Three-loop Matching Coefficient of the Vector Current

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Outline

1. Introduction
2. Matching Coefficient
3. Conclusion
The matching coefficient of the vector current comprises an important building block for

- Measurement of the top-quark mass at a future linear collider
  \[ \Delta M_t < 100 \text{ MeV} \]

- Measurements of the top-quark mass at hadron colliders limited by systematic errors
  \[ \Delta M_t \approx 1 \text{ GeV} \]
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3. Measurement of the bottom-quark mass from \( \Upsilon \) sum rules
The top-quark mass can be obtained from a scan of the top anti-top threshold at a linear collider

\[ M_t = 175 \text{ GeV} \]
Introduction

The top-quark mass can be obtained from a scan of the top anti-top threshold at a linear collider

\[ M_t = 175 \text{ GeV} \pm 200 \text{ MeV} \]
Threshold expansion $\leftrightarrow$ effective field theories

QCD
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QCD

$\downarrow$

integrate out hard modes

$\downarrow$

non-relativistic QCD (NRQCD)
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QCD

$\downarrow$

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non-relativistic QCD (NRQCD)

$\downarrow$

integrate out soft/potential modes

potential NRQCD (PNRQCD)
Theoretical Status

two-loop: calculation done by several groups

[Hoang, Teubner; Melnikov, Yelkhovsky; Yakovlev; Beneke, Signer, Smirnov; Nagano, Ota, Sumino; Penin, Pivovarov ]
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**three-loop:**

- matching coefficient $c_V$
  - $n_l$ ✓
  - $n_f$ ✓
  - $n_f^0 \rightarrow$ this talk

- heavy-quark potential $a_3$ ✓
  - [Smirnov, Smirnov, Steinhauser; Anzai, Kiy, Sumino]

- potential contributions ✓
  - [Beneke, Kiy, Schuller]

- ultrasoft contributions ✓
  - [Beneke, Kiy, Penin]
**QCD vector current**

\[ j^\mu_v = \bar{Q} \gamma^\mu Q \]

**NRQCD vector current**

\[ \tilde{j}^k_v = \phi^\dagger \sigma^k \chi \]

\[ j^k_v = c_v \tilde{j}^k_v + \mathcal{O} \left( \frac{1}{M^2} \right) \]
**QCD vector current**

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\[ \tilde{j}^k_V = \phi^\dagger \sigma^k \chi \]

\[ j^k_V = c_V \tilde{j}^k_V + \frac{d_V}{6M^2} \phi^\dagger \sigma^k D^2 \chi + \cdots \]
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NRQCD vector current

\[ \tilde{j}^k_v = \phi^\dagger \sigma^k \chi \]

\[ j^k_v = c_v \tilde{j}^k_v + O \left( \frac{1}{M^2} \right) \]

c_v can be extracted by calculating vertex corrections involving \( j_v \) and \( \tilde{j}_v \)

\[ Z_2 \Gamma_v = c_v \tilde{Z}_2 \tilde{Z}_v^{-1} \tilde{\Gamma}_v + \cdots \]
full and effective theory contain the same soft, ultra-soft and potential contributions ⇒ sufficient to calculate vertex functions at threshold

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(full theory) ✓
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- wave-function renormalization (full theory) ✓
- wave-function renormalization (effective theory) \( \tilde{Z}_2 = 1 \) ✓
- renormalization of the vector current (effective theory) ✓
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wave-function renormalization
(effective theory) \( \tilde{Z}_2 = 1 \ ✓ \)

renormalization of the vector current
(effective theory) ✓
Setup of the Calculation

- Feynman diagrams generated using QGRAF
- mapped onto 78 topologies using Q2E/EXP
- Feynman integrals reduced to master integrals with CRUSHER
- master integrals in different topologies have to be identified
- \( \mathcal{O}(100) \) master integrals calculated analytically/numerically using various techniques
- numerical errors added in quadrature
Calculation of Master Integrals

- some simple (propagator-type) master integrals known analytically
- others can be calculated precisely using Mellin-Barnes methods
- difficult (vertex-type) integrals calculated numerically using FIESTA (Feynman Integral Evaluation by a Sector decomposiTion Approach)

[Smirnov, Tentyukov]
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\[ \mathcal{N} \left( + \frac{0.411236(3)}{\epsilon^2} + \frac{3.4860(1)}{\epsilon} + 34.520(2) + 339.68(4)\epsilon + \mathcal{O}(\epsilon^2) \right) \]

[Smirnov, Tentyukov]
Checks

- Renormalization constant $\tilde{Z}_v$ of the NRQCD current can be reproduced
  - $\tilde{Z}_v$ analytically known, $1/\epsilon$ part numerically small
  - agreement within the error estimate at the percent level
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Gauge independence: terms linear in $\xi$ vanish after renormalization
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Gauge independence: terms linear in $\xi$ vanish after renormalization

Change basis of master integrals and compare
Residue of the QCD two-point function

\[ \Pi(q^2) \rightarrow \sum_{E \rightarrow E_n} \frac{N_c}{2m_Q} \frac{Z_n}{E_n - (E + i0)} \]
Phenomenology

Residue of the QCD two-point function

\[ \Pi(q^2) \xrightarrow{E \rightarrow E_n} \frac{N_c}{2m_Q} \frac{Z_n}{E_n - (E + i0)} \]
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Graph showing the variation of $Z_t(\mu)/Z_t(\mu_S)$ with $\mu$ (GeV) for different orders of approximation (LO, NLO, NNLO, NNNLO), with NNNLO (ferm.) indicated specifically.
Conclusion

- Calculated the final (?) missing piece for the complete NNNLO theory prediction for $e^+e^- \rightarrow t\bar{t}$ at threshold at a future linear collider.
- Numerical result with an error in the permille range.
- Sizable corrections, but matching coefficient alone not physical
- Full analysis and final checks of other building blocks still missing.