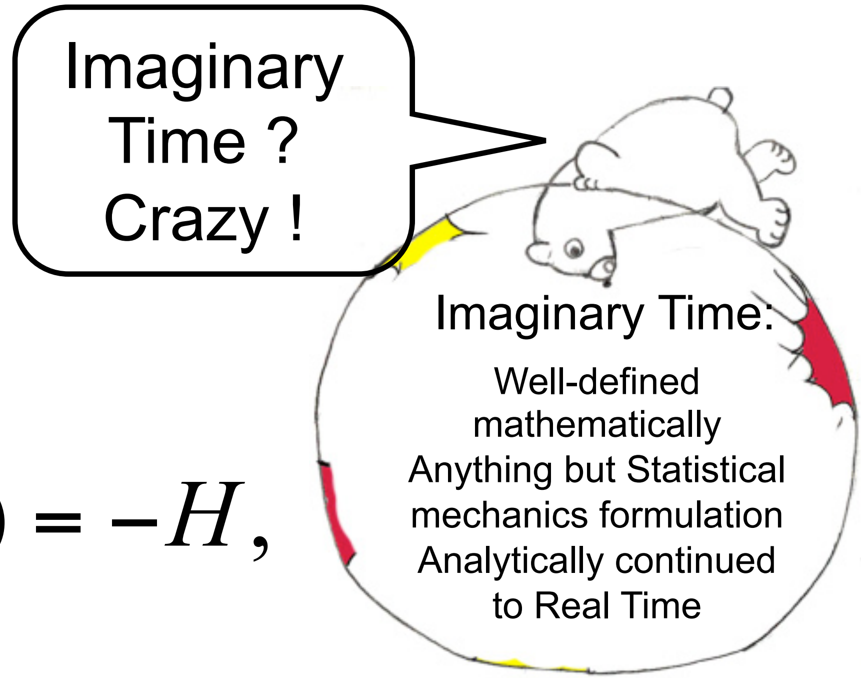
$$L = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x),$$

$$Dx = \lim_{n \to \infty} dx_1 dx_2 \cdots dx_n$$



$$x_0 \equiv x(t_i), x_1 \equiv x(t_1), x_2 \equiv x(t_2), \ldots, x_{n-1} \equiv x(t_{n-1}), x_n \equiv x(t_f)$$

Euclidean World (Imaginary Time)



$$t \to -i\tau,$$

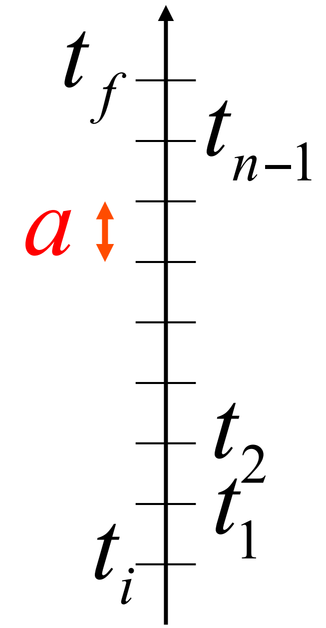
$$L \to -\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 - V(x) = -H,$$

$$Z \to \int Dx e^{\frac{i}{\hbar}\int(-id\tau)(-H)} = \int Dx e^{-\frac{1}{\hbar}\int d\tau H} = \int Dx e^{-\frac{1}{\hbar}S}$$

Discretize

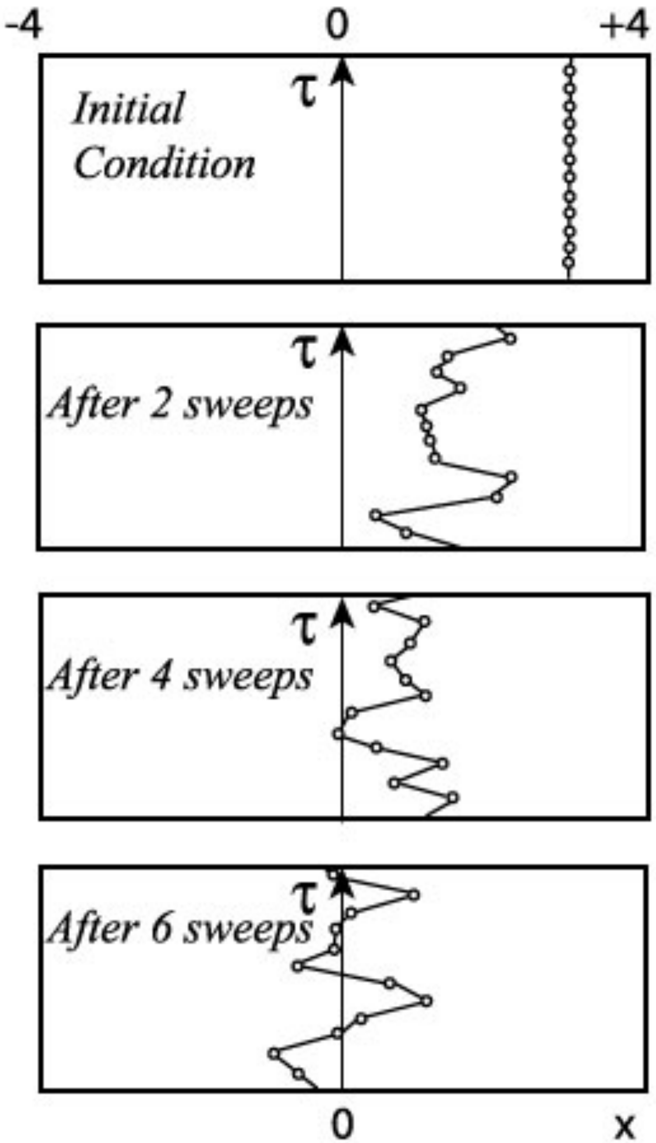
$$Z \sim \int dx_1 dx_2 \cdots dx_{n-1} e^{-S/\hbar},$$

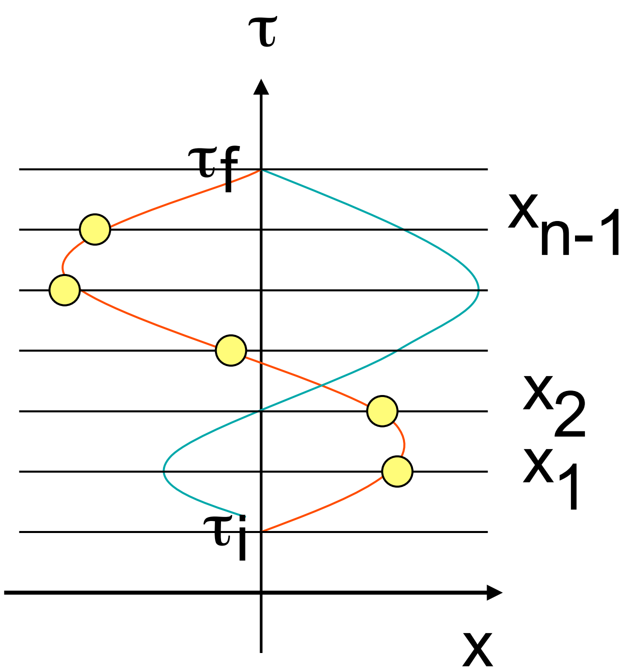
$$S = \sum_j a \left[\frac{m}{2} \left(\frac{x_{j+1} - x_j}{a} \right)^2 + V(x_j) \right]$$

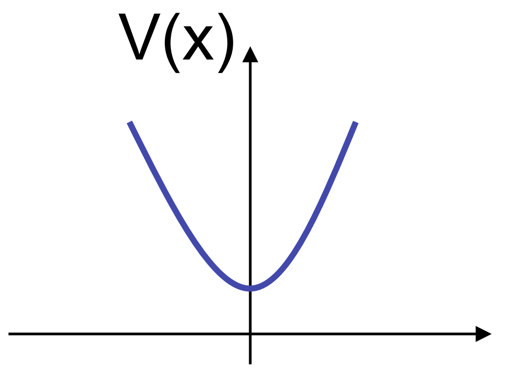


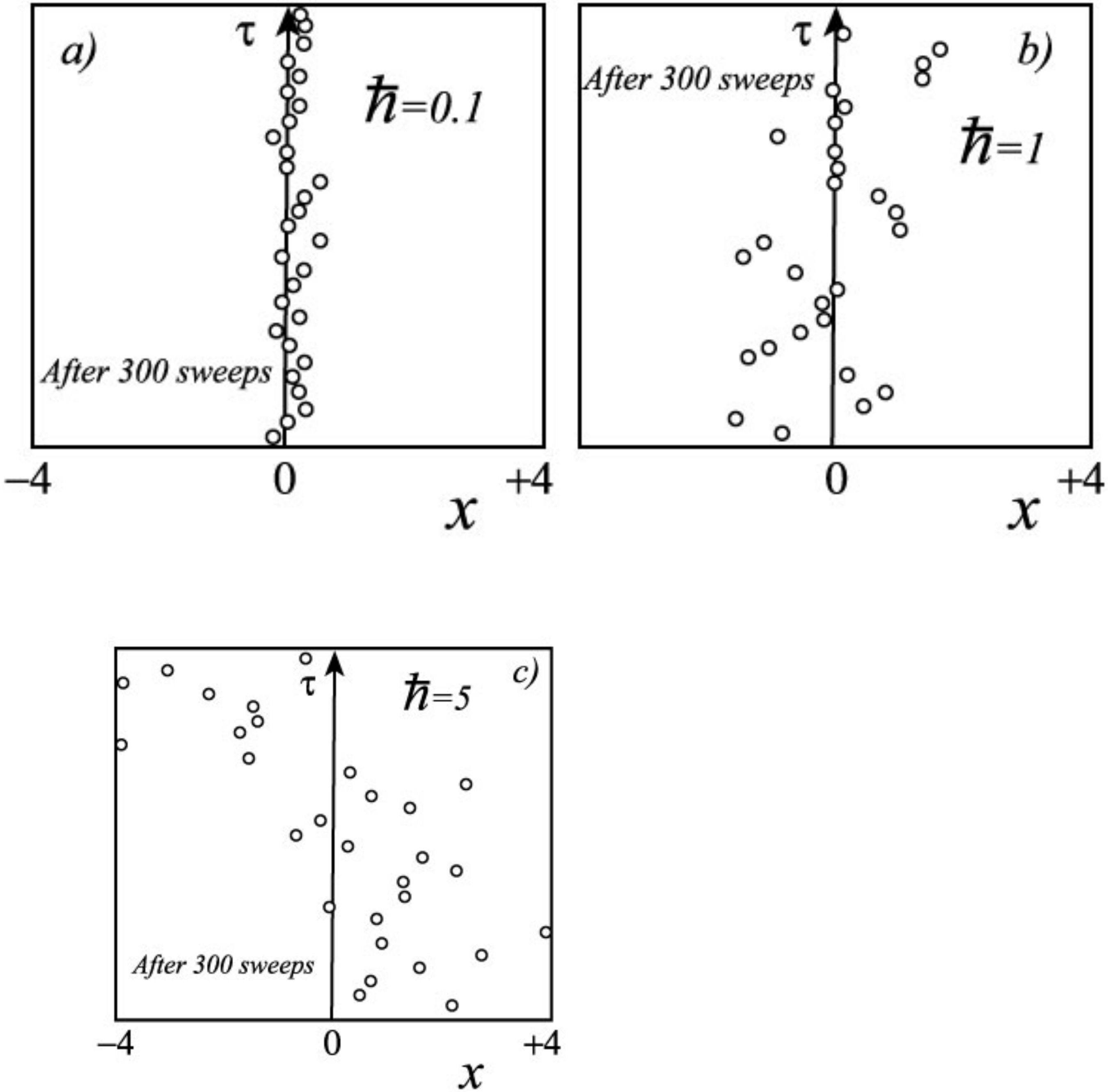
$$x_0 \equiv x(t_i), x_1 \equiv x(t_1), x_2 \equiv x(t_2), \ldots, x_{n-1} \equiv x(t_{n-1}), x_n \equiv x(t_f)$$

Simulation Results

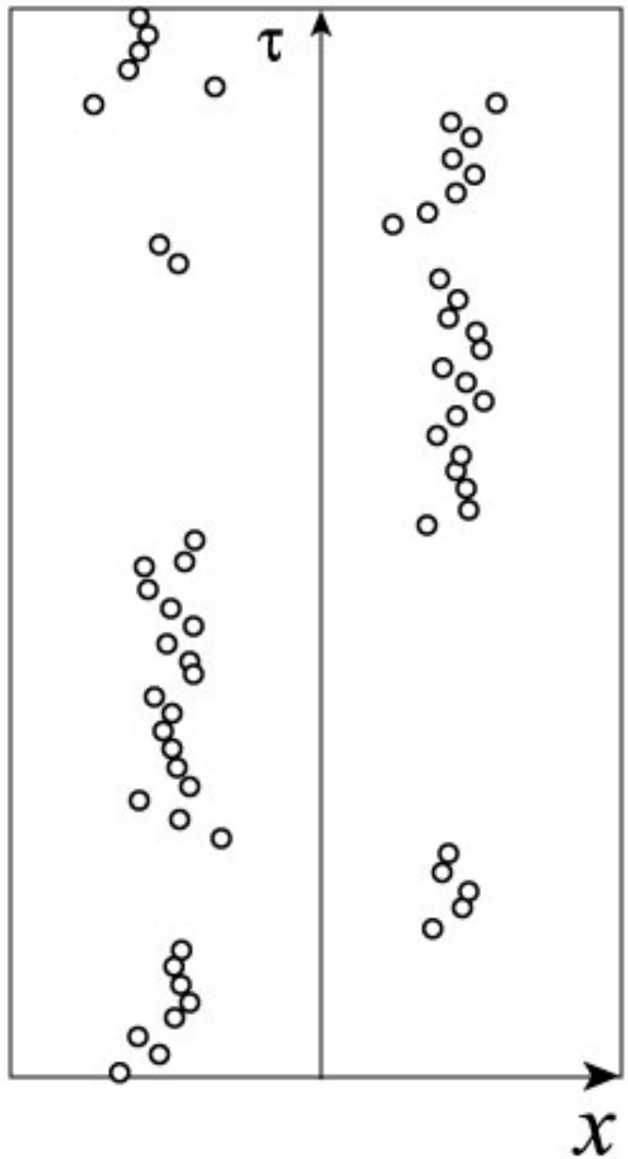


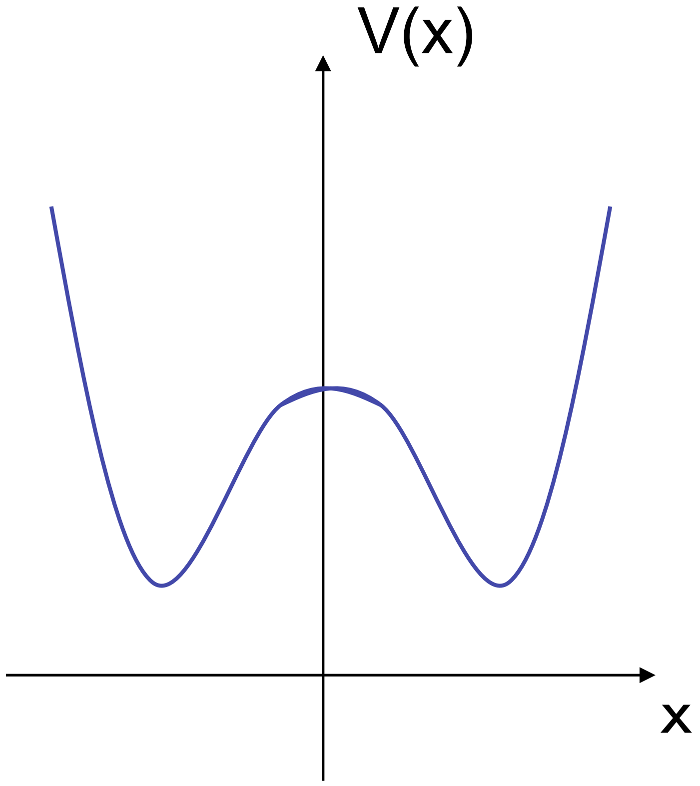




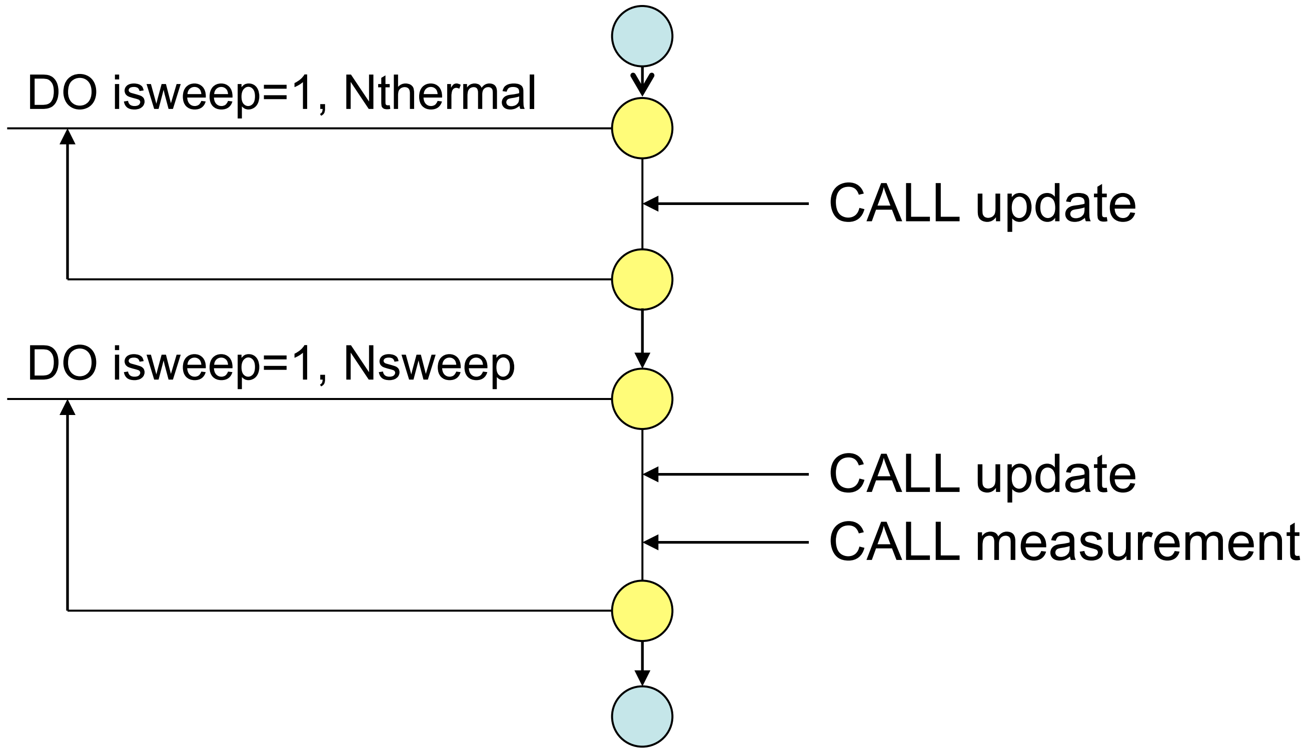

a)
τ
ħ=0.1
After 300 sweeps
–4
0
+4
x
b)
τ
After 300 sweeps
ħ=1
0
+4
x
c)
τ
ħ=5
After 300 sweeps
–4
0
+4
x

Anharmonic Oscillator

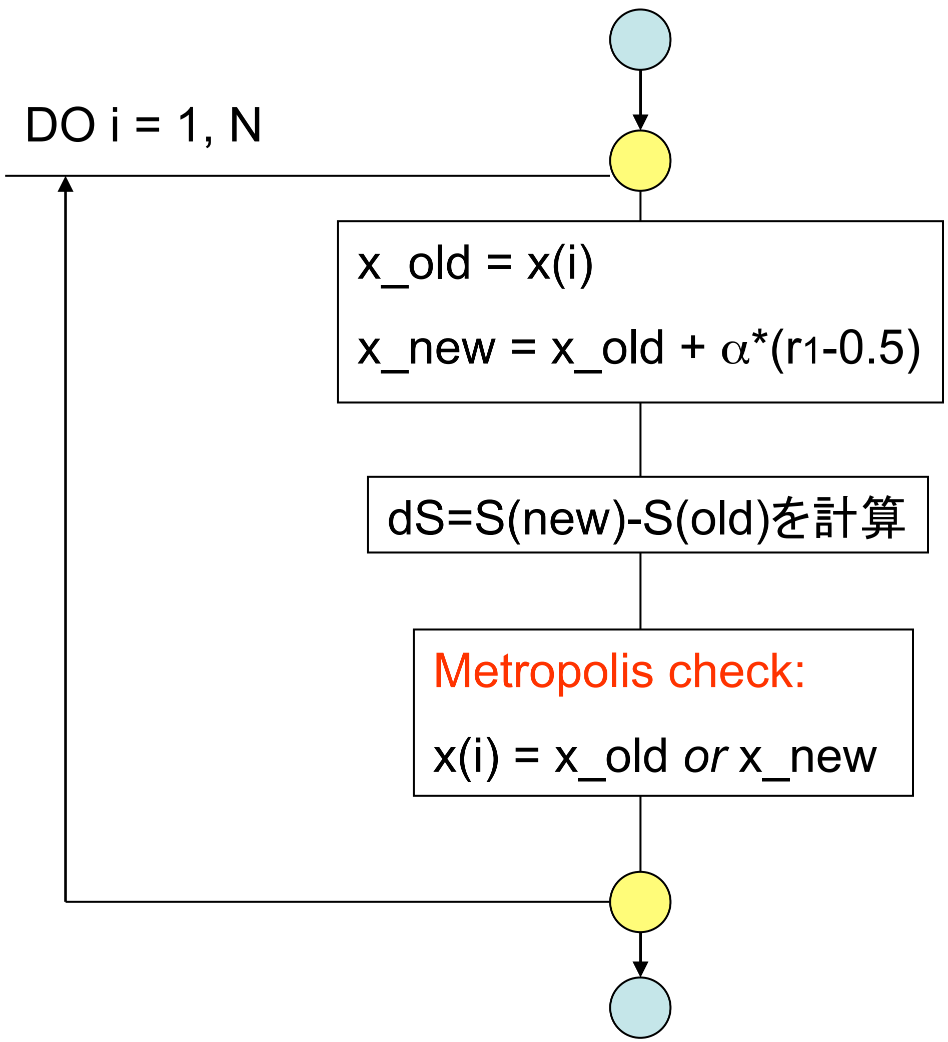




Flow-Chart
(1) MAIN



Flow Chart
（2）update



Boundary Conditions

- Periodic: $x(N+1)=x(1),\ x(0)=x(N)$
- Anti-Periodic: $x(N+1)=-x(1),\ x(0)=-x(N)$

```
DO i = 1, N
  ia = i + 1
  ib = i  - 1
  IF( i==N ) ia = 1
  IF( i==1 )  ib = N
  xa = x(ia)
  xb = x(ib)
  . . .
```

```
REAL, DIMENSION(0:N+1) :: x
x(0) = x(N)
x(N+1) = x(1)
DO i = 1, N
  xa = x(i+1)
  xb = x(i -1)
  . . .
```

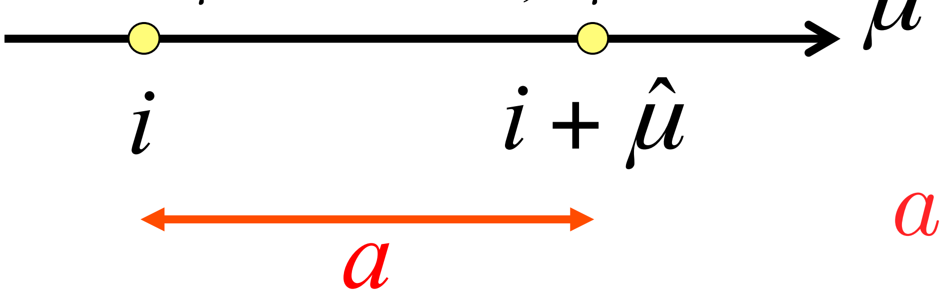
How to treat the boundary conditions

```
INTEGER,  DIMENSION(N,2) :: inn

DO i = 1, N
  xa = x(inn(i,1))
  xb = x(inn(i,2))
  . . .
```

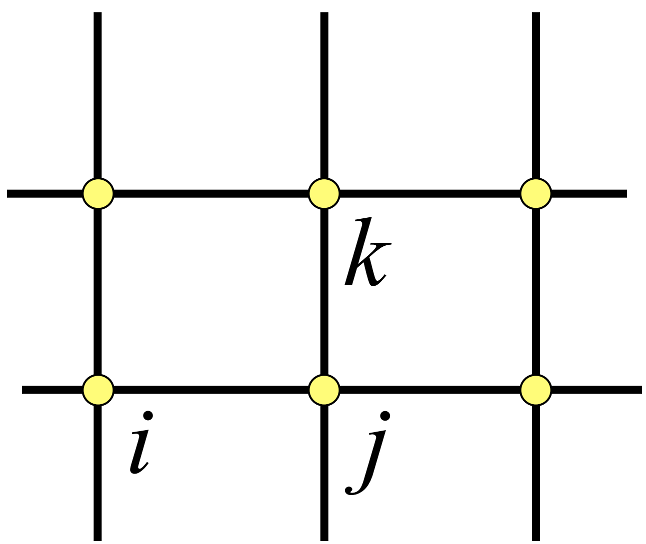
```
SUBROUTINE MakeTable

DO i = 1, N
  ia = i + 1;  ib = i – 1
  IF( i==N ) ia = 1
  IF( i==1  ) ib = N
  inn(i,1) = ia
  inn(I,2) = ib
ENDDO
RETURN
END
```

Lattice QCD Lagrangian (Preparation)

$$U_\mu(i) = U_{i,i+\hat{\mu}}$$



a : lattice spacing



$$U_{i,j}U_{j,k}$$

$$\bar{\psi}_i U_{i,j} \psi_j$$

$$U_{j,i} = U_{i,j}^{\dagger}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

$$U^\dagger = {}^t U^*$$

$$UU^\dagger = I \qquad \det UU^\dagger = \det U (\det U)^*$$

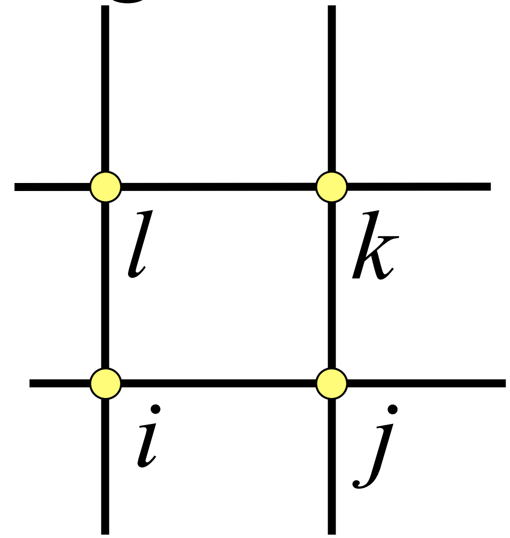
$$\det U = 1$$

$$U = e^{iA}$$

$$A^\dagger = A,$$

$$\det U = e^{Tr \log U} = e^{iTrA} = 1$$

Lattice QCD Lagrangian



- K.G.Wilson
  - Phys. Rev. D10, 2445 (1974)
  - Erice Lecture Note 1977

$$S = S_G + S_F$$

$$S_G = \beta \sum_{plaquette} \left\{ 1 - \frac{1}{N_c} Tr(U_{ij} U_{jk} U_{kl} U_{li}) \right\}$$

$$\beta \equiv \frac{2N_c}{g^2} \qquad U_{i,j} \in SU(N_c)$$

Exercise : How many plaqeuts exist on a lattice of size NxNyNzNt ?

Fermion (Quark) Action

$$S_F = \sum_{i,j} \bar{\psi}_i \Delta(i,j) \psi_j$$

$$\Delta(i,j) = I - \kappa \sum_{\mu=1}^{4} \left\{ (1-\gamma_\mu) U_{i,j} \delta_{i+\hat{\mu},j} + (1+\gamma_\mu) U_{i,j} \delta_{i-\hat{\mu},j} \right\}$$

$$\Delta^{ab}_{\alpha,\beta}(i,j) = \delta_{\alpha\beta}\delta_{ab}\delta_{ij} - \kappa \sum_{\mu=1}^{4} \left\{ (1-\gamma_\mu)_{\alpha\beta} U_{i,j}{}^{ab} \delta_{i+\hat{\mu},j} + (1+\gamma_\mu)_{\alpha\beta} U_{i,j}{}^{ab} \delta_{i-\hat{\mu},j} \right\}$$

κ : hopping parameter

(Classical) Continuum limit

$$U_\mu(n) = e^{igaA_\mu(na)}$$

$$\psi_n = \sqrt{\frac{a^3}{2\kappa}}\psi(na)$$

$$\lim_{a\to 0} S_G = \frac{1}{2}\int d^4x Tr\left\{F_{\mu\nu}{}^2\right\}$$

$$\lim_{a\to 0} S_F = -\int d^4x\left\{m\overline{\psi}(x)\psi(x) + \overline{\psi}(x)\gamma_\mu(\partial_\mu + igA_\mu(x))\psi(x)\right\}$$

Warming up : U(1) case

$$P_{\mu\nu}(x) \equiv U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu{}^\dagger(x+\hat{\nu}) U_\nu{}^\dagger(x)$$

$$= e^{iagA_\mu(x)} e^{iagA_\nu(x+\hat{\mu})} e^{-iagA_\mu(x+\hat{\nu})} e^{-iagA_\nu(x)}$$

$$= e^{ia^2 g\left(\frac{A_\nu(x+\hat{\mu})-A_\nu(x)}{a} - \frac{A_\mu(x+\hat{\nu})-A_\mu(x)}{a}\right)}$$

$$= e^{ia^2 g F_{\mu\nu}(x)} = 1 + ia^2 g F_{\mu\nu} - \frac{1}{2} a^4 g^2 F_{\mu\nu}^2 + \ldots.$$

$$\sum_{plaquette} P_{\mu\nu}(x) = \sum_x \left(1 - \frac{1}{2} a^4 g^2 F_{\mu\nu}^2 \ .\right)$$

Formulae we need

$$e^X e^Y = e^F$$

$$F = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \ldots$$

$$f(x + \hat{\mu}) = f(x) + a\partial_\mu f(x) + O(a^2)$$

$$\kappa = \frac{1}{8 + 2ma} \qquad \psi_n = \sqrt{\frac{a^3}{2\kappa}}\psi(na)$$

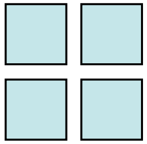
Exercise

- Show that S_G and S_F become a standard dontinuum QCD action when $a \to 0$.

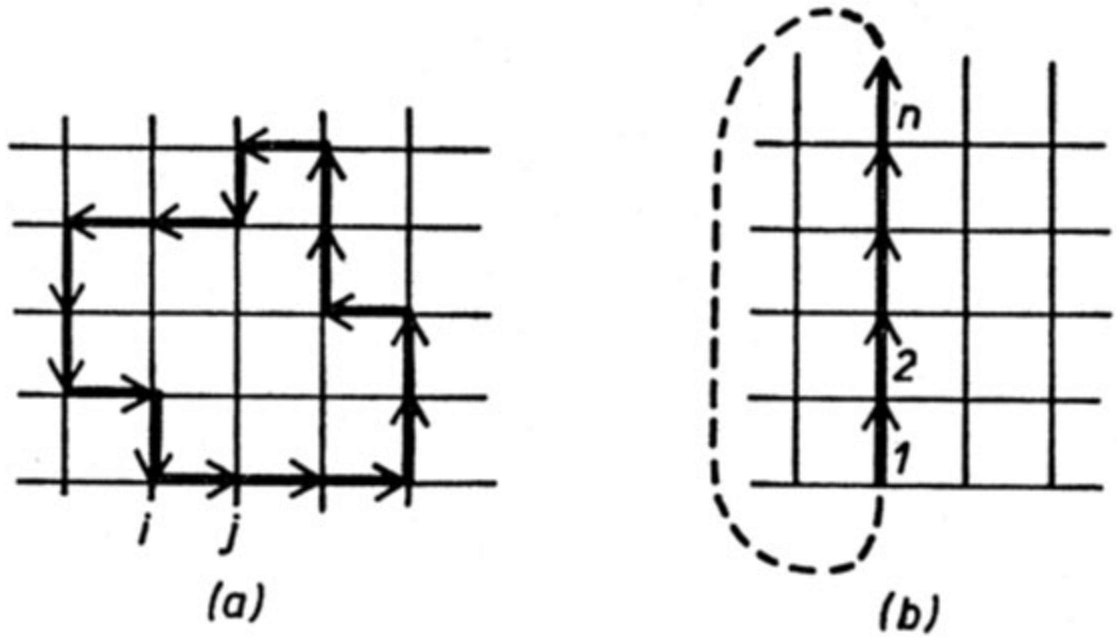
Lattice Actions are not unique

- Requirement : it will go to QCD action in naïve classical limit.
- All gauge invariant expressions
- $TrU_{ij}U_{jk}U_{kl}\ldots U_{xi}, \qquad \bar{\psi}\ldots\ldots\psi$
  are OK.
- Improved action: Higher order terms of *a* are suppressed
  – Standard ones
    - Gauge：Iwasaki、Syzmanzik, DBW2

      $$\beta(C_0 \square + C_1 \square\square)$$

      
    - Fermions：Wilson fermions with Clover term
      Smeared KS (Kogut-Susskind, or staggered) fermions

Wilson Loop and Polyakov Line



$$W = \frac{1}{N_c} Tr(U_{ij} U_{jk} \cdots U_{li})$$

$$L = \frac{1}{N_c} Tr(U_{12} U_{23} \cdots U_{n-1,n})$$

Add an external source of Gauge $j_\mu = g\delta^3(x_\mu - x_\mu(t))$

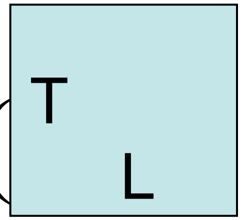
⇨ Free energy increases $i\int d^4x j_\mu A_\mu = ig\int dx_\mu A_\mu$

$$e^{-S_G} \rightarrow e^{-ig\int dx_\mu A_\mu - S_G}$$

$$= e^{igaA_n} e^{igaA_{n-1}} \cdots e^{igaA_1} e^{-SG}$$

$$= We^{-SG} \text{ or } Le^{-SG}$$

$$\frac{e^{-(F+\Delta F)}}{e^{-F}} = \frac{\int dU e^{-S_G} W}{Z} = < W >$$

$< Tr($ 

$) >= e^{-TV(L)}$

Polyakov Line

- Polyakov line: Free energy increases when one quark line is added

$$< L >= e^{-\Delta F}$$

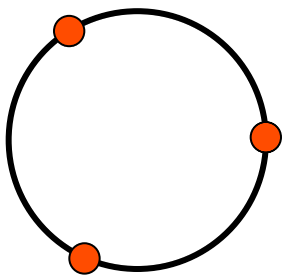
Confinement: $\Delta F = \infty$

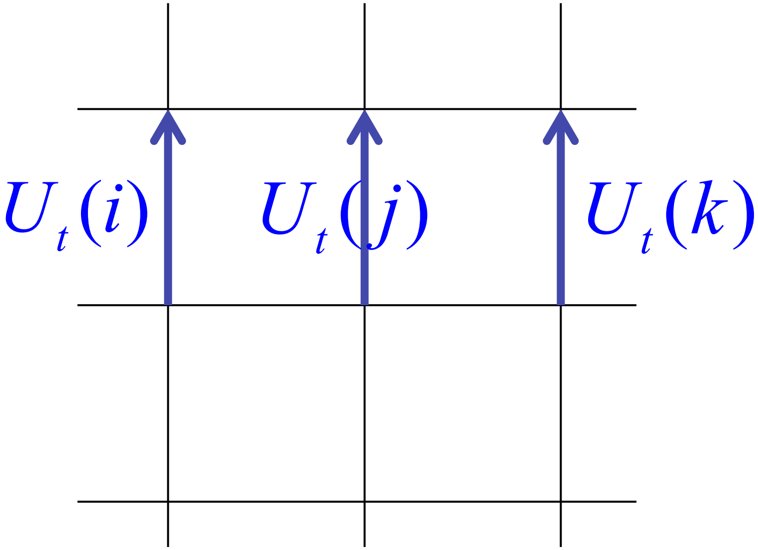
$$\Rightarrow \quad < L >= 0$$

McLerran and Svetisky, Phys.Rev. D24,450, 1981
This is one of the references that you should read.

Z3 Symmetry

- Three elements of SU(3) $1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}$ are commutable with all other elements





$$U_t(i), U_t(j), \cdots U_t(k)$$

$$\rightarrow zU_t(i), zU_t(j), \cdots zU_t(k)$$

$$z \in Z_3$$

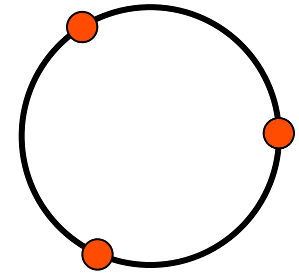
S_G invariant

$$L \rightarrow zL$$

(In Quench Approximation)

$< L > \neq 0$ ⇨ Spontaneous symmetry breaking of Z3

$z \in Z_3$



$$L \to zL$$

S_G invariant

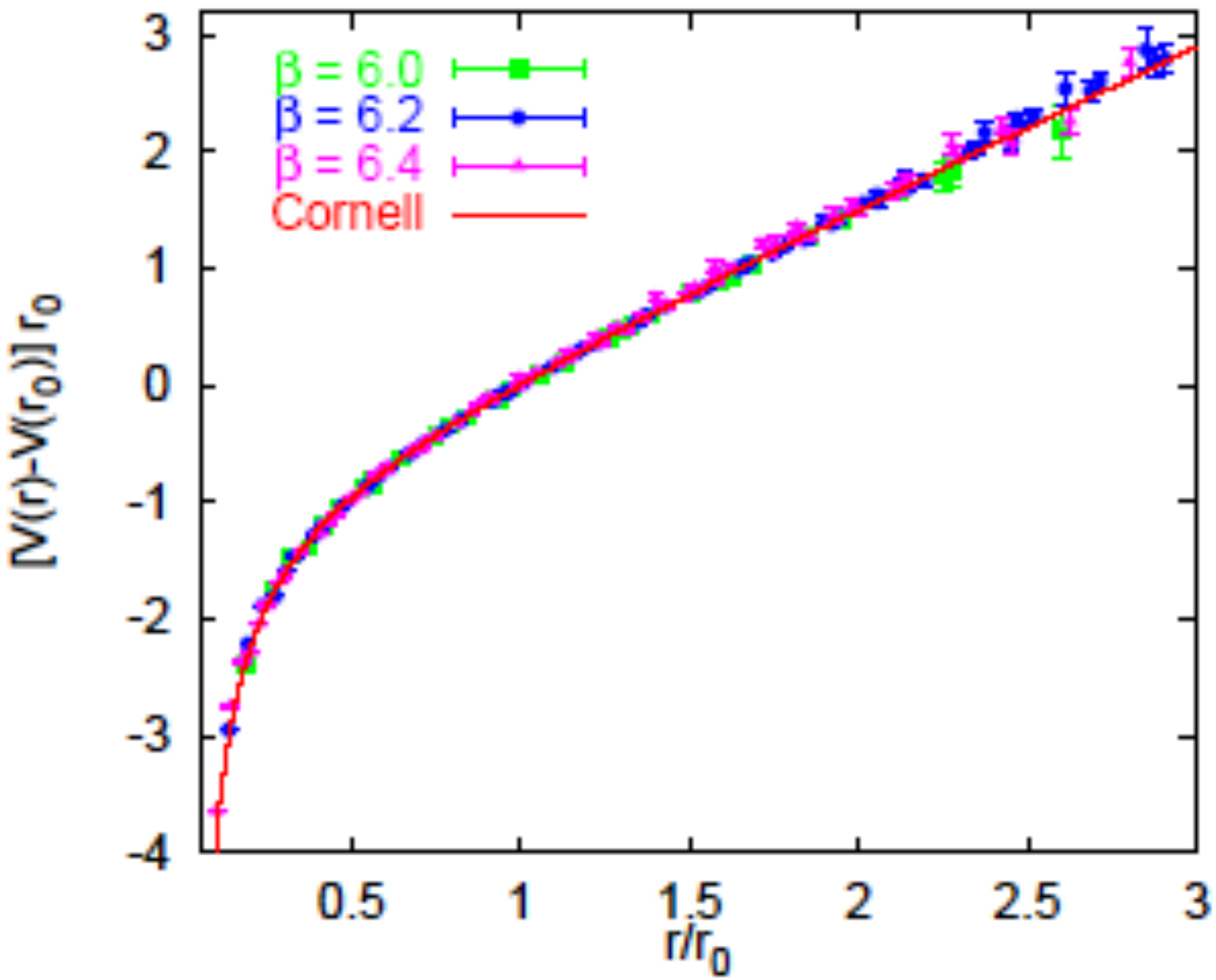
(In Quench Approximation)

$$\langle L \rangle = z \langle L \rangle$$

$$\langle L \rangle = 0$$

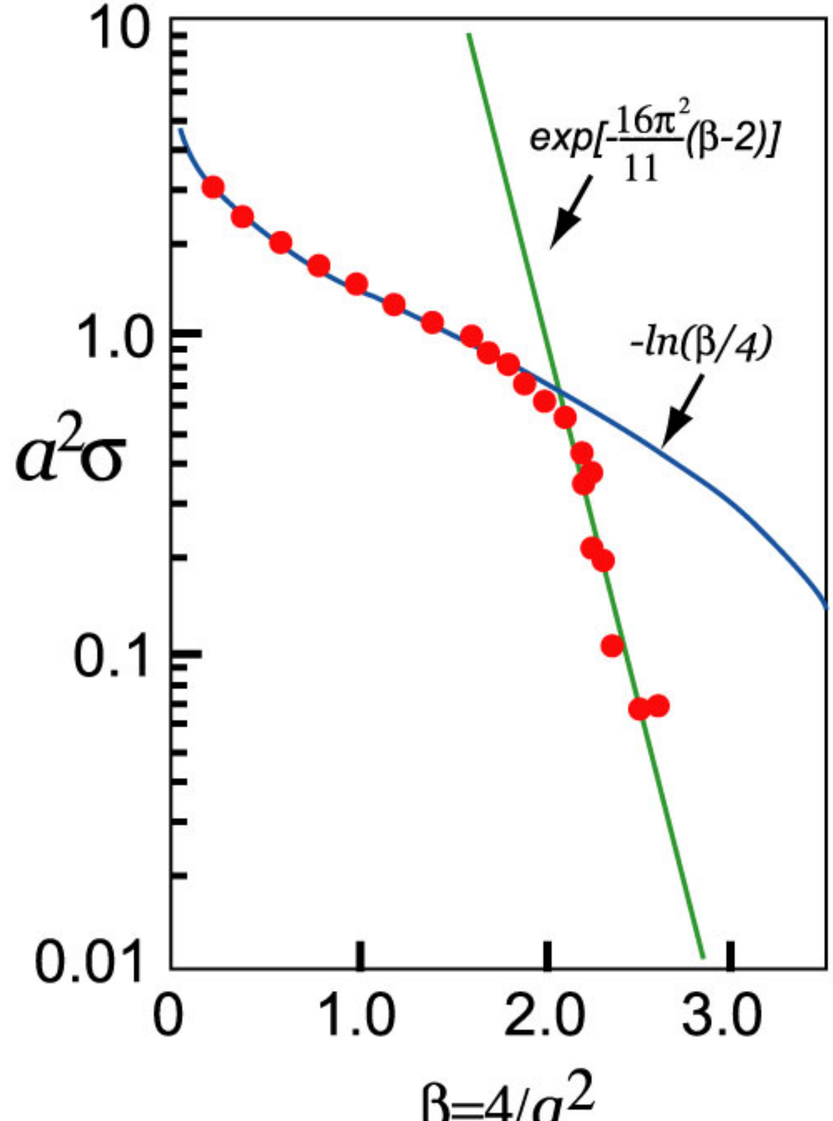
If $< L > \neq 0$ → Spontaneous symmetry breaking of Z3

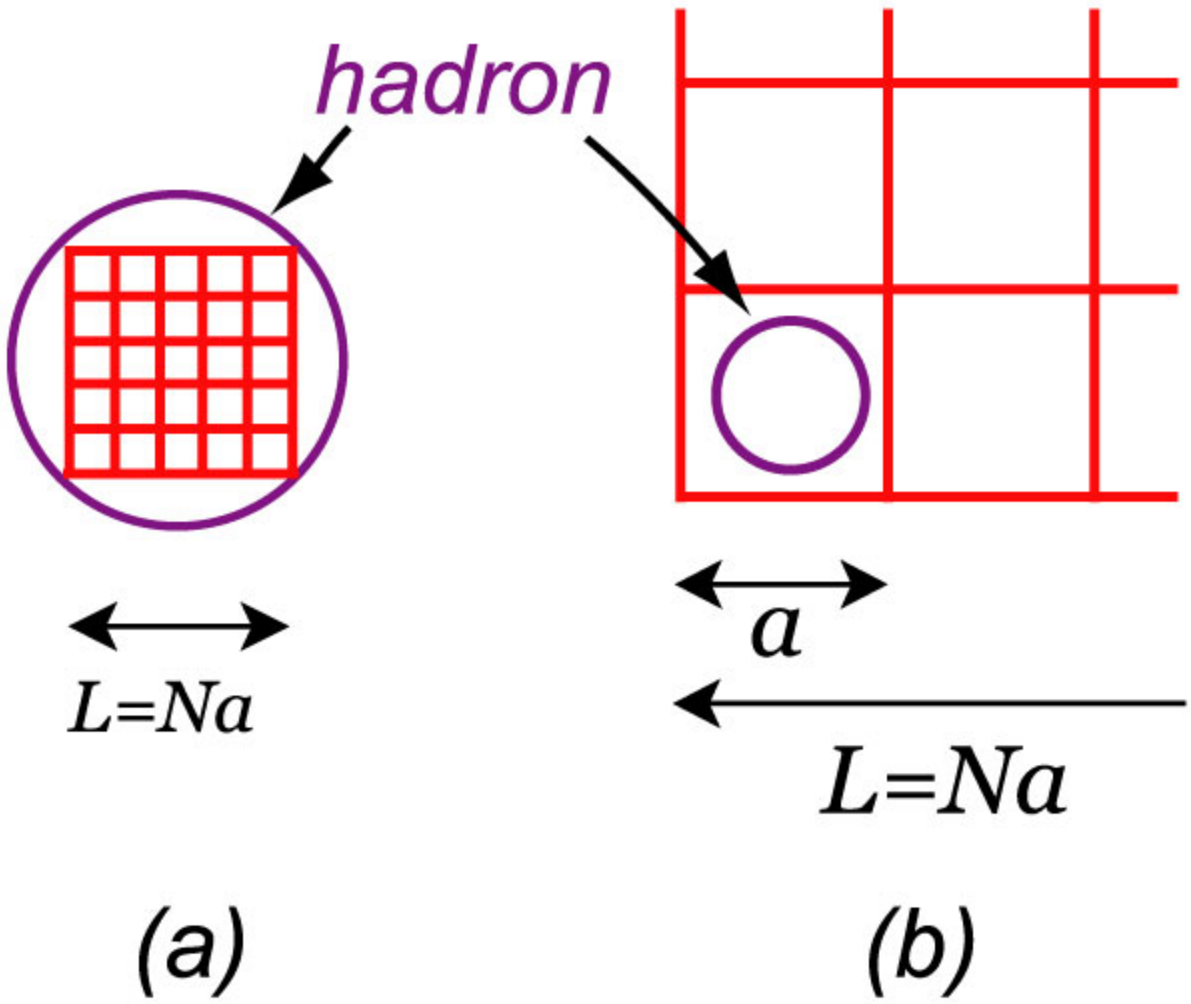
Heavy Quark Potential



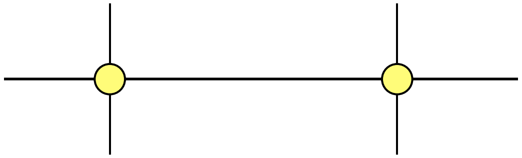
Lattice spacing and the coupling constant

- M.Creutz,
  - Phys.Rev.D21, 2308 (1980)
  - SU(2)



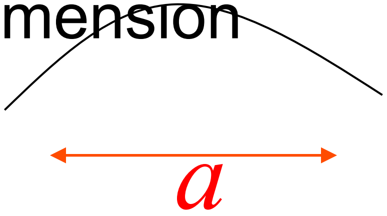

hadron
L=Na
a
L=Na
(a)
(b)

- Lattice : $(\text{Cut off}) = \dfrac{\pi}{a}$



$$m = \frac{1}{a} F(g)$$

m : Quantity of mass dimension



$$\frac{d}{da} m = 0$$

$$\Rightarrow \quad F = a \frac{dF}{da} = a \frac{dg}{da} \frac{dF}{dg} = -\beta(g) \frac{dF}{dg}$$

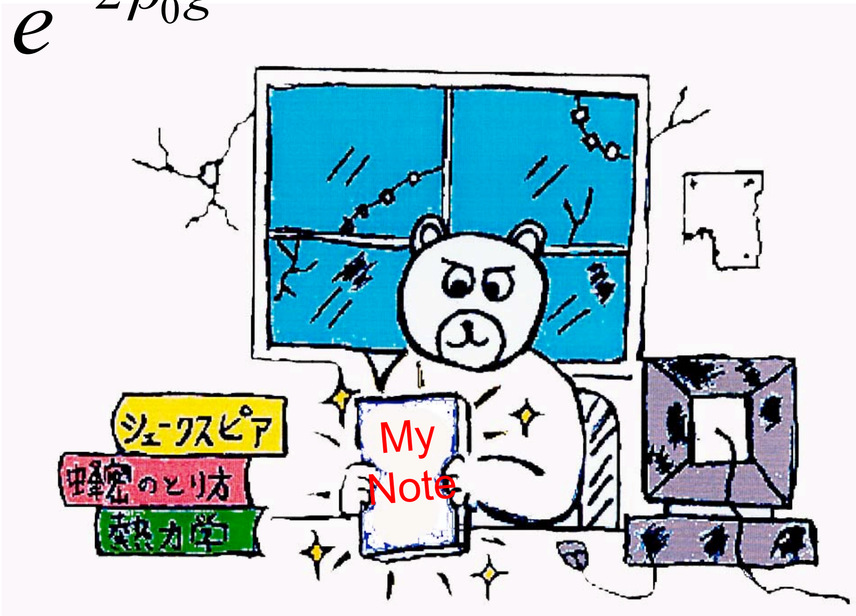
$$\beta(g) = -a \frac{dg}{da}$$

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \dots$$

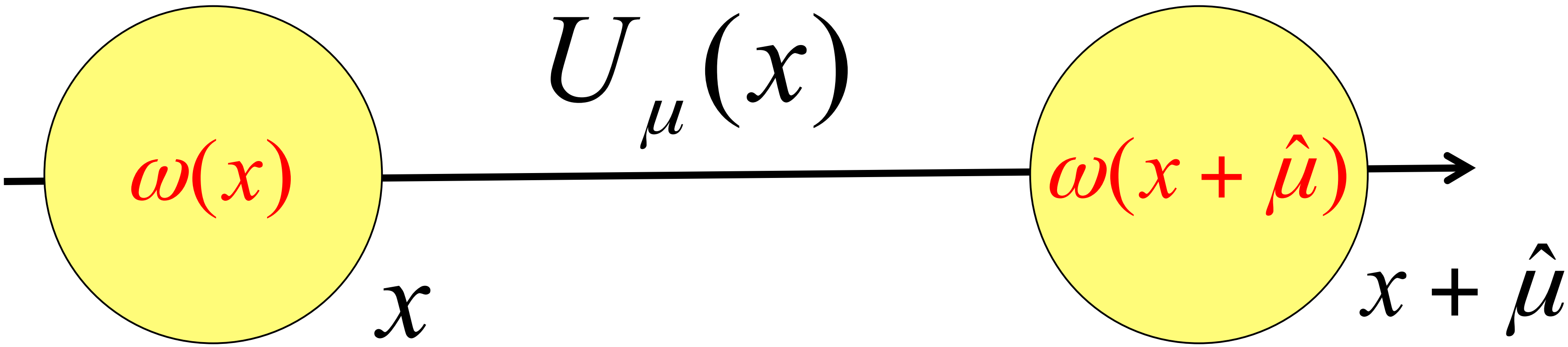
$$\int \frac{da}{a} = \int \frac{dg}{-\beta(g)} = \int \frac{dg}{\beta_0 g^3 + \beta_1 g^5}$$

$$a = \frac{1}{\Lambda}\left(\frac{1}{\beta_0 g^2}\right)^{\frac{\beta_1}{2\beta_0^2}} e^{-\frac{1}{2\beta_0 g^2}}$$

You should check this calculation once.



Gauge Transformation on Lattice



$$U_\mu(x) \to \omega(x)^\dagger U_\mu(x)\,\omega(x+\hat{\mu})$$

$$\bar{\psi}(x) \to \bar{\psi}(x)\,\omega(x)$$

$$\psi(x) \to \omega(x)^\dagger \psi(x)$$

$$\bar{\psi}(x)\psi(x)$$

$$\bar{\psi}(x)U_\mu(x)\psi(x+\hat{\mu})$$

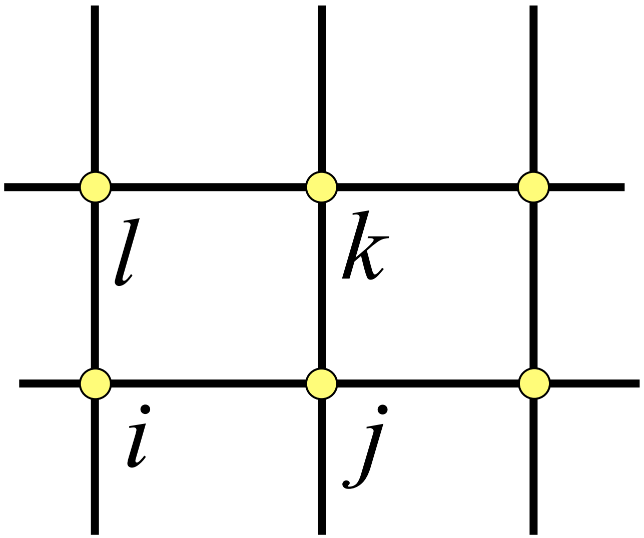
Invariant

$$U_{ij}U_{jk}U_{kl}U_{li} = U_{ij}U_{jk}U_{lk}^{\dagger}U_{il}^{\dagger}$$

$$\to (\omega_i^{\dagger}U_{ij}\omega_j)(\omega_j^{\dagger}U_{jk}\omega_k)(\omega_l^{\dagger}U_{lk}\omega_k)^{\dagger}(\omega_i^{\dagger}U_{il}\omega_l)^{\dagger}$$

$$= (\omega_i^{\dagger}U_{ij}\omega_j)(\omega_j^{\dagger}U_{jk}\omega_k)(\omega_k^{\dagger}U_{lk}^{\dagger}\omega_l)(\omega_l^{\dagger}U_{il}^{\dagger}\omega_i)$$

$$= \omega_i^{\dagger}U_{ij}U_{jk}U_{lk}^{\dagger}U_{il}^{\dagger}\omega_i$$



$$Tr\,U_{ij}U_{jk}U_{kl}U_{li}$$

invariant

Gauge Transformation in Continume limit

$$\omega(x)^\dagger U_\mu(x)\omega(x+\hat{\mu})$$

- U(1)ケース

$$\omega(x) = e^{i\chi(x)} \qquad U_\mu(x) = e^{iaA_\mu(x)}$$

$$U_\mu(x) = e^{iaA_\mu(x)} \longrightarrow e^{-i\chi(x)} e^{iaA_\mu(x)} e^{i\chi(x+\hat{\mu})}$$

$$A_\mu(x) \to A_\mu(x) + \frac{\chi(x+\hat{\mu}) - \chi(x)}{a}$$

$$= A_\mu(x) + \partial_\mu \chi + O(a)$$

- SU(N)

$$e^{iaA_\mu(x)} \to \omega(x)^\dagger e^{iaA_\mu(x)} \omega(x+\hat{\mu})$$

$$(1 + iaA_\mu(x) + ..) \to \omega(x)^\dagger (1 + iaA_\mu(x) + ..)(\omega(x) + a\partial_\mu \omega(x) + ..)$$

$$A_\mu(x) \to \omega(x)^\dagger A_\mu(x) \omega(x) - i\omega(x)^\dagger \partial_\mu \omega(x) + O(a)$$

Quark Propagators

- Quark Propagators = Inverse of the Fermion matrix Δ
- Gauss elimination ?
  - N^3 Operations （N：rank of the matrix Δ）
  - Δ is Sparse, but we cannot use this advantage.
- In many cases, it is enough if we can solve

$$\Delta\vec{x}=\vec{b}$$

$\vec{b}$ is a unit vector

$$\vec{x}=\Delta^{-1}\vec{b}$$

Conjugate Gradient (CG) Method

$$A\vec{x} = \vec{b}$$

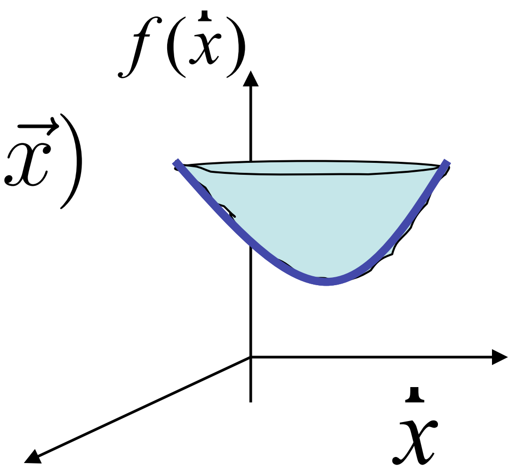
A:symetric, positive definite

$$(\vec{x}, A\vec{x}) \geq 0 \quad \text{for } \forall \vec{x}$$

If not, we rewrite it as $\;{}^{t}\!A A\,\vec{x} = {}^{t}\!A\,\vec{b}$

We minimize

$$f(\vec{x}) = \frac{1}{2}(\vec{x}, A\vec{x}) - (\vec{b}, \vec{x})$$



The solution is obtained at the bottom, where

$$\nabla f(\vec{x}) = A\vec{x} - \vec{b} = \vec{0}$$

CG Method

$$\vec{p}^{(0)} = \vec{r}^{(0)} = \vec{b} - A\vec{x}^{(0)}$$

DO i

$$\alpha^{(i)} = \frac{(\vec{p}^{(i)}, \vec{r}^{(i)})}{(\vec{p}^{(i)}, A\vec{p}^{(i)})}$$

$$\vec{x}^{(i+1)} = \vec{x}^{(i)} + \alpha^{(i)}\vec{p}^{(i)}$$

$$\vec{r}^{(i+1)} = \vec{r}^{(i)} - \alpha^{(i)}A\vec{p}^{(i)}$$

$$\beta^{(i)} = \frac{(\vec{r}^{(i+1)}, A\vec{p}^{(i)})}{(\vec{p}^{(i)}, A\vec{p}^{(i)})}$$

$$\vec{p}^{(i+1)} = \vec{r}^{(i+1)} + \beta^{(i)}\vec{p}^{(i)}$$

$$\vec{r}^{(i)} = \vec{b} - A\vec{x}^{(i)}$$

Residue

$\vec{p}^{(1)}, \vec{p}^{(2)}, \vec{p}^{(3)}, \cdots$ independent

$\vec{r}^{(1)}, \vec{r}^{(2)}, \vec{r}^{(3)}, \cdots$ independent

Maximal iteration to converge is N

(Matrix)x(Vector) and the inner product of vectors

Grassman Variables

$$\bar{\psi}_i \psi_j + \psi_j \bar{\psi}_i = \delta_{ij},$$
$$\bar{\psi}_i \bar{\psi}_j + \bar{\psi}_j \bar{\psi}_i = 0,$$
$$\psi_i \psi_j + \psi_j \psi_i = 0$$

$$\int d\bar{\psi}_i = \int d\psi_i = 0,$$
$$\int \bar{\psi}_i d\bar{\psi}_i = \int \psi_i d\psi_i = 1$$

Berezin（1966）

$$\int D\bar{\psi} D\psi \, e^{-\bar{\psi} A \psi} = \det A,$$
$$\int D\bar{\psi} D\psi (\bar{\psi}_i \psi_j) e^{-\bar{\psi} A \psi} = \left(A^{-1}\right)_{ji} \det A,$$
$$\int D\bar{\psi} D\psi (\bar{\psi}_i \psi_j \bar{\psi}_k \psi_l) e^{-\bar{\psi} A \psi} = \left\{\left(A^{-1}\right)_{ji}\left(A^{-1}\right)_{lk} - \left(A^{-1}\right)_{jk}\left(A^{-1}\right)_{li}\right\} \det A$$

Matthews-Salam

Exercise

For $\bar{\psi} A \psi = \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$

Show that $\int d\bar{\psi}_1 d\psi_1 d\bar{\psi}_2 d\psi_2 e^{-\bar{\psi} A \psi} = \det A$

$$e^{-\bar{\psi} A \psi} = 1 + (\bar{\psi}_1 A_{11} \psi_1 + \bar{\psi}_1 A_{12} \psi_2 + \bar{\psi}_2 A_{21} \psi_2 + \bar{\psi}_2 A_{22} \psi_2)$$
$$+ \frac{1}{2} (\bar{\psi}_1 A_{11} \psi_1 + \bar{\psi}_1 A_{12} \psi_2 + \bar{\psi}_2 A_{21} \psi_2 + \bar{\psi}_2 A_{22} \psi_2)^2 + ..$$

Only these terms contribute

Meson Propagators

- Example 1

$$\pi(x) = \bar{u}(x)\gamma_5 d(x) = \bar{u}^a_\alpha(x)(\gamma_5)_{\alpha\beta} d^a_\beta(x)$$

$$\frac{1}{Z}\int \mathcal{D}U\mathcal{D}\bar{u}\mathcal{D}u\mathcal{D}\bar{d}\mathcal{D}d e^{-S_G-\bar{u}\Delta u-\bar{d}\Delta d}\pi(x)\pi(y)^\dagger$$

$$\bar{u}^a_\alpha(x)(\gamma_5)_{\alpha\beta} d^a_\beta(x)(-\bar{d}^b_{\alpha'}(y)(\gamma_5)_{\alpha'\beta'} u^b_{\beta'}(y))$$

$$= \frac{1}{Z}\int \mathcal{D}U e^{-S_G} \det \Delta^{(u)} \det \Delta^{(d)}$$

$$\times G^{(u)ba}{}_{\beta'\alpha}(y,x)(\gamma_5)_{\alpha\beta} G^{(d)ab}{}_{\beta\alpha'}(x,y)(\gamma_5)_{\alpha'\beta'}$$

$$Tr\left(G^{(u)}(y,x)\gamma_5 G^{(d)}(x,y)\gamma_5\right)$$

$$= \frac{1}{Z} \int \mathcal{D}U e^{-S_G} \det \Delta^{(u)} \det \Delta^{(d)}$$

$$\times Tr\left(G^{(u)}(y,x)\gamma_5 G^{(d)}(x,y)\gamma_5 \right)$$

$$G^{(u)} \equiv \left(\Delta^{(u)}\right)^{-1} \quad G^{(d)} \equiv \left(\Delta^{(d)}\right)^{-1}$$

- Example 2 $\sigma(x) = \frac{\bar{u}(x)u(x) + \bar{d}(x)d(x)}{\sqrt{2}}$

$$= \frac{\bar{u}^a_\alpha(x)u^a_\alpha(x) + \bar{d}^a_\alpha(x)d^a_\alpha(x)}{\sqrt{2}}$$

$$\frac{1}{Z}\int DUD\bar{u}DuD\bar{d}Dde^{-S_G-\bar{u}Wu-\bar{d}Wd}\sigma(x)\sigma(y)^\dagger$$

$$\frac{\bar{u}^a_\alpha(x)u^a_\alpha(x) + \bar{d}^a_\alpha(x)d^a_\alpha(x)}{\sqrt{2}} \times \frac{\bar{u}^b_\beta(y)u^b_\beta(y) + \bar{d}^b_\beta(y)d^b_\beta(y)}{\sqrt{2}}$$

$$\Rightarrow (G^{(u)aa}{}_{\alpha\alpha}(x,x)G^{(u)bb}{}_{\beta\beta}(y,y) - G^{(u)ab}{}_{\alpha\beta}(x,y)G^{(u)ba}{}_{\beta\alpha}(y,x))$$

$$+G^{(d)aa}{}_{\alpha\alpha}(x,x)G^{(u)bb}{}_{\beta\beta}(y,y) +$$

$$\frac{1}{Z}\int \mathcal{D}U e^{-S_G} \det \Delta^{(u)} \det \Delta^{(d)}$$

$$Tr\left(G^{(u)}(x,x)\right)Tr\left(G^{(u)}(y,y)\right) - Tr\left(G^{(u)}(x,y)G^{(u)}(y,x)\right)$$

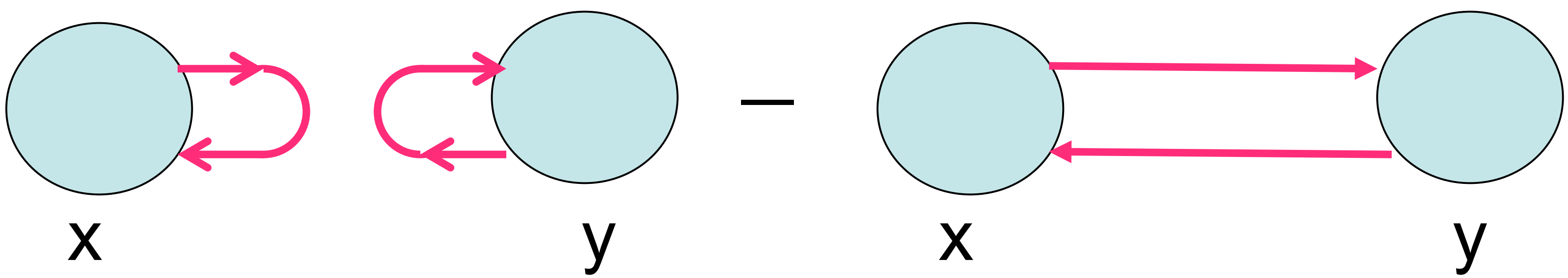
$$+Tr\left(G^{(d)}(x,x)\right)Tr\left(G^{(u)}(y,y)\right) + Tr\left(G^{(u)}(x,x)\right)Tr\left(G^{(d)}(y,y)\right)$$

$$+Tr\left(G^{(d)}(x,x)\right)Tr\left(G^{(d)}(y,y)\right) - Tr\left(G^{(d)}(x,y)G^{(d)}(y,x)\right)$$

Set $G^{(u)} = G^{(d)}$

$$2Tr\left(G(x,x)\right)Tr\left(G(y,y)\right) - 2Tr\left(G(x,y)G(y,x)\right)$$

$$+2Tr\left(G(x,x)\right)Tr\left(G(y,y)\right)$$



Enjoy！

Thank you！ 謝謝