New results on the 3-loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering

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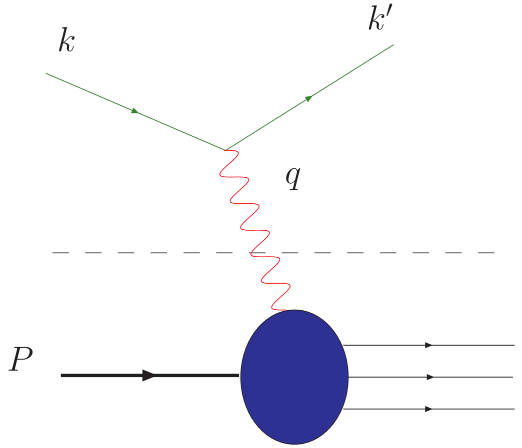
- Calculation methods.
- Ladder diagrams.
- The $O(n_f T_f^2 \alpha_s^3)$ contributions to OMEs.
- Convergent diagrams.
- Conclusions.

Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q} \quad \text{Bjorken–x}$$

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \ .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x, Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x, \mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz\, \delta(x - yz) f(y) g(z) \ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx\, x^{N-1} f(x) \ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten, 1976 Nucl.Phys.B; Babcock, Sivers, 1978 Phys.Rev.D; Shifman, Vainshtein, Zakharov, 1978 Nucl.Phys.B; Leveille, Weiler, 1979 Nucl.Phys.B; Glück, Reya, 1979 Phys.Lett.B; Glück, Hoffmann, Reya, 1982 Z.Phys.C.]

Next-to-Leading Order : [Laenen, van Neerven, Riemersma, Smith, 1993 Nucl. Phys. B]
[Large Q^2/m^2: Buza, Matiounine, Smith, Migneron, van Neerven, 1996 Nucl.Phys.B] IBP
[Bierenbaum, Blümlein, Klein, 2007 Nucl.Phys.B] via ${}_pF_q$'s, more compact results
[Bierenbaum, Blümlein, Klein 2008 Nucl.Phys.B, 2009 Phys.Lett.B]:$O(\alpha_s^2\varepsilon)$ contributions (all N)

NNLO:[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] Moments for F_2: $N = 2...10(14)$
[Blümlein, Klein, Tödtli 2009 Phys. Rev. D] contrib. to transversity: $N = 1...13$
[Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl.Phys.B] contrib. $\propto n_f$ to F_2 (all N):

At 3-loop order known:

- $A_{qq,Q}^{\rm PS}$, $A_{qg,Q}$: complete.
- A_{Qg}, $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS}$, $A_{gq,Q}$, $A_{gg,Q}$: all terms of $O(n_f T_F^2 C_{A/F})$
- $A_{Qq}^{\rm PS}$, $A_{qq,Q}^{\rm NS}$: all terms of $O(T_F^2 C_{A/F})$
- Ladder and Benz topologies with a single massive line: first results this talk.

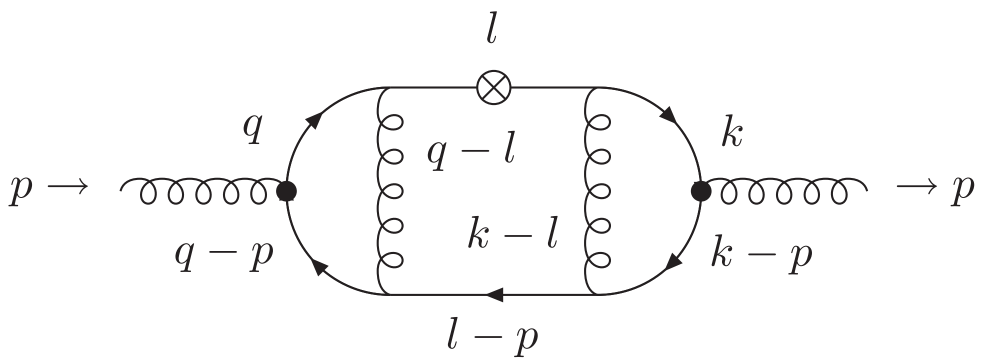
Calculation Methods

- Generation of diagrams with QGRAF [Nogueira 1993 J. Comput. Phys].
- Summation methods based on Zeilberger's algorithm, implemented in the Mathematica program Sigma [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion the summands in ε.
  - Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  - Harmonic sums are algebraically reduced using the package HarmonicSums (Ablinger) [Ablinger, Blümlein, Schneider 2011].
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008].
- Mellin-Barnes representations.
- Integration by parts identities.

Ladder Diagrams for Quarkonic OMEs

[Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock; arXiv:1206.2252]

Let's consider the scalar integral with all powers of the propagators equal to one.



After Feynman parametrization, and performing the momentum integrals, we obtain

$$I_{1a} = \frac{i(\Delta.p)^N a_s^3 S_\epsilon^3}{(m^2)^{2-\frac{3}{2}\epsilon}} \hat{I}_{1a}\,,$$

where S_ϵ is the spherical factor $S_\epsilon = \exp\left[\frac{\epsilon}{2}(\gamma_E - \ln(4\pi))\right]$, and

$$\hat{I}_{1a} = -\exp\left(-\frac{3}{2}\epsilon\gamma_E\right)\Gamma(2-3\epsilon/2)\prod_{i=1}^{7}\int_0^1 dw_i \frac{\theta(1-w_1-w_2)w_1^{-\epsilon/2}w_2^{-\epsilon/2}(1-w_1-w_2)}{\left(1+w_1\frac{w_3}{1-w_3}+w_2\frac{w_4}{1-w_4}\right)^{2-3\epsilon/2}}$$
$$\times w_3^{\epsilon/2}(1-w_3)^{-1+\epsilon/2}w_4^{\epsilon/2}(1-w_4)^{-1+\epsilon/2}(1-w_5w_1-w_6w_2-(1-w_1-w_2)w_7)^N$$

We see that by doing a binomial expansion for the polynomial raised to the Nth power (which arises due to the operator insertion), the resulting integrals in w_1 and w_2 can be written in terms of Appell hypergeometric functions:

$$\int_0^1 dw_1 \int_0^1 dw_2 \, \frac{\theta(1-w_1-w_2) w_1^{b-1} w_2^{b'-1} (1-w_1-w_2)^{c-b-b'-1}}{(1-w_1 x - w_2 y)^a}$$
$$= \Gamma \begin{bmatrix} b, b', c-b-b' \\ c \end{bmatrix} F_1 \left[a; b, b'; c; x, y\right] \, .$$

In our case, the parameters x, y correspond to $w_3/(1-w_3)$ and $w_4/(1-w_4)$, respectively. To obtain a series-representation of the integral, we carry out the following analytic continuation:

$$\begin{aligned} F_1 \left[a; , b, b'; c; \frac{x}{1-x}, \frac{y}{1-y}\right] &= (1-x)^b (1-y)^{b'} F_1 \left[c-a; b, b'; c; x, y\right] \\ &= (1-x)^b (1-y)^{b'} \sum_{m,n}^{\infty} \frac{(c-a)_{m+n} (b)_n (b')_m}{m! n! (c)_{m+n}} x^m y^n \end{aligned}$$

Applying this to our integral $\hat{I}_{1a}$, we obtain

$$\begin{aligned}
\hat{I}_{1a} \quad = \quad & \frac{\exp\left(-\frac{3}{2}\epsilon\gamma_E\right)\Gamma(2-3\epsilon/2)}{(N+1)(N+2)(N+3)} \sum_{m,n=0}^{\infty} \Bigg\{ \\
& \sum_{t=1}^{N+2} \binom{N+3}{t} \frac{(t-\epsilon/2)_m (N+2+\epsilon/2)_{m+n} (N+3-t-\epsilon)_n}{(N+4-\epsilon)_{m+n}} \\
& \times \Gamma\left[\begin{matrix} t, t-\epsilon/2, m+1+\epsilon/2, n+1+\epsilon/2, N+3-t, N+3-t-\epsilon/2 \\ N+4-\epsilon, m+1, n+1, m+t+1+\epsilon/2, N+n-t+4+\epsilon/2 \end{matrix}\right] \\
& - \sum_{s=1}^{N+3} \sum_{r=1}^{s-1} \binom{s}{r} \binom{N+3}{s} (-1)^s \frac{(r-\epsilon/2)_m (s-1+\epsilon/2)_{m+n} (s-r-\epsilon/2)_n}{(s+1-\epsilon)_{m+n}} \\
& \times \Gamma\left[\begin{matrix} r, r-\epsilon/2, s-r, m+1+\epsilon/2, n+1+\epsilon/2, s-r-\epsilon/2 \\ m+1, n+1, m+r+1+\epsilon/2, s-r+n+1+\epsilon/2, s+1-\epsilon \end{matrix}\right] \Bigg\}
\end{aligned}$$

After expanding in ϵ, the summation can be performed using `Sigma`.

The result for this and other integrals can be written in terms of harmonic sums $S_{\vec{a}}$ and their generalizations $S_{\vec{a}}(\vec{\xi};N)$:

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{\mathrm{sign}(b)^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset}(k) = 1$$

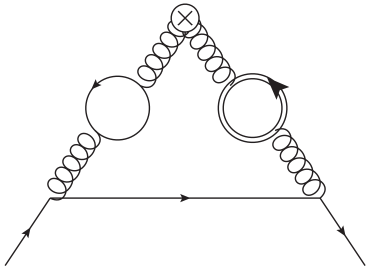
$$S_{b,\vec{a}}(\eta,\vec{\xi};N) = \sum_{k=1}^{N} \frac{\eta^k}{k^b} S_{\vec{a}}(\vec{\xi};k), \quad S_{\emptyset} = 1, \quad \eta,\xi \in \Re$$

The threefold and fourfold sums in $\hat{I}_{1a}$ give

$$\begin{aligned}\hat{I}_{1a} \quad = \quad & -\frac{4(N+1)S_1+4}{(N+1)^2(N+2)}\zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)} \frac{1}{(N+1)(N+2)(N+3)} \Bigg\{ \\ & -2(3N+5)S_{3,1} - \frac{S_1^4}{4} + \frac{4(N+1)S_1 - 4N}{N+1} S_{2,1} 2\left[(2N+3)S_1 + \frac{5N+6}{N+1}\right] S_3 \\ & + \frac{9+4N}{4} S_2^2 + \left[2\frac{7N+11}{(N+1)(N+2)} + \frac{5N}{N+1}S_1 - \frac{5}{2}S_1^2\right] S_2 + \frac{2(3N+5)S_1^2}{(N+1)(N+2)} \\ & + \frac{N}{N+1}S_1^3 + \frac{4(2N+3)}{(N+1)^2(N+2)}S_1 - \frac{1}{2}(2N+3)S_4 + 8\frac{2N+3}{(N+1)^3(N+2)} \Bigg\}\end{aligned}$$

This result was checked using `MATAD` for the fixed moments $N = 1 \ldots 10$.

The $O(n_f T_F^2 \alpha_s^3)$ contributions to $A_{gq,Q}$



The all-ε result constituting the color factor $T_F^2 n_f C_F$ [arXiv:1205.4184, to appear in Nucl. Phys. B]

$$\hat{\hat{A}}^{(3)}_{gq,T_F^2 n_f} = -96 a_s^3 T_F^2 n_f C_F \left(\frac{m^2}{\mu^2}\right)^{\frac{3\varepsilon}{2}} S_\varepsilon^3 \frac{1+(-1)^N}{2} e^{-\frac{3\varepsilon}{2}\gamma} \frac{(\varepsilon-1)^2(\varepsilon+2)\left(\varepsilon+N^2+N+2\right)}{\varepsilon(\varepsilon+1)(\varepsilon+3)}$$
$$\times\Gamma(1-\varepsilon)^2\Gamma\left(-\frac{\varepsilon}{2}-4\right)\Gamma\left(\frac{\varepsilon}{2}+2\right)\frac{\Gamma\left(\frac{\varepsilon}{2}+5\right)\Gamma\left(-\frac{3\varepsilon}{2}\right)\Gamma(N-1)}{\Gamma(4-2\varepsilon)\Gamma\left(\frac{\varepsilon}{2}+N+2\right)}$$

yields the renormalized contribution

$$A_{gq,Q}^{(3),n_f T_F^2,\overline{\mathsf{MS}}} = n_f T_F^2 \frac{1+(-1)^N}{2}\Biggl\{ \mathbf{C_F}\frac{32\left(N^2+N+2\right)}{9(N-1)N(N+1)} \mathbf{ln^3}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right) + \mathbf{C_F}\Biggl[-\frac{16\left(N^2+N+2\right)}{3(N-1)N(N+1)}\left(S_1^2+S_2\right)$$
$$+\frac{32\left(8N^3+13N^2+27N+16\right)}{9(N-1)N(N+1)^2}S_1+\frac{32\left(19N^4+81N^3+86N^2+80N+38\right)}{27(N-1)N(N+1)^3}\Biggr]\mathbf{ln}\left(\frac{\bar{\mathbf{m}}^2}{\boldsymbol{\mu}^2}\right)$$
$$+\mathbf{C_F}\Biggl[\frac{32\left(N^2+N+2\right)}{27(N-1)N(N+1)}\left(S_1^3+3S_2S_1+2\,S_3-24\zeta_3\right)-\frac{32\left(8N^3+13N^2+27N+16\right)}{27(N-1)N(N+1)^2}\left(S_1^2+S_2\right)$$
$$+\frac{64\left(4N^4+4N^3+23N^2+25N+8\right)}{27(N-1)N(N+1)^3}S_1+\frac{64\left(197N^5+824N^4+1540N^3+1961N^2+1388N+394\right)}{243(N-1)N(N+1)^4}\Biggr]\Biggr\}$$

Here we confirm the n_f contribution to the anomalous dimension:
[Moch, Vermaseren, Vogt 2004 Nucl.Phys.B]

$$\hat{\gamma}_{gq}^{(2),n_f} = n_f T_F^2 C_F \left(\frac{64\,(N^2+N+2)}{3(N-1)N(N+1)} - \left(S_1^2 + S_2\right) + \frac{128\,(8N^3+13N^2+27N+16)}{9(N-1)N(N+1)^2} S_1 \right.$$
$$\left. - \frac{128\,(4N^4+4N^3+23N^2+25N+8)}{9(N-1)N(N+1)^3} \right)$$

in an independent calculation.

Furthermore we are able to check a result for the combination

$$\tilde{\gamma}_{gg}^{(2)} + \frac{\tilde{\gamma}_{gq}^{(2)} \gamma_{qg}^{(0)}}{\tilde{\gamma}_{gg}^{(0)} n_f}$$

of 3-loop anomalous dimensions, derived from the large n_f expansion in QCD by [Bennett, Gracey 1997]; where we denote with $\tilde{\gamma}_{ij}^{(k)}$ the leading n_f coefficient of $\gamma_{ij}^{(k)}$.

Calculation of Convergent Massive 3-Loop Graphs

Many of the Feynman integrals appearing in the calculation of the massive 3-loop operator matrix elements are finite.

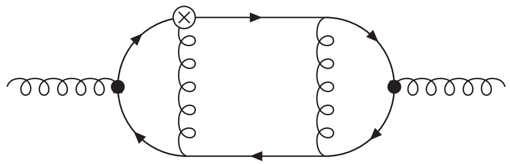
We have generalized a method originally proposed by F. Brown [Comm. Math. Phys. 2008] to the case where we have masses and operator insertions in order to find general N representations for all convergent 3-loop topologies.

Here we work in the α-representation to calculate the integrals.

The corresponding graph polynomials of a graph G are given by

- $U = \sum_T \prod_{l \notin T} \alpha_l$, where T denotes the spanning trees of G
- $V = \sum_{l \in massive} \alpha_l$
- Dodgson polynomials arise from the operator insertions.

Calculation of Convergent Massive 3-Loop Graphs



$$\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\ d\alpha_2\ d\alpha_3\ d\alpha_4\ d\alpha_5\ d\alpha_6\ d\alpha_7\ d\alpha_8\ \frac{\sum_{j=0}^{N} T_{4\alpha}^{N-j} T_{4b}^{j}}{U^2 V^2} \\
T_{4\alpha} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad +\alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad +\alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad +\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad +\alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad +\alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}$$

- The integral above is a projective integral, one α-parameter may be set 1
- The operators sit on on-shell diagrams which obey specific symmetries. These are generally not obeyed by the operator insertion.
- For the above example : after applying symmetry transformations $\alpha_1 \to x_1 - \alpha_2$, $\alpha_3 \to x_2 - \alpha_4$, $\alpha_5 \to x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.

Calculation of Convergent Massive 3-Loop Graphs

- Feynman parameter integrals are performed in terms of Hyperlogarithms,
  [Brown 2008 Comm. Math. Phys.]
  $L(\overrightarrow{w}, z) : \mathbb{C}\backslash\Sigma \to \mathbb{C}$, where
  - $\Sigma = \{\sigma_0, \sigma_1, ..., \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables
  - $\overrightarrow{w}$ is a word over the alphabet $\mathfrak{A} = \{a_0, a_1, ..., a_N\}$ where each letter a_i corresponds to a point σ_i
- $L(\overrightarrow{w}, z)$ is uniquely defined by the following properties

1. $L(\{\}, z) = 1$, and $L(0^n, z,) = \frac{1}{n!}\log^n(z)$ for $n \geq 1$
2. $\frac{\partial}{\partial z} L(\{a_i \overrightarrow{w}\}, z) = \frac{1}{z-\sigma_i} L(\overrightarrow{w}, z)$ for $z \in \mathbb{C}\backslash\Sigma$
3. If $\overrightarrow{w}$ is not of the form $w = (0, 0, \cdots, 0)$, then $\lim_{z\to 0} L(\overrightarrow{w}, z) = 0$.

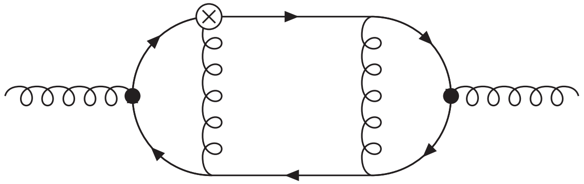
- e.g. $L(a_i, z) = \log(z - \sigma_i) - \log(\sigma_i)$

- The hyperlogarithms satisfy shuffle relations $L(\overrightarrow{w_1}, z)\, L(\overrightarrow{w_2}, z) = L(\overrightarrow{w_1} \sqcup\!\sqcup \overrightarrow{w_2}, z)$, e.g.:
$L(\{a_1, a_2\}, z)\, L(\{a_3\}, z) = L(\{a_3, a_1, a_2\}, z) + L(\{a_1, a_3, a_2\}, z) + L(\{a_1, a_2, a_3\}, z)$
- The indices a_i contain further integration variables.
- Using these properties after partial fractioning and integration by parts, one can express any primitive for expressions consisting of rational and hyperlogarithmic functions in terms of different hyperlogarthmic functions. These primitives have to be evaluated at the respective integration limits
- Due to the operator-insertions leading to power-type functions, the integrals do not fit directly into the framework of the algorithm for general values of N.
- In order to use the algorithm also on integrals with general values of N, a generating function is constructed e.g. by the mapping

$$p(\alpha_1, \cdots, \alpha_n)^N \to \frac{1}{1 - x\, p(\alpha_1, \cdots, \alpha_n)} \, .$$

- Performing the Feynman-parameter integrations then leads to an expression which contains hyperlogarithms $L_w(x)$ in the variable x.
- Finally the Nth coefficient of this expression in x has to be extracted analytically. This has been done with the package `HarmonicSums` by J.Ablinger. [Ablinger, Blümlein, Schneider; 2012]

Six Massive Lines and Vertex Insertion

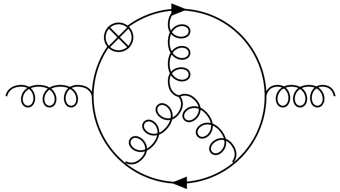


$$
\begin{aligned}
\hat{I}_4 \;=\;& \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
&+\frac{(-24-5N+2N^2)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2 S_2 \\
&+\frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3 S_{-2} - S_{-2,1}S_{-2} \\
&+\frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{(59+42N+6N^2)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3 S_1(2) \\
&-\frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3 S_2 - \frac{3}{2}S_3 S_2 - 2S_{2,1}S_2 + \frac{(99+225N+190N^2+65N^3+7N^4)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
&+\frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3 S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2 S_1 - S_{2,3} \\
&+\frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3 S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1 S_{2,1} + \frac{(-79-40N+N^2)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
&+S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{(-7+2N^2)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
&+5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
&-\frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

The 2^N factors cancel in the large N limit:

$$\begin{aligned}
\hat{I}_4 \approx\ & \zeta_2^2\left[\frac{1115231}{20N^{10}}-\frac{74121}{4N^9}+\frac{122951}{20N^8}-\frac{40677}{20N^7}+\frac{13391}{20N^6}-\frac{873}{4N^5}+\frac{1391}{20N^4}-\frac{417}{20N^3}+\frac{101}{20N^2}\right]\\
&+\zeta_3\left[\left(-\frac{95855}{2N^{10}}+\frac{31525}{2N^9}-\frac{10295}{2N^8}+\frac{3325}{2N^7}-\frac{1055}{2N^6}+\frac{325}{2N^5}-\frac{95}{2N^4}+\frac{25}{2N^3}-\frac{5}{2N^2}\right)\ln(N)\right.\\
&\left.-\frac{23280115}{2016N^{10}}+\frac{2093041}{1008N^9}-\frac{177251}{1008N^8}-\frac{25843}{336N^7}+\frac{2569}{48N^6}-\frac{155}{8N^5}+\frac{91}{24N^4}+\frac{2}{3N^3}-\frac{11}{12N^2}\right]\\
&+\zeta_2\left[\left(\frac{19171}{N^{10}}-\frac{6305}{N^9}+\frac{2059}{N^8}-\frac{665}{N^7}+\frac{211}{N^6}-\frac{65}{N^5}+\frac{19}{N^4}-\frac{5}{N^3}+\frac{1}{N^2}\right)\ln^2(N)\right.\\
&+\left(\frac{103016863}{2520N^{10}}-\frac{3091261}{315N^9}+\frac{2571839}{1260N^8}-\frac{6215}{21N^7}-\frac{293}{20N^6}+\frac{2071}{60N^5}-\frac{103}{6N^4}+\frac{67}{12N^3}-\frac{1}{N^2}\right)\ln(N)\\
&\left.+\frac{292993001621}{302400N^{10}}-\frac{4402272031}{30240N^9}+\frac{22261739}{840N^8}-\frac{78507473}{14112N^7}+\frac{180961}{144N^6}-\frac{111807}{400N^5}+\frac{629}{12N^4}-\frac{319}{72N^3}-\frac{7}{4N^2}\right]\\
&+\left(\frac{249223}{6N^{10}}-\frac{145015}{12N^9}+\frac{10295}{3N^8}-\frac{11305}{12N^7}+\frac{1477}{6N^6}-\frac{715}{12N^5}+\frac{38}{3N^4}-\frac{25}{12N^3}+\frac{1}{6N^2}\right)\mathbf{\ln^3(N)}\\
&+\left(\frac{193493767}{10080N^{10}}+\frac{210658237}{10080N^9}-\frac{21541697}{2520N^8}+\frac{243269}{96N^7}-\frac{30539}{48N^6}+\frac{2123}{16N^5}-\frac{59}{3N^4}+\frac{5}{8N^3}+\frac{1}{2N^2}\right)\ln^2(N)\\
&+\left(-\frac{2207364771673}{4233600N^{10}}+\frac{1390655509}{352800N^9}+\frac{285594061}{22050N^8}-\frac{67234111}{14400N^7}+\frac{8617073}{7200N^6}-\frac{35209}{144N^5}+\frac{116}{3N^4}-\frac{119}{24N^3}+\frac{1}{N^2}\right)\ln(N)\\
&+\frac{1344226725047831}{889056000N^{10}}-\frac{165849841805771}{889056000N^9}+\frac{808151260279}{27783000N^8}-\frac{708430537}{120960N^7}+\frac{304474703}{216000N^6}\\
&-\frac{606811}{1728N^5}+\frac{1867}{24N^4}-\frac{1813}{144N^3}+\frac{1}{N^2}+O(N^{-11})
\end{aligned}$$

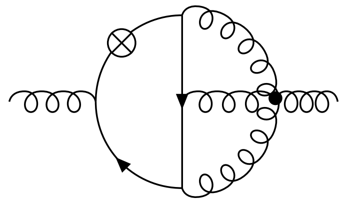
General Values of N: Higher Topologies



$$I(x) = \frac{1}{(1+N)(2+N)x}\Big\{\zeta_3\Big[2L(\{-1\},x) - 2(-1+2x)L(\{1\},x) - 4L(\{1,1\},x)\Big] - 3L(\{-1,0,0,1\},x)$$
$$+2L(\{-1,0,1,1\},x) - 2xL(\{0,0,1,1\},x) + 3xL(\{0,1,0,1\},x) - xL(\{0,1,1,1\},x)$$
$$+(-3+2x)L(\{1,0,0,1\},x) + 2xL(\{1,0,1,1\},x) - (-1+5x)L(\{1,1,0,1\},x) + xL(\{1,1,1,1\},x)$$
$$-2L(\{1,0,0,1,1\},x) + 3L(\{1,0,1,0,1\},x) - L(\{1,0,1,1,1\},x) + 2L(\{1,1,0,0,1\},x)$$
$$+2L(\{1,1,0,1,1\},x) - 5L(\{1,1,1,0,1\},x) + L(\{1,1,1,1,1\},x)\Big\}$$

$$I(N) = \frac{1}{(N+1)(N+2)(N+3)}\Big\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3}$$
$$-\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)}S_1^3 + \frac{1}{24}S_1^4$$
$$-\frac{\left(7+22N+10N^2\right)}{2(1+N)^2(2+N)}S_2 - \frac{19}{8}S_2^2 - \frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2 + \frac{9}{4}S_2 - \frac{(-9+4N)}{3(1+N)}S_3$$
$$-\frac{1}{4}S_4 - 2(-1)^N S_{-2,1} + \frac{(-1+6N)}{(1+N)}S_{2,1} + \frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1$$
$$+4\zeta_3 S_1 - \frac{(-2+7N)}{2(1+N)}S_2 S_1 + \frac{13}{3}S_3 S_1 - 7S_{2,1}S_1 - 7S_{3,1} + 10S_{2,1,1}\Big\}$$

General Values of N: Higher Topologies



$$
\begin{aligned}
I(N) \quad = \quad & \frac{1}{(N+1)(N+2)}\Bigg\{ \frac{2\left(1-13(-1)^N+(-1)^N 2^{3+N}+N-7(-1)^N N+3(-1)^N 2^{1+N} N\right)}{(1+N)(2+N)}\zeta_3 \\
& +\frac{1}{(2+N)}S_3+\frac{(-1)^N}{2(2+N)}S_1^3-\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_2+\frac{5(-1)^N}{2}S_2^2 \\
& +\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_1^2-\frac{(-1)^N}{2}S_2S_1^2+\frac{3(-1)^N(4+3N)}{(1+N)(2+N)}S_3+3(-1)^N S_4+\frac{2}{(2+N)}S_{-2,1} \\
& +2(-1)^N\zeta_3 S_1\,(2)+\frac{2(-1)^N(3+N)}{(1+N)(2+N)}S_{2,1}-12(-1)^N S_1\zeta_3 \\
& +\frac{(-1)^N(5+7N)}{2(1+N)(2+N)}S_1S_2+3(-1)^N S_1S_3+4(-1)^N S_{2,1}S_1-4(-1)^N S_{3,1} \\
& -\frac{4\left((-1)^N 2^{2+N}-3(-2)^N N+3(-1)^N 2^{1+N} N\right)}{(1+N)(2+N)}S_{1,2}\left(\frac{1}{2},1\right)-5(-1)^N S_{2,1,1} \\
& +\frac{2\left(-(-1)^N 2^{2+N}-13(-2)^N N+5(-1)^N 2^{1+N} N\right)}{(1+N)(2+N)}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& -2(-1)^N S_{1,1,2}\left(2,\frac{1}{2},1\right)-(-1)^N S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)\Bigg\}
\end{aligned}
$$

Conclusions

- There has been substantial progress in the calculation of massive 3-loop operator matrix elements for general values of the Mellin variable N in the last few years.
- The quarkonic 3-loop contributions of $O(n_f T_F^2 C_{A,F})$ to A_{qq} and A_{qg} were calculated in [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl. Phys. B]. Now also $A_{gg,Q}$ and $A_{gq,Q}$ have been obtained for these color coefficients at general N [arXiv:1205.4184].
- Ladder topologies, including poles, are currently calculated using `Sigma` and the method of hyperlogarithms. [Brown, Comm. Math. Phys. 2008]
- As a consequence of the complicated nature of sums arising in due course of the calculation, considerable upgrades have been made to the package `Sigma`.
- With the help of hyperlogarithms non-divergent 3-loop graphs can be calculated. For general values of N first analytic results have been obtained, including Benz-topologies, performing the calculation automatically.
- 3-loop moments of polarized massive OMEs up to the constant terms have been calculated.