Calculating repetitively

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- **Part 1 : How**

  &

  Part 2 : Why
- **This talk addresses a problem set in stationary planar curved space(cf.);**
- **S.Deser, R.Jackiw and G. ’tHooft, “Three-dimensional Einstein gravity: Dynamics of Flat space” Ann.Phys.120,220(1984)**
- **G.Clement, “Stationary solutions in three –dimensional general relativity”, Int.J.Theor.Phys. 24, 267(1985)**

- Consider the two integrals

$$K_0\left(\vec{p}\right) = -\frac{1}{2}\left(\frac{\lambda}{2\pi}\right)^3 \int r_1 \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{\left(\vec{r} - \vec{q}\right)^2} e^{-xr^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{\left(\vec{q} - \vec{p}\right)^2} e^{-zq^2}$$

$$K_1\left(\vec{p}\right) = -\frac{1}{2}\left(\frac{\lambda}{2\pi}\right)^3 \int q_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{\left(\vec{p} - \vec{r}\right)^2} e^{-xr^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{\left(\vec{q} - \vec{p}\right)^2} e^{-zq^2}$$

with $\int$ in each of the above being short for

$\int d^2r\, d^2q \quad , \quad q^2 = q_1^2 + q_2^2 \quad , \quad r^2 = r_1^2 + r_2^2$

and x and z being real and non - negative.

Of these only K_1 is easily evaluated as

$$K_1 = -\frac{\lambda^3}{64\pi}\frac{p_1 e^{-(x+z)p^2}}{bc^2}(-1+e^{xp^2})\left\{c\left(p_2^2 e^{zp^2} - p_1^2\right) + z\left(p_1^2 - p_2^2\right)\left(1-e^{zp^2}\right)\right\}$$

with $c = z^2 p^2, b = x^2 p^2$.For K_0 : Rewrite

$$K_0 = -\frac{1}{2}\left(\frac{\lambda}{2\pi}\right)^3 \int \delta(\vec{q}-\vec{s}) r_1 \frac{(r_2 - s_2)(r_2 s_1 - r_1 s_2)}{\left(\vec{r}-\vec{s}\right)^2} e^{-xr^2}$$

$$\times \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{\left(\vec{q}-\vec{p}\right)^2} e^{-zq^2}$$

and use

$$(2\pi)^2\,\delta(\vec{q}-\vec{s}) = \iint d\alpha\, d\beta\; e^{i\alpha(q_1-s_1)+i\beta(q_2-s_2)}$$

Completing the r and s integrations gives

$$\frac{\pi^2}{8c^2x^3}e^{-\frac{c}{x}}\left[\frac{x}{2}(\alpha^2-\beta^2)-c\alpha^2\right],\, 4c \equiv \alpha^2+\beta^2$$

And the α, β integrations lead to

$$K_0 = \pi\left(-\frac{\lambda}{4\pi x}\right)^3 \int F \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{\left(\vec{q} - \vec{p}\right)^2} e^{-z q^2}$$

$$F \equiv \left\{ \frac{x}{q^2}\left(q_2^2 - q_1^2\right) + \frac{2\left(q_2^2 - q_1^2\right)\left(-1 + e^{-xq^2}\right)}{q^4} - \frac{2xq_1^2 e^{-xq^2}}{q^2} \right\}$$

with only the integration over q remaining.

The answer here is cumbersome to display but another example is

$$16\pi^3 K_2(\vec{p}) = -\lambda^3 \int p_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{\left(\vec{p} - \vec{r}\right)^2} \times$$

$$e^{-xr^2} \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{\left(\vec{r} - \vec{q}\right)^2} e^{-zq^2}$$

;using

$$\left(2\pi\right)^2 \delta(\vec{r} - \vec{s}) = \iint d\alpha d\beta \, e^{i\alpha(r_1 - s_1) + i\beta(r_2 - s_2)}$$

one completes the q and s integration to get

- $-\frac{i\alpha}{cz^2}\left(\frac{\pi}{2}\right)^2 e^{-c/z}$ and that over α and β

- To obtain $$\left(\frac{\pi}{2}\right)^3 \frac{16r_1}{r^2 z^2}\left(1-e^{-zr^2}\right)$$

- And the final answer consists of two parts viz.

$$16\pi^3 K_2(\vec{p}) = -\frac{\pi^4\lambda^3}{2z^2 x}A - \frac{\pi^4\lambda^3}{2z^2(x+z)}B$$

- With

$$A \equiv \left\{ -\left(1 + \frac{p_1^2 - p_2^2}{xp^4} \right) e^{-x p^2} + \frac{1}{p^4} \left(2 p^2 p_2^2 + \frac{p_1^2 - p_2^2}{x} \right) + xp^2 \Gamma(0, xp^2) \right\}$$

$$B \equiv \left\{ \left(1 + \frac{p_1^2 - p_2^2}{(x+z) p^4} \right) e^{-(x+z) p^2} - \frac{1}{p^4} \left(2 p^2 p_2^2 + \frac{p_1^2 - p_2^2}{(x+z)} \right) - (x+z) p^2 \Gamma(0, (x+z) p^2) \right\}$$

Part 2 : Some References:

- F.Antonsen and K.Bormann, “ Propagators in curved space,” arXiv:hep-th/9608141v1.
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- S.G.Kamath, “A derivation of the scalar propagator in a planar model in curved space”, presented at FFP10 , AIP Conf.Proc. **1246**,174(2010)
- S.G.Kamath, “Reworking the Antonsen – Bormann idea” ,presented at FFP11 ,AIP Conf.Proc.(to appear)

- S.G.Kamath, “Operator Regularization ,scale and conformal anomalies for the Landau problem”,Mod.Phys.Lett.A**14**,1391(1999)
- S.G.Kamath, “Zeta – function regularization and scale and conformal anomalies for the Landau problem”, Mod.Phys.Lett.A **12**, 2631(1997).
- D.G.C.McKeon and T.N.Sherry, “Operator regularization and one loop Green’s functions”, Phys.Rev.D**35**,3854(1987)
- J.Schwinger, “On gauge invariance and vacuum polarization”, Phys.Rev.**82**,664(1951)

- S.Deser, R.Jackiw and G. ’tHooft, “Three-dimensional Einstein gravity: Dynamics of Flat space” Ann.Phys.**120**,220(1984)
- G.Clement, “Stationary solutions in three –dimensional general relativity”, Int.J.Theor.Phys. **24**, 267(1985)
- S.G.Kamath, “Reworking the Antonsen-Bormann idea”, Presented at QTS7,published in 2012 J.Phys.: Conf.Ser.343 012051

Consider the Lagrangian density

$$L = \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} m^2 \phi^2$$

The operator associated with L viz.

$$B \equiv -\partial_\alpha (g^{\alpha\beta} \partial_\beta) - m^2$$

can be reworked as $B = H_0 + H_I$,with

$$H_I \equiv - e_\alpha^m \partial_m (e_n^\alpha) \partial^n$$

and

$$H_0 \equiv \eta^{\mu\nu} p_\mu p_\nu - m^2$$

With the vierbeins e_α^m defined by

$$g^{\alpha\beta} = \eta^{mn} e_m^\alpha e_n^\beta \,,\, g_{\alpha\beta} = \eta_{mn} e_\alpha^m e_\beta^n \,,$$

$$\eta^{mn} = diag(1, -1, -1)$$

and the stationary solutions of the Einstein field eqns.(Deser et al., Ann.Phys.**120**,220(1984), Clement, Int.J.Theor.Phys. **24**,267(1985)) defining $g^{\mu\nu}$ as

$$g^{00} = 1 - \frac{\lambda^2}{r^2}, g^{01} = -\frac{\lambda y}{r^2}, g^{02} = \frac{\lambda x}{r^2},$$

$$g^{11} = -1, g^{12} = 0, g^{22} = -1, r = |\vec{r}|, \lambda = 4GJ$$

one can choose the vierbeins as

$$e_a^\mu : \qquad\qquad\qquad\qquad e_\mu^a :$$

$$e_0^0 = 1, \quad e_1^0 = \frac{\lambda x}{r^2}, e_2^0 = \frac{\lambda y}{r^2} \qquad e_0^0 = 0, \quad e_1^0 = -\frac{i}{\sqrt{2}}, \; e_2^0 = \frac{i}{\sqrt{2}}$$

$$e_0^1 = 0, \quad e_1^1 = 0 \quad , e_2^1 = 1 \qquad e_0^1 = -i, \quad e_1^1 = \frac{i\lambda y}{r^2} \quad , e_2^1 = -\frac{i\lambda x}{r^2}$$

$$e_0^2 = 0, \quad e_1^2 = 0 \quad , e_2^2 = 1 \qquad e_0^2 = 0, \quad e_1^2 = \frac{1}{\sqrt{2}} \quad , e_2^1 = \frac{1}{\sqrt{2}}$$

and these are time independent. With

$$H_I \equiv -e_\alpha^m \partial_m (e_n^\alpha)\, \partial^n$$

one has

- for this set

$$H_I = -\frac{\lambda}{r^4}\left((y^2 - x^2)\, p_1 - 2x\, y\; p_2\right)$$

This makes the Schwinger expansion a power series in λ; write

$$B = \eta^{\alpha\beta} p_\alpha p_\beta - m^2 + H_I$$

$$\Rightarrow \quad p_0^2 - m^2 + \vec{p}^{\,2} + H_I$$

in Euclidean space; now use the labels

$$C_0 \equiv p_0^2 - m^2, C_1 \equiv \vec{p}^{\,2} + H_I$$

The Schwinger expansion

and define after McKeon and Sherry PRD**35**, 3854 (1987)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t\, t^{s-1} e^{-Bt}$$

With

$$e^{-Bt} = e^{-(C_0 + C_1)t} = e^{-C_0 t} e^{-C_1 t}$$

since the vierbeins are time independent.

- The Schwinger expansion is now worked from

$$e^{-C_1 t} = e^{-(\vec{p}^{\,2} + H_I)t}$$

- as

$$= e^{-\vec{p}^2 t} + (-t)\int_0^1 du\, e^{-t(1-u)\vec{p}^2} H_I\, e^{-tu\vec{p}^2} + ..$$

Following McKeon and Sherry it is easy show that the matrix element of the second term above in momentum space will yield zero. The second order term is therefore of interest ,it being

$$(-t)^2 \int_0^1 u\,du \int_0^1 du_1\; e^{-t(1-u)p^2} \left\{ \int_r \langle p | H_I | r \rangle e^{-ut(1-u_1)r^2} \right.$$

$$\left. \langle r | H_I | p \rangle e^{-u\,t u_1 p^2} \right\}$$

With $\vec{r}$ a momentum vector and

$$\langle r | H_I | p \rangle = -\frac{\lambda}{4\pi} \left\{ p_1 - 2\frac{(r_2 - p_2)(r_2 p_1 - r_1 p_2)}{(\vec{r} - \vec{p})^2} \right\}$$

Only three apparently non-zero integrals contribute to the second order term above namely,

$$J_0 \equiv -2\int_r r_1 \frac{(r_2 - p_2)(r_2 p_1 - r_1 p_2)}{(\vec{r} - \vec{p})^2} e^{-u\,t(1-u_1)\vec{r}^2}$$

$$J_1 \equiv -2\int_r p_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\vec{r} - \vec{p})^2} e^{-u\,t(1-u_1)\vec{r}^2}$$

and

$$J_2 \equiv (-2)^2 \int_r \left\{ \frac{(r_2 - p_2)(r_2 p_1 - r_1 p_2)}{(\vec{r} - \vec{p})^2} e^{-u\,t(1-u_1)\vec{r}^{\,2}} \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\vec{r} - \vec{p})^2} \right\}$$

With $a = z^2 p^2, z = ut(1-u_1)$

- one gets for the Schwinger expansion to order
- λ^2 the answer

$$e^{-C_1 t} = e^{-t p^2} \left\{ \begin{aligned} & 1 + \left(\frac{\lambda t}{4\pi}\right)^2 \pi\left(p_1^2 - p_2^2\right) \times \\ & \int_0^1 u du \int_0^1 du_1 \frac{1}{a}\left[\left(1 + e^{z p^2}\right) + \frac{2z}{a}\left(1 - e^{z p^2}\right)\right] \end{aligned} \right\}$$

- The third order term in the Schwinger expansion is

$$(-t)^3 \int_0^1 u^2 du \int_0^1 u_1 du_1 \int_0^1 du_2 \left\{ e^{-t(1-u)p^2} \int_r \int_s \langle p | H_I | r \rangle e^{-tu(1-u_1)r^2} \right.$$

$$\left. \langle r | H_I | s \rangle e^{-tuu_1(1-u_2)s^2} \langle s | H_I | p \rangle e^{-tuu_1u_2p^2} \right\}$$

There are only 5 apparently non-zero terms to reckon with now, they being

$$16\pi^3 K_0 = -\lambda^3 \int_{qr} r_1 \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{(\vec{r} - \vec{q})^2} e^{-xr^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(\vec{q} - \vec{p})^2} e^{-zq^2}$$

$$16\pi^3 K_1 = -\lambda^3 \int_{qr} q_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\vec{p} - \vec{r})^2} e^{-xr^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(\vec{q} - \vec{p})^2} e^{-zq^2}$$

$$16\pi^3 K_2 = -\lambda^3 \int_{q} p_1 \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\vec{p} - \vec{r})^2} e^{-xr^2} \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{(\vec{r} - \vec{q})^2} e^{-zq^2}$$

$$8\pi^3 K_3 = \lambda^3 \int_{q} \int_{r} \left\{ \frac{(p_2 - r_2)(p_2 r_1 - p_1 r_2)}{(\vec{p} - \vec{r})^2} e^{-xr^2} \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{(\vec{r} - \vec{q})^2} \right.$$

$$\left. e^{-zq^2} \frac{(q_2 - p_2)(q_2 p_1 - q_1 p_2)}{(\vec{q} - \vec{p})^2} \right\}$$

And, the easiest of them all to calculate being

$$K_4(\vec{p}) = \left(-\frac{\lambda}{4\pi}\right)^3 (-2)\int e^{-xr^2} \frac{(r_2 - q_2)(r_2 q_1 - r_1 q_2)}{\left(\vec{r} - \vec{q}\right)^2} q_1 p_1 e^{-zq^2}$$

$$= \frac{1}{2\pi}\left(\frac{\lambda}{4}\right)^3 \frac{p_1}{xz(x+z)}$$

Thank you