3D Ising Model
and
Conformal Bootstrap

Slava Rychkov

(ENS Paris & CERN)

Lecture 4

CERN Winter School on Supergravity, Strings, and Gauge Theory 2013

see also "*Lectures on CFT in D≥3*" @ sites.google.com/site/slavarychkov

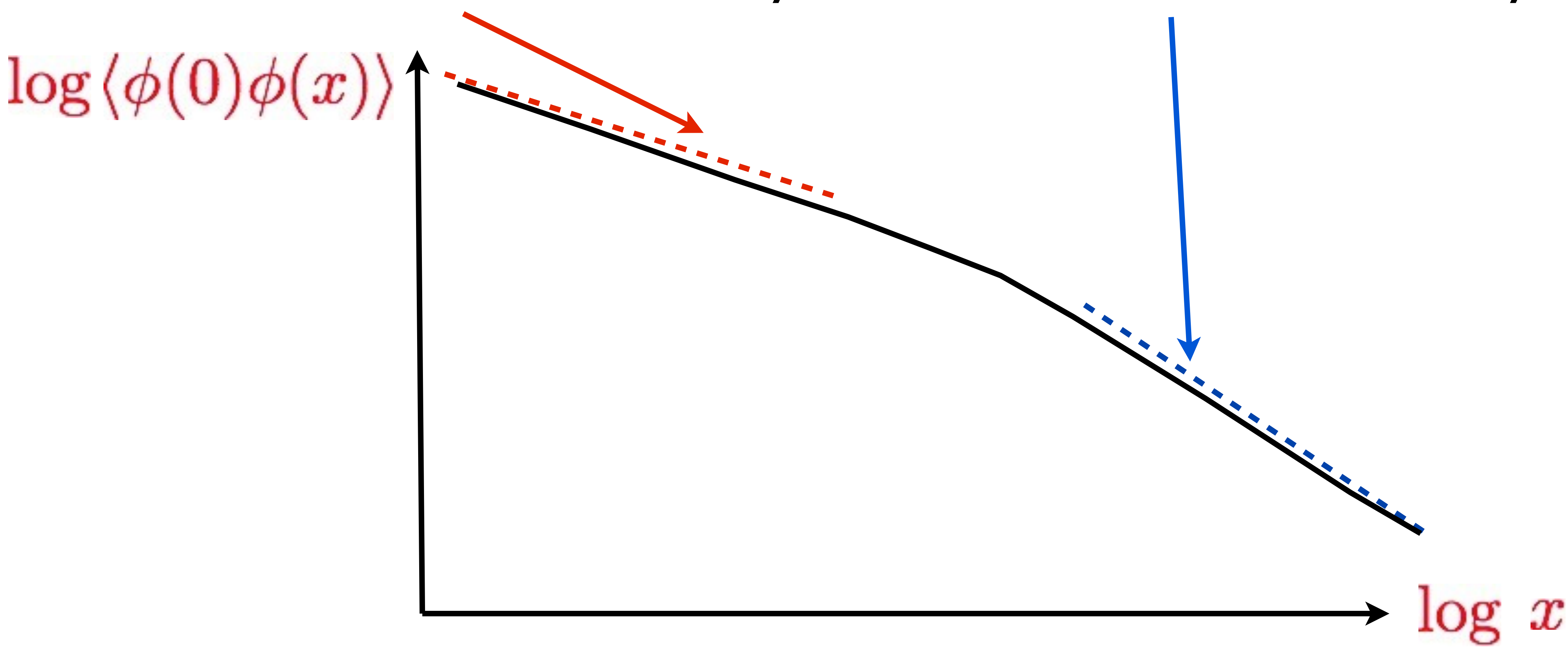
Recap of Lectures 1-3

1. RG flows

a) take CFT_{UV}
b) perturb by a relevant scalar operator
c) flow to CFT_{IR}

Problem for future CERN String schools:
find a method to solve any flow

To solve a flow means to find how every correlation function interpolates between the UV tail described by CFT_{UV} and IR tail described by CFT_{IR}



Recap of Lectures 1-3

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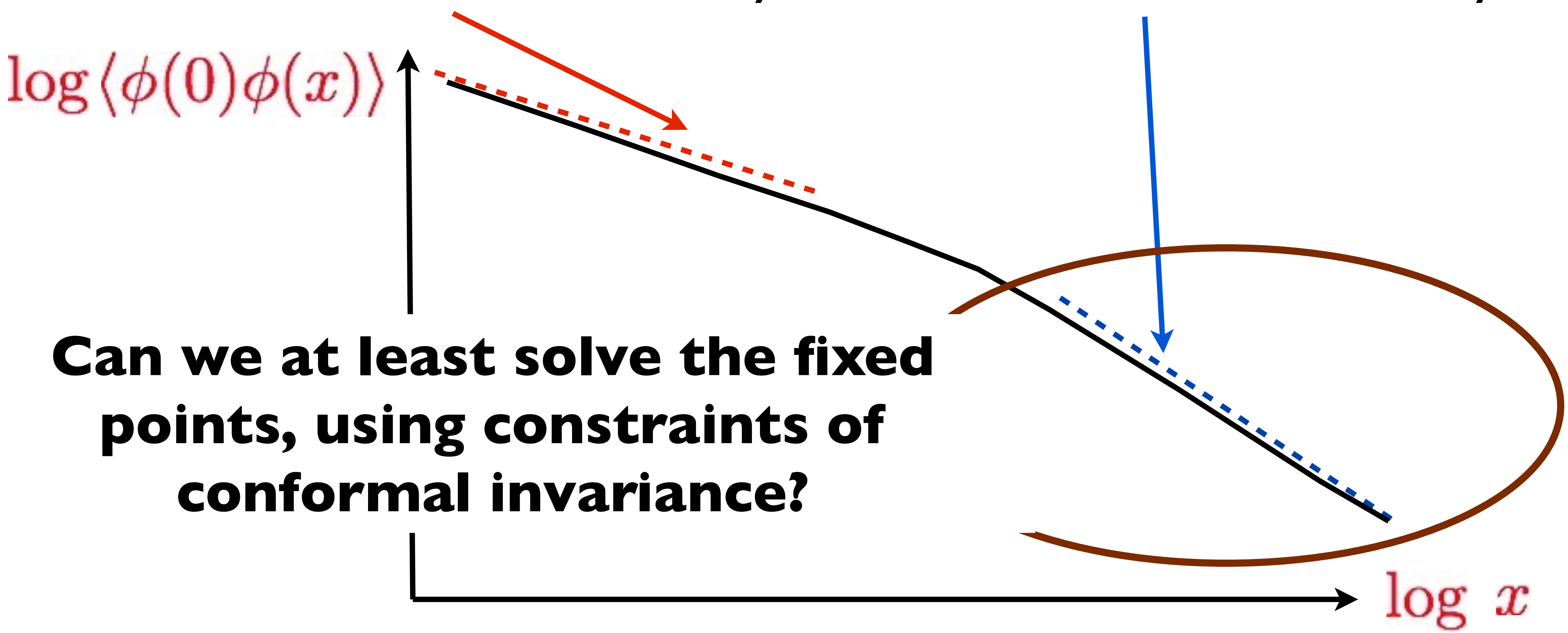


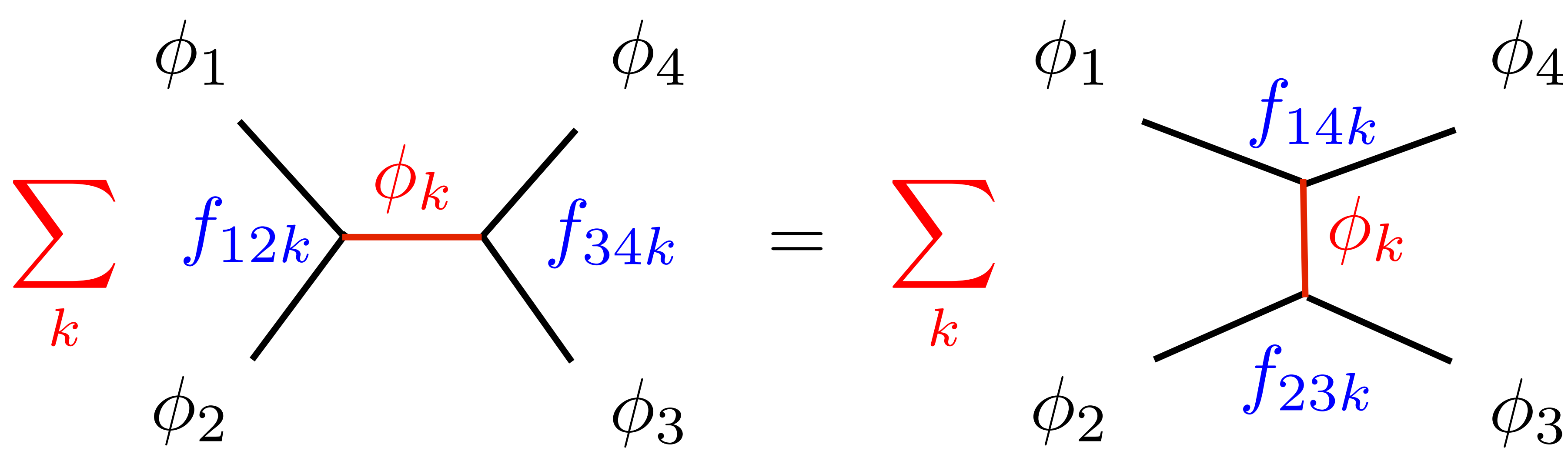
Can we at least solve the fixed points, using constraints of conformal invariance?

2. “Axiomatic definition of CFT”

Think in terms of local operators forming multiplets under
1) conformal algebra (primary + its derivatives)
2) global symmetry group (if any)

Relevant parameters= $\underbrace{\text{spectrum } \{\Delta_i, \ell_i\} \quad + \text{ OPE coeffs } f_{ijk}}_{\text{Conformal data}}$

Subject to the condition of crossing symmetry:



E.g. for 3D Ising some dimensions are approximately known from other techniques:

Operator	Spin l	$\mathbb{Z}_2$	Δ	Exponent
σ	0	$-$	0.5182(3)	$\Delta = 1/2 + \eta/2$
σ'	0	$-$	$\gtrsim 4.5$	$\Delta = 3 + \omega_A$
ε	0	$+$	1.413(1)	$\Delta = 3 - 1/\nu$
ε'	0	$+$	3.84(4)	$\Delta = 3 + \omega$
ε''	0	$+$	4.67(11)	$\Delta = 3 + \omega_2$
$T_{\mu\nu}$	2	$+$	3	n/a
$C_{\mu\nu\kappa\lambda}$	4	$+$	5.0208(12)	$\Delta = 3 + \omega_{\mathrm{NR}}$

Can we do better using conformal invariance?

Simplest bootstrap setup

4-pt function of the spin field:

$$\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\sigma}|x_{34}|^{2\Delta_\sigma}}$$

Conformal block decomposition:

$$g(u,v) = 1 + \sum f_i^2\, G_{\Delta_i,\ell_i}(u,v)$$

"known" functions

Crossing symmetry constraint:

$$v^{\Delta_\sigma} g(u,v) = u^{\Delta_\sigma} g(v,u)$$

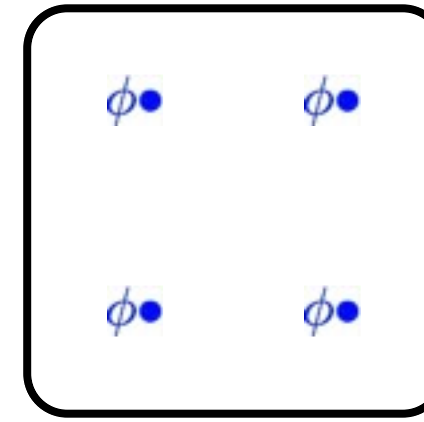
$$\Updownarrow$$

$$\sum f_i^2 [v^{\Delta_\sigma} G_{\Delta,l}(u,v) - v \leftrightarrow u] = u^{\Delta_\sigma} - v^{\Delta_\sigma}$$

Numerical bootstrap [Rattazzi,S.R,Tonni,Vichi,2008]

Taylor-expand around the square configuration

$$\sum f_i^2[v^{\Delta_\sigma} G_{\Delta,l}(u,v) - v \leftrightarrow u] = u^{\Delta_\sigma} - v^{\Delta_\sigma}$$



$$\sum f_i^2 \vec{R}_{\Delta_i,\ell_i} = \vec{R}_0$$

vectors of derivatives around $z = \bar{z} = 1/2$ [O(100) components]

- truncate to $\Delta \leq \Delta_{\max} = O(50)$
- allow all dimensions in the interval $[\ell + D - 2, \Delta_{\max}]$

discretized with $\delta\Delta = O(0.01)$

- get a system of linear equations for O(10^4) unknowns $f_i^2 \geq 0$

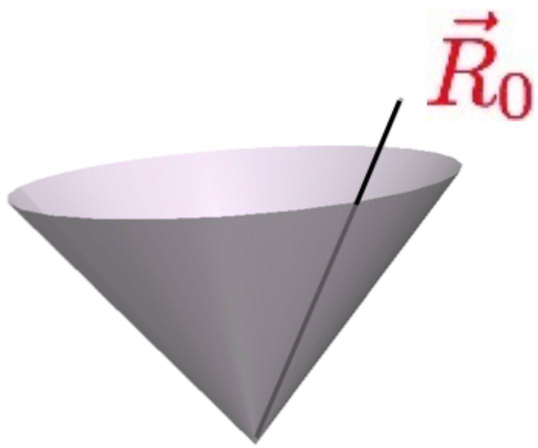
Problem of linear algebra for which efficient algorithms exist
(e.g. simplex method; look it up in *Numerical recipes*)

Geometric interpretation

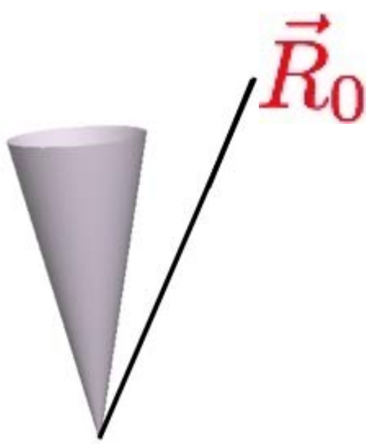
The set $$\left\{\sum f_i^2 \vec{R}_{\Delta_i,\ell_i} \ : \ f_i^2 \geq 0\right\}$$ is a **convex cone**

The bootstrap equation has a solution iff this cone contains vector R_0

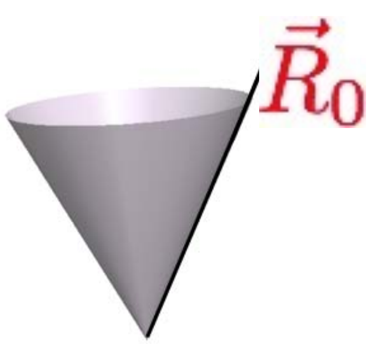
There is a solution (not unique):



Now reduce the basis
(e.g. by raising the threshold for scalars in the σ x σ OPE)
Cone shrinks and might eventually arrive at no solution:



Critical situation; the solution exists and is unique:

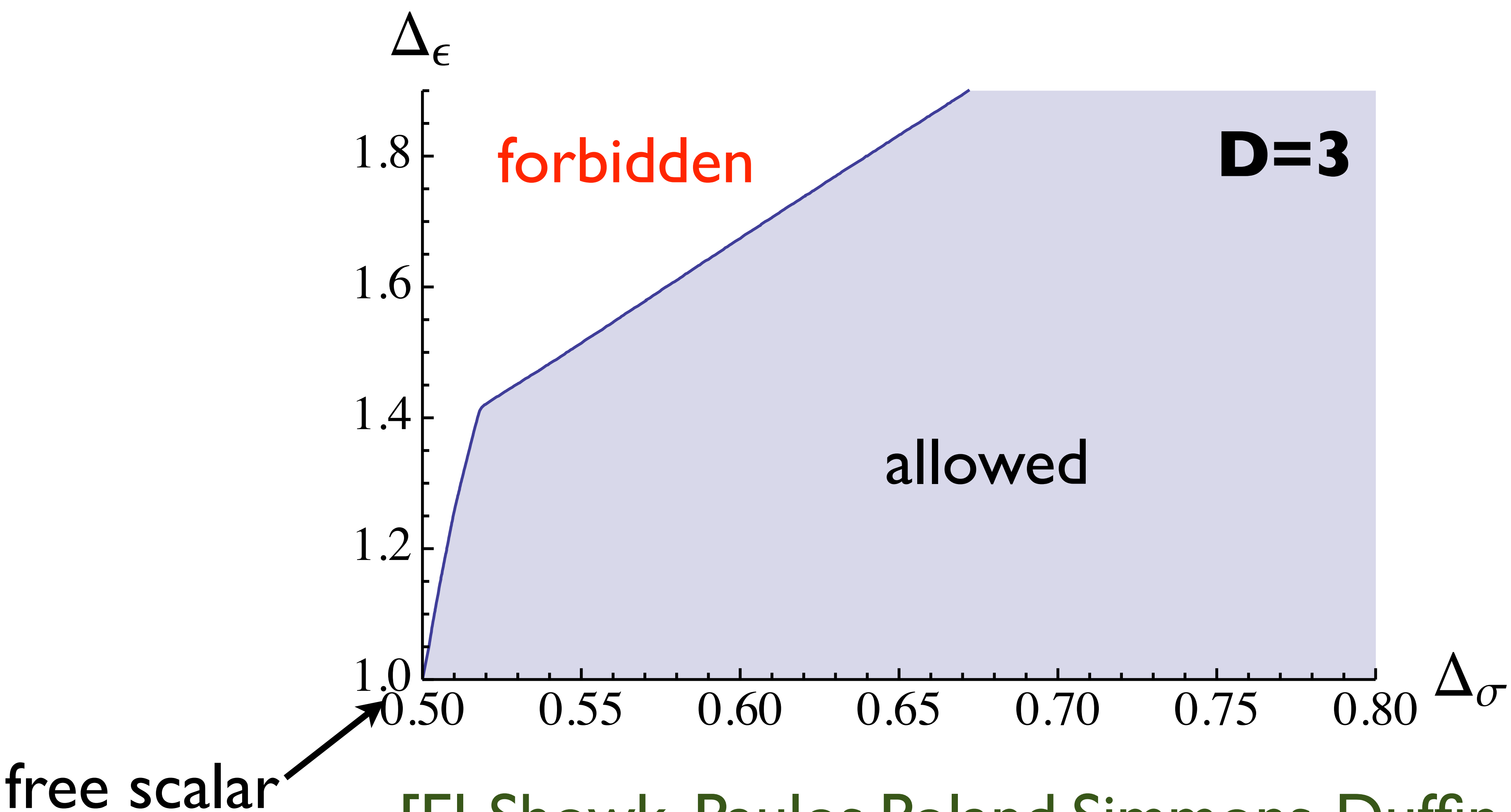


Constraints on Dims of σ and ε via conf. bootstrap.

Δ_σ enters the equations as the parameter in $v^{\Delta_\sigma} g(u,v) = u^{\Delta_\sigma} g(v,u)$

Δ_ε enters as the lower threshold on the allowed scalars in σ x σ

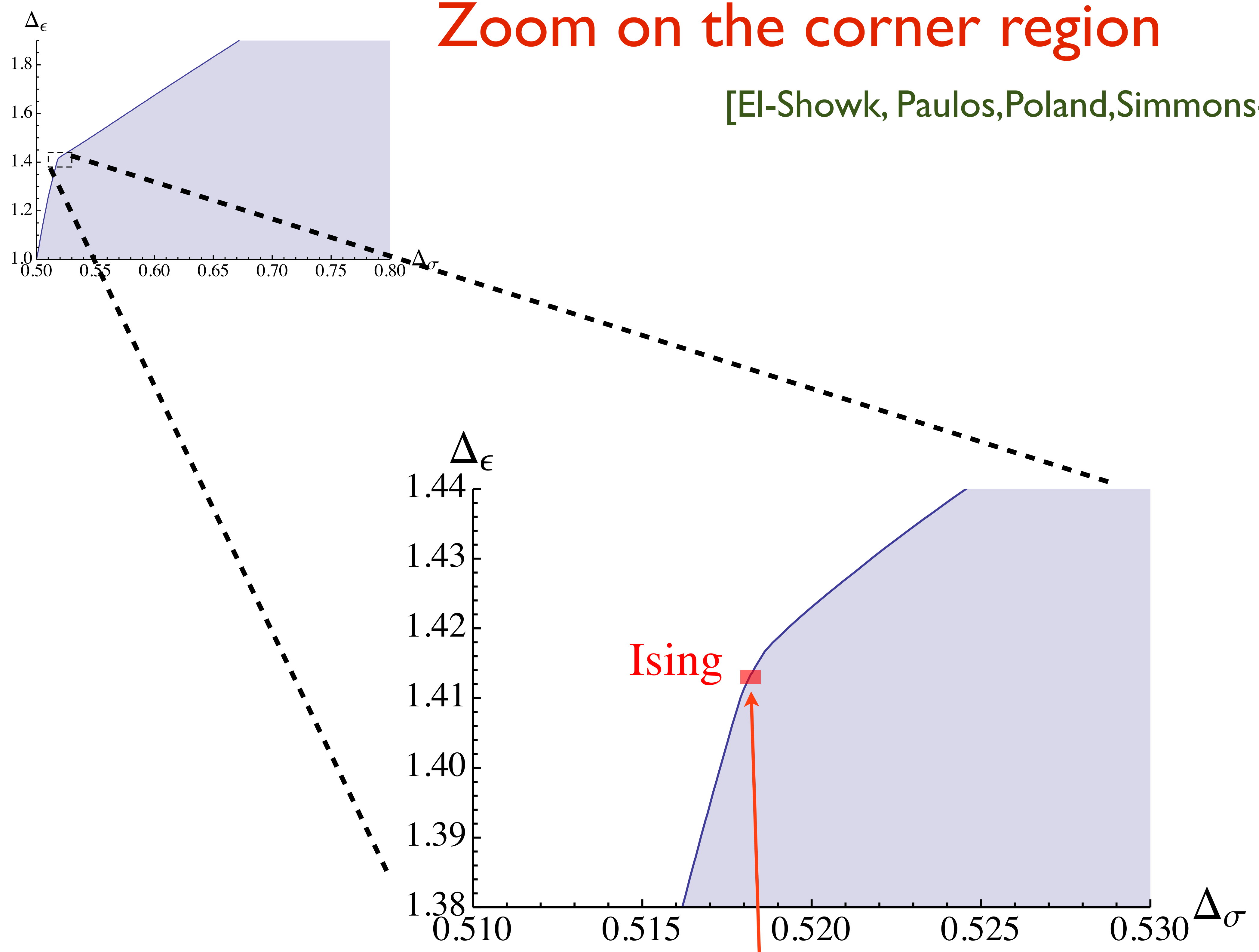
As Δ_ε is increased, cone shrinks; might eventually run out of solutions.
Numerical analysis ⇒ this indeed happens; gives a lower bound on Δ_ε as $f(\Delta_\sigma)$:



[El-Showk, Paulos,Poland,Simmons-Duffin,S.R.,Vichi'2012]

Zoom on the corner region

[El-Showk, Paulos,Poland,Simmons-Duffin,S.R.,Vichi'2012]

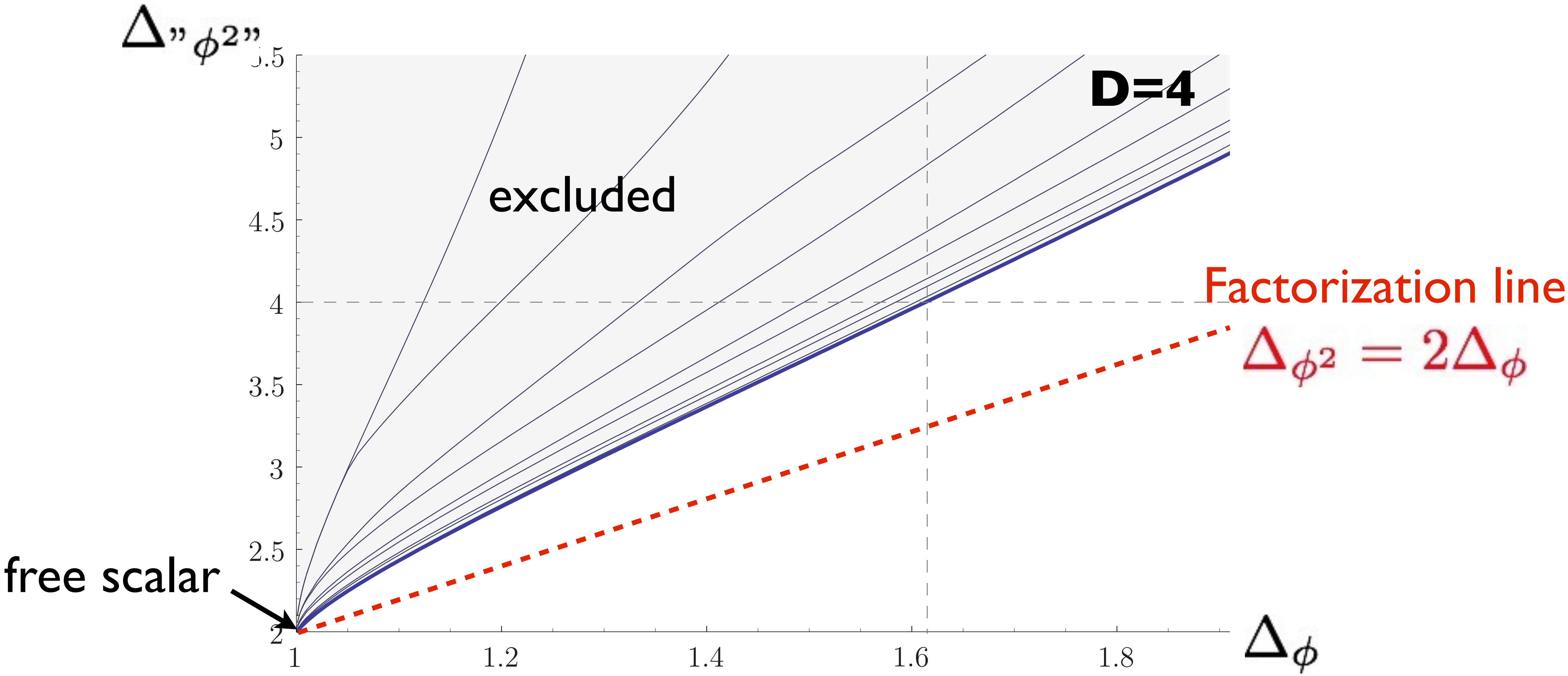


best available determination of 3D Ising dims by ε-expansion and other RG techniques

Analogous plot in D=4

Rattazzi, S.R., Tonni, Vichi 2008
S.R., Vichi 2009
…, Poland, Simmons-Duffin, Vichi 2011

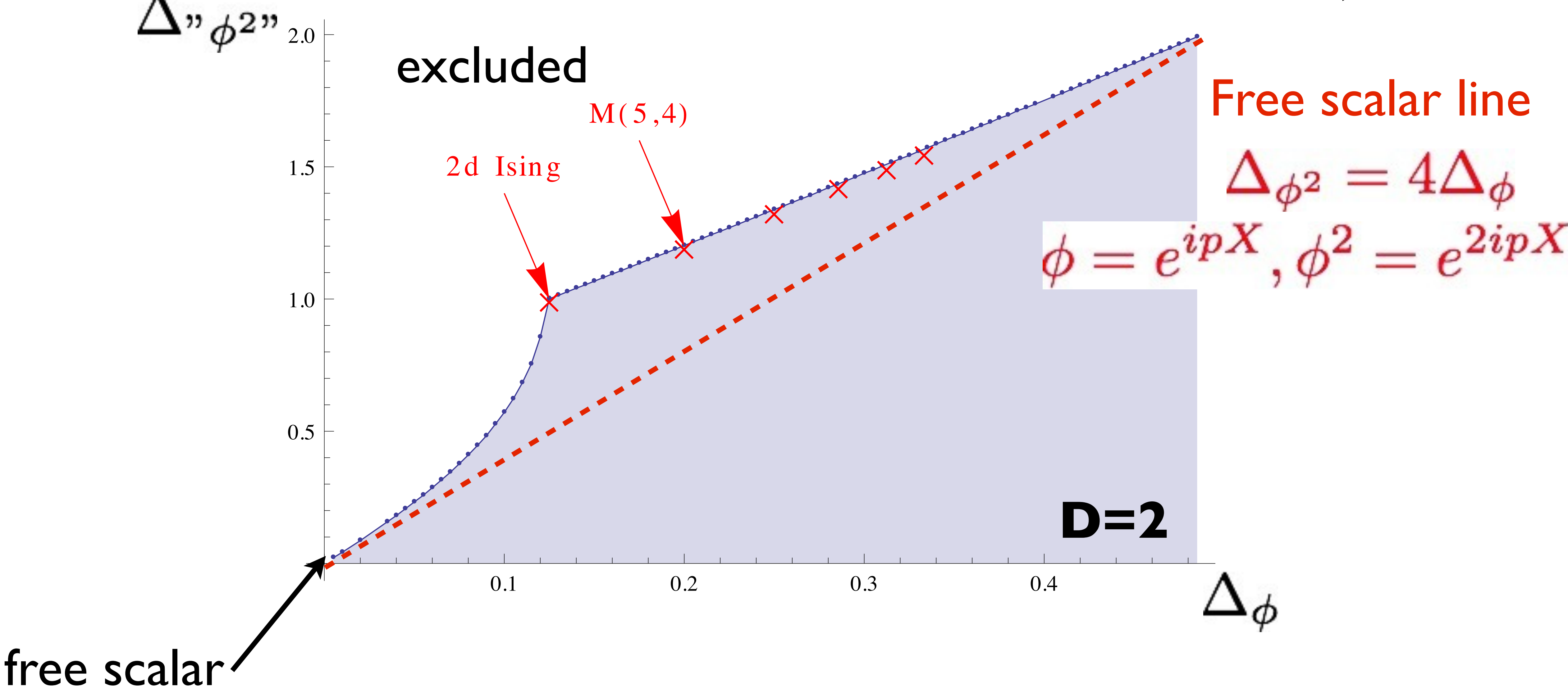
$$\phi \times \phi = 1 + "\phi^2" + \ldots$$



No interesting theories are known to saturate this bound

Analogous plot in D=2

S.R., Vichi 2009
El-Showk, Paulos 2012



Conjecture

2D and 3D Ising models correspond to very special solutions of conformal bootstrap

(a) they maximize Δ_ε for fixed Δ_σ
(b) the bound has a kink at the Ising vale of Δ_σ

- Probably true for all Wilson-Fisher fixed points in $2 \leq D < 4$

- Opens the way to the determination of the full Ising spectrum (the solution on the boundary is unique!)

Future problems 1. Ising-related

1. What is the origin of kink? (partially understood)

2. Recover full spectrum along the boundary and at the kink

   - By analyzing the spectrum for unexpected degeneracies hope to get evidence for integrability (or not)

[El-Showk, Paulos, Poland, Simmons-Duffin, S.R., Vichi, work in progress]

3. Look at several correlation functions simultaneously, e.g.

$$\langle\sigma\sigma\sigma\sigma\rangle \qquad \langle\epsilon\epsilon\sigma\sigma\rangle \qquad \langle\epsilon\epsilon\epsilon\epsilon\rangle$$

Future problems I. General

1. SUSY - way to learn about unprotected operators
& isolated theories (like (2,0) in D=6)

2. Crossing for Non-scalar external operators
- Stress tensor, currents, fermions

3. Bootstrap in presence of the boundary and for defect CFTs
[Liendo, Rastelli, van Rees 2012]

4. Bootstrap for 2D CFTs for c>1 using "long" conformal blocks
- compute them via Al. Zamolodchikov's recursion relations

5. Bootstrap on the lightcone
[Fitzpatrick,Kaplan,Poland,Simmons-Duffin'2012]
[Komargodski,Zhiboedov'2012]

6. Analytic methods?

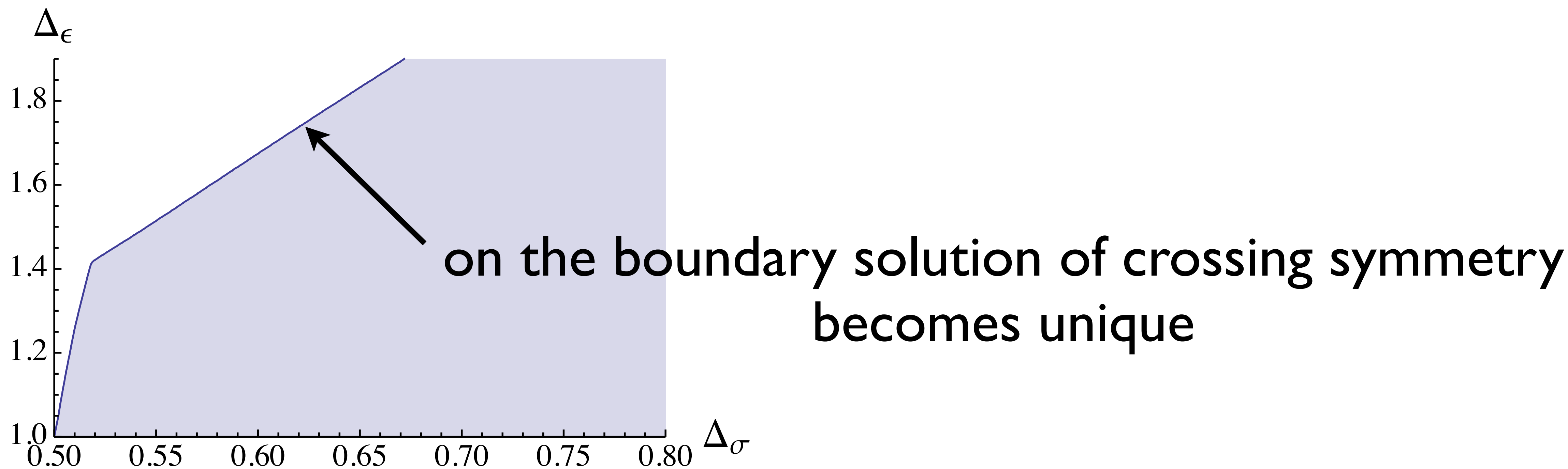
7. Bootstrap as a machine for generating ε-expansion?

BOOTSTRAP

IT WORKS !

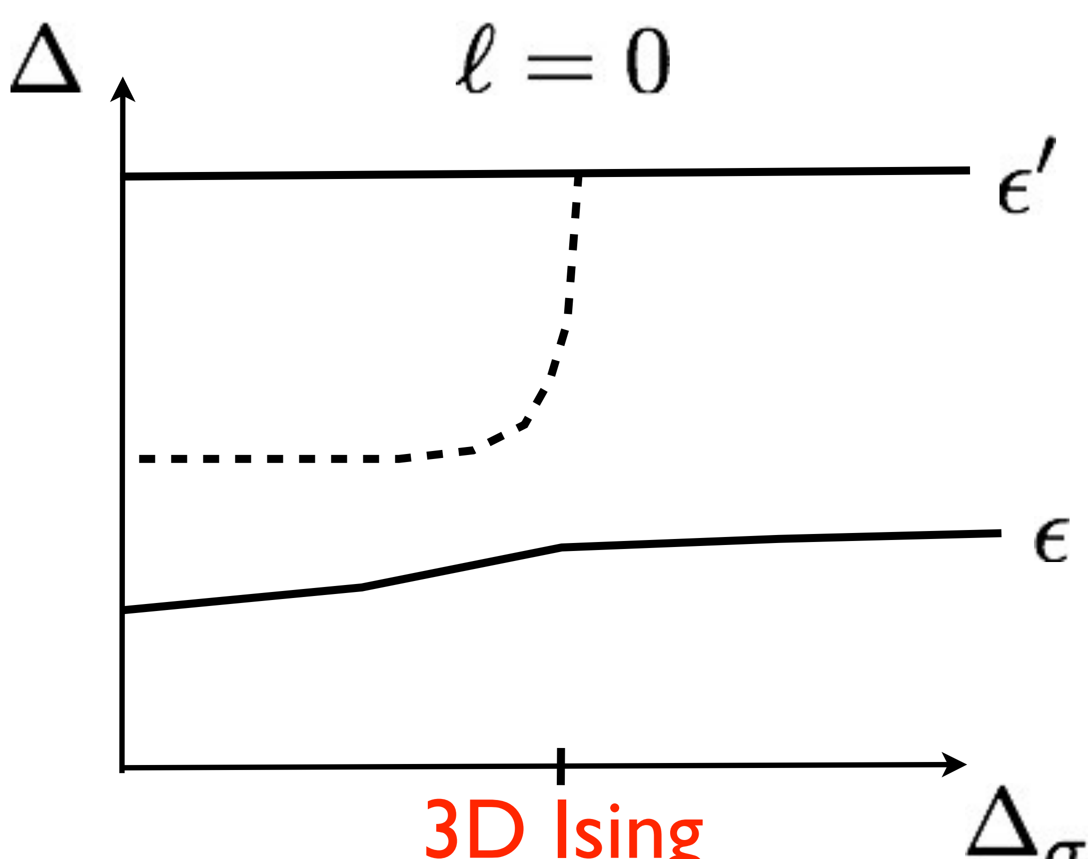
Backup slides

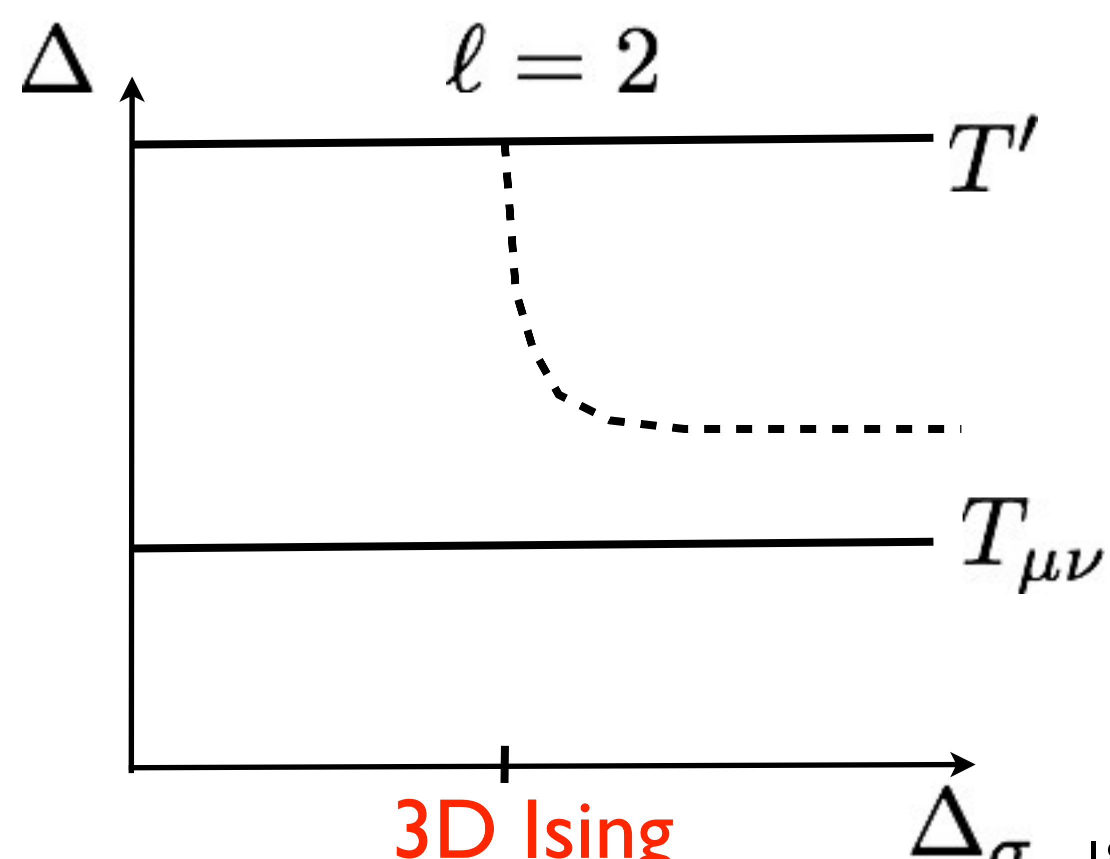
Full spectrum on the boundary



Instructive to compute and plot spectrum as f(Δ_σ)

Schematically:





Bootstrap is not limited for 3D Ising

Bootstrap in SUSY theories is one of the few tools to learn about the unprotected quantities

E.g. in 6D (2,0) theory or in N=4 SYM away from large N

Take the lowest dimension chiral primary $\mathcal{O} = \mathrm{Tr}[\Phi^{\{a}\Phi^{b\}}] \in 20'$

$$\mathcal{O} \times \mathcal{O} \supset \mathrm{Konishi}$$

- Bootstrap likely puts an upper bound (λ-independent) on Konishi dim

  from experience with non-SUSY with continuous global sym.
  and N=1 SUSY bootstrap [Rattazzi,S.R.,Vichi'2010]
  [Poland,Simmons-Duffin,2010]
  [Poland,Simmons-Duffin,Vichi,2011],...

- Analogously for 6D (2,0) theory...