

Progress in $N=2$ theories

In the last few years, big progress in "exact results in $N=2$ theories."

My aim: Describe one aspect of this progress — new methods for computing the spectrum of BPS states.

- (1) What is a BPS state? What is the "Kontsevich-Sokolman wall-crossing formula"?
- (2) How to understand/prove the KS formula — using supersymmetric line defects
- (3) How to determine the BPS spectrum using SUSY surface defects.

Most of the story I will tell is part of work of Gaiotto, Moore.

[\exists another parallel approach using quiver quantum mech
[Denef, Cecotti-Vafa et al.]]

$N=2$ QFT in $d=4$

$N=2$ theories typically have a cts moduli space of vacua.

Focus on one branch, "Coulomb branch".

Ex $N=2$ SYM with gauge gp $G = SU(2)$.

Fields: $\left. \begin{array}{l} \text{gauge field } A^\mu \\ \text{C scalar } \varphi \\ \text{fermions } \psi \end{array} \right\}$ all in adjoint of $su(2)$.

Action contains a potential $V = \text{Tr} [\varphi, \varphi^\dagger]^2$ which has flat directions:

take $\varphi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, then $V(\varphi) = 0$. $a \sim -a$

So classically we have a moduli space of vacua,
parameterized by $u := \text{Tr} \varphi^2 = 2a^2$.

If $a \neq 0$ then the gauge symmetry is Higgsed:
at low energies find $U(1)$ gauge theory, not $SU(2)$.

• This is what happens generically: Coulomb branch \mathcal{B} is a complex manifold
and the IR physics at a generic point $u \in \mathcal{B}$ is abelian gauge theory,
gauge group $U(1)^r$.

• States carry electric+magnetic charges: charge lattice $\Gamma \simeq \mathbb{Z}^{2r}$
(plus possible flavor charges, neglect these)

Hilbert space $\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$, 1-particle subspace $\mathcal{H}^1 = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma^1$

• SUSY algebra: $so(3,1) \oplus \mathbb{C} \oplus$ (odd part)

Odd part generated by spinors Q_α^A : $\alpha = 1, 2$ $A = 1, 2$

$$\{Q^A, \bar{Q}^B\} = \delta^{AB} P$$

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = \delta^{AB} P^\mu \Gamma_{\mu\alpha\beta}$$

$$\{Q^A, Q^B\} = \varepsilon^{AB} Z$$

$$\{Q_\alpha^A, Q_\beta^B\} = \varepsilon^{AB} \varepsilon_{\alpha\beta} Z$$

Z is the "central charge": it acts by a scalar $Z_\gamma \in \mathbb{C}$ on \mathcal{H}_γ

with $Z_\gamma + Z_{\gamma'} = Z_{\gamma+\gamma'}$

Let's study representations.

Consider massive 1-particle states, at rest: $P = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\mathcal{H}'_{rest} \subset \mathcal{H}' \subset \mathcal{H}$

\mathcal{H}'_{rest} Invariant under $SO(3)$ "little group".

Fix $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ and define

$$\begin{aligned} R_\vartheta &= \frac{1}{\sqrt{2}} \left[e^{-i\vartheta/2} Q' - e^{i\vartheta/2} \bar{Q}' \right] \\ T_\vartheta &= \frac{1}{\sqrt{2}} \left[e^{-i\vartheta/2} Q' + e^{i\vartheta/2} \bar{Q}' \right] \end{aligned} \Rightarrow \left[\begin{aligned} \{R, \bar{R}\} &= (M - \operatorname{Re}(e^{i\vartheta} Z)) \\ \{T, \bar{T}\} &= (M + \operatorname{Re}(e^{i\vartheta} Z)) \\ \{R, \bar{T}\} &= \operatorname{Im}(e^{i\vartheta} Z) \\ \{\bar{R}, T\} &= \operatorname{Im}(e^{i\vartheta} Z) \end{aligned} \right]$$

Choose $\vartheta = -\arg Z$, then have $\{R, \bar{R}\} = M - |Z|$ all other brackets vanish.
 $\{T, \bar{T}\} = M + |Z|$

2 Clifford algebras. Each has a 4-dimensional representation, with $SO(3)$ content $2[0] \oplus [\frac{1}{2}]$.

NB: $\langle \psi | \{R, \bar{R}\} | \psi \rangle = (M - |Z|) \|\psi\|^2$

$$\|R\psi\|^2 + \|\bar{R}\psi\|^2 \geq 0$$

So $M \geq |Z|$.

Moreover, if $M = |Z|$ then $R\psi = \bar{R}\psi = 0$.

So, states at rest come in 2 different types:

• $M > |Z|$: "long representations" —

• $M = |Z|$ "short rep" —

$$\begin{aligned} [j] \otimes \underbrace{(2[0] \oplus [\frac{1}{2}])}_{\text{from } T} \otimes \underbrace{(2[0] \oplus [\frac{1}{2}])}_{\text{from } R} &= L_j \\ [j] \otimes (2[0] \oplus [\frac{1}{2}]) &= S_j \end{aligned}$$

The states with $M=|Z|$ are called "BPS states."
 They have the minimum energy consistent with their charges.

$$\mathcal{H}_{\text{BPS}} \subset \mathcal{H}$$

Can we determine the BPS states?

}

Q: How does \mathcal{H}_{BPS} depend on the parameters of the theory?

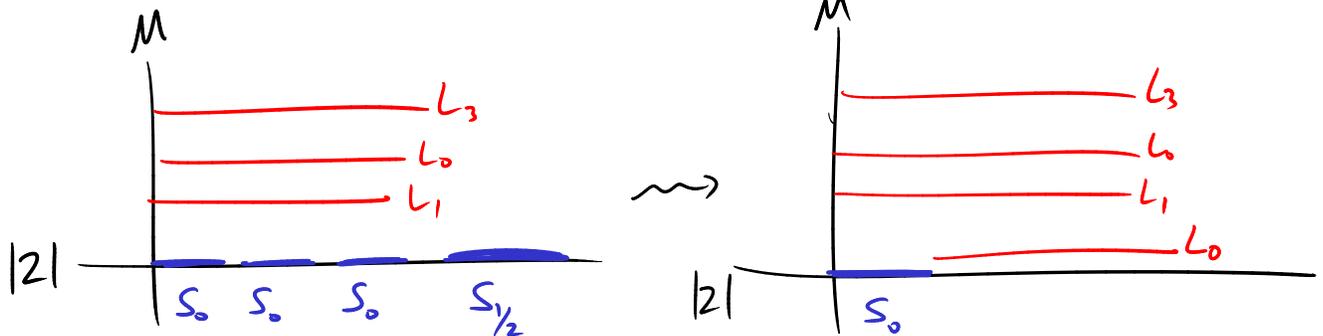
$$\mathcal{H}_{\text{BPS}}(t, u)$$

↑ coupling constants ↑ IR vacuum choice

2 ways \mathcal{H}_{BPS} could change:

① States could be "lifted": move continuously from $M=|Z|$ to $M>|Z|$.

Can only happen in way consistent with $SO(3)$ symm.



$$S_0 \oplus S_0 \oplus S_{1/2} \simeq L_0$$

To sidestep this problem: define "helicity supertrace"

$$\Omega(\gamma) = -\frac{1}{2} \text{Tr}_{\mathcal{H}'_{\gamma, \text{rest}}} (J_3)^2 (-1)^{2J_3}$$

This receives 0 contribution from the L_j .

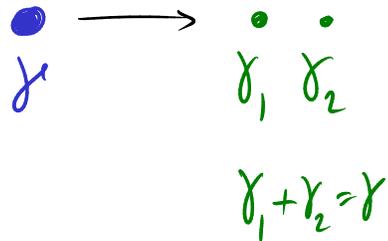
It counts S_j with weight

$$(-1)^{2j} (2j+1)$$

e.g. $+1$ for S_0 (massive hyper)
 -2 for $S_{1/2}$ (massive vector)
 \vdots

② BPS States could decay.
 When can this happen?

Consider 2-particle decay:



$$|Z_{\gamma_1 + \gamma_2}| = |Z_\gamma| = M \geq M_1 + M_2 \geq |Z_{\gamma_1}| + |Z_{\gamma_2}|$$

$$\text{But, } \Delta_{\text{neg}} \Rightarrow |Z_{\gamma_1 + \gamma_2}| \leq |Z_{\gamma_1}| + |Z_{\gamma_2}|$$

$$\text{So, all } \geq \text{ must be } = : \quad M_1 = |Z_{\gamma_1}| \quad |Z_{\gamma_1 + \gamma_2}| = |Z_{\gamma_1}| + |Z_{\gamma_2}| \\ M_2 = |Z_{\gamma_2}|$$

So: \cdot the decay products are BPS

\cdot $\arg(Z_{\gamma_1}) = \arg(Z_{\gamma_2})$ (i.e. the decay products are mutually BPS)

Similar constraint for multiparticle decays: $\arg(Z_{\gamma_1}) = \dots = \arg(Z_{\gamma_n})$

So: $\Omega(\gamma)$ is well defined and deformation invt, except
 at places in param. space where $\gamma = \gamma_1 + \gamma_2$
 $\arg(Z_{\gamma_1}) = \arg(Z_{\gamma_2})$

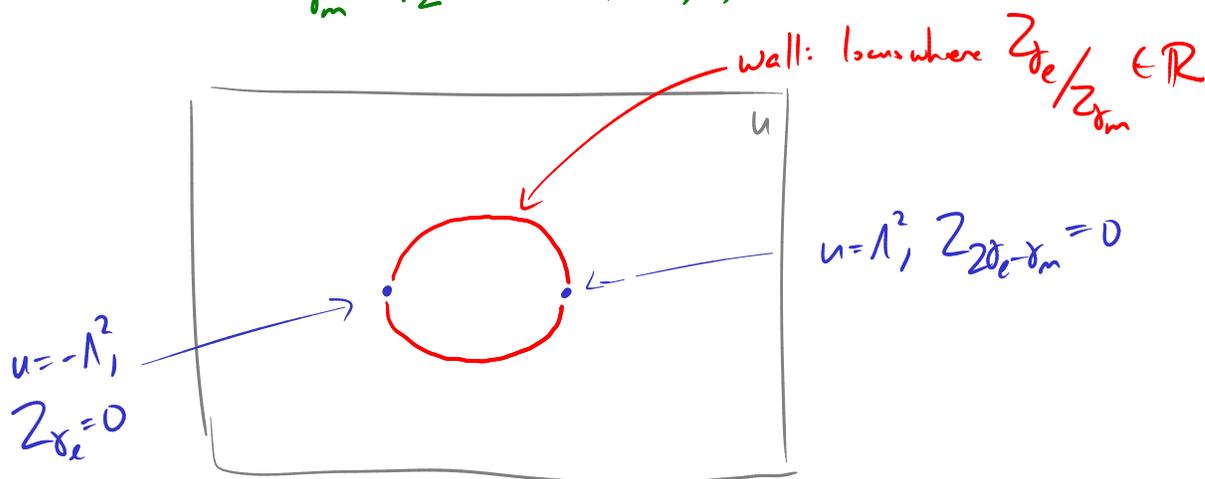


E_x $\mathcal{N}=2$ SYM, $G=SU(2)$

\mathcal{B} = complex plane, coordinate $u = \langle \text{Tr } \varphi^2 \rangle$. $\Lambda = \text{QCD scale}$

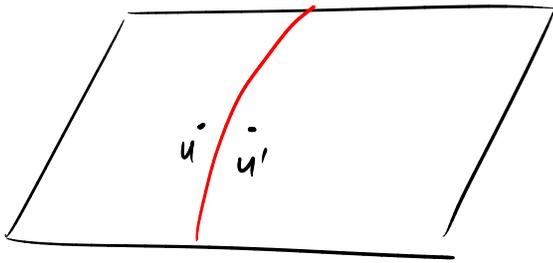
Expand charges $\gamma = q\gamma_e + p\gamma_m$
 \uparrow \uparrow
 "electric" "magnetic"

Then $Z_{\gamma_e} = \frac{i}{4} \Lambda (\alpha-1) {}_2F_1\left(\frac{3}{4}, \frac{3}{4}, 2; 1-\alpha\right)$ $\left(\alpha = \frac{u^2}{\Lambda^4}\right)$ [Seiberg-Witten]
 $Z_{\gamma_m} = \frac{1}{\sqrt{2}} \Lambda \alpha^{1/4} {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}, 1; \alpha\right)$



At large $|u|$ have $\Omega(\pm 2\gamma_e) = -2$ W bosons
 $\Omega(n\gamma_e \pm \gamma_m) = 1$ ($n \in \mathbb{Z}$) (monopoles + dyons)
 other $\Omega(\gamma) = 0$

At small $|u|$ have $\Omega(\pm \gamma_m) = 1$ monopole
 $\Omega(\pm (2\gamma_e - \gamma_m)) = 1$ dyon [Batal-Ferrari]
 other $\Omega(\gamma) = 0$



Wall-crossing: if we know all $\Omega(\gamma)_{\gamma \in T}$ at u , to determine all $\Omega(\gamma)$ at u' .

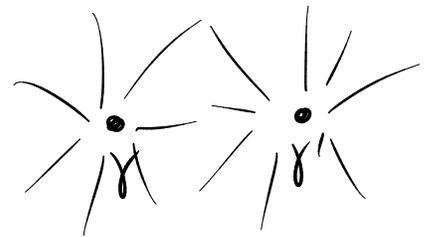
Wall-crossing formula

Written by Kontsevich-Sokalman in context of "generalized Donaldson-Thomas invariants."

T charge lattice

$$\langle, \rangle : T \times T \rightarrow \mathbb{Z} \quad \text{"DSZ pairing"}$$

(e.g. for $r=1$, $\langle \underset{(p,q)}{\gamma}, \underset{(p',q')}{\gamma'} \rangle = pq' - q'p'$)

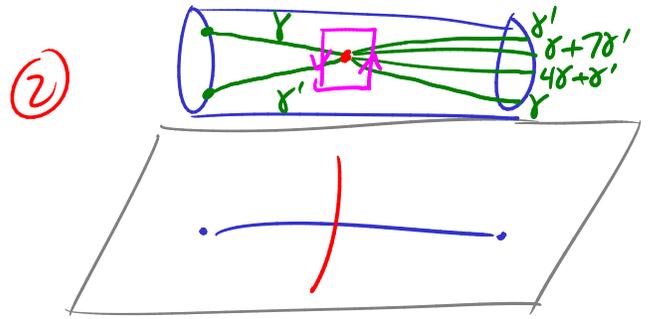
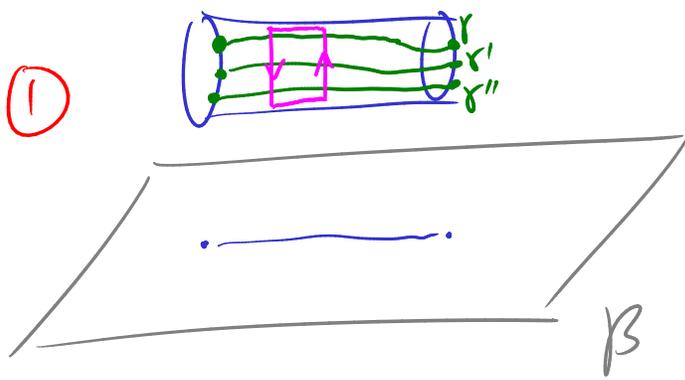


Consider an algebra $X_{\gamma} X_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} X_{\gamma + \gamma'}$

For each $\gamma \in T$, Define an automorphism K_{γ} of this algebra (change of variables):

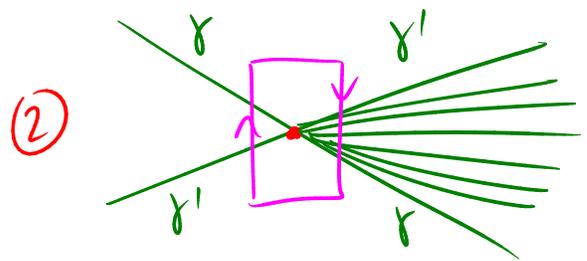
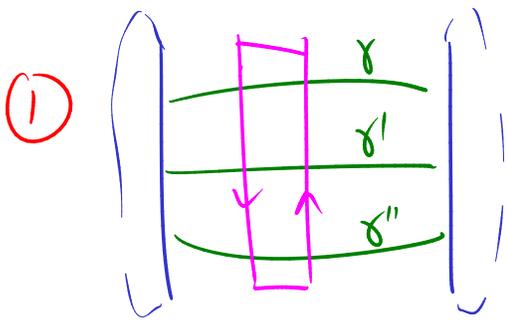
$$K_{\gamma}: X_{\gamma'} \rightarrow X_{\gamma'} (1 - X_{\gamma})^{\langle \gamma', \gamma \rangle}$$

Now, consider $\beta \times S^1$. On this space, mark a codim-1 locus for each charge γ with $\Omega(\gamma) \neq 0$, $C_{\gamma} = \{(u, \arg -Z_{\gamma}(u))\}$



Now, consider any closed loop $P \subset B \times S^1$.

Define $A(P) = \prod_{P \cap (U \times \gamma)} K_\gamma^{\pm \Omega(\gamma)}$ (sign depends on which way we cross)



$$A(P) = K_{\gamma''} K_{\gamma'} K_\gamma K_\gamma^{-1} K_{\gamma'}^{-1} K_{\gamma''}^{-1} \quad A(P) = K_{\gamma'} K_\gamma K_{\gamma'}^{-1} \prod_{\substack{m, n \\ m, n > 0}} K_{m\gamma + n\gamma'}^{-\Omega(m\gamma + n\gamma')} K_\gamma^{-1}$$

KS wall-crossing formula: $A(P) = \mathbb{1}$ for any contractible P !

So, in ② above, $K_{\gamma'} K_\gamma = K_\gamma \left[\prod_{\substack{m, n \\ m, n > 0}} K_{m\gamma + n\gamma'}^{\Omega(m\gamma + n\gamma')} \right] K_{\gamma'}^{-1}$

This determines all the $\Omega(\gamma)$ on the RHS!

Reduces wall-crossing to purely algebraic exercise.

Ex If $\langle \gamma, \gamma' \rangle = 1$ then

$$\underbrace{K_{\gamma'} K_{\gamma}}_{2 \text{ BPS hypermultiplets}} = \underbrace{K_{\gamma} K_{\gamma+\gamma'} K_{\gamma'}}_{3 \text{ BPS hypermultiplets}}$$

$$\Omega(\gamma) = \Omega(\gamma') = 1 \quad \Omega(\gamma) = \Omega(\gamma') = \Omega(\gamma+\gamma') = 1$$

e.g. act with both sides on X_{γ} :

LHS: $X_{\gamma} \xrightarrow{K_{\gamma}} X_{\gamma} \xrightarrow{K_{\gamma'}} (1 - X_{\gamma'}) X_{\gamma} = X_{\gamma} + X_{\gamma+\gamma'}$

RHS: $X_{\gamma} \xrightarrow{K_{\gamma'}} X_{\gamma} + X_{\gamma+\gamma'} \xrightarrow{K_{\gamma+\gamma'}} X_{\gamma} + X_{2\gamma+\gamma'} + X_{\gamma+\gamma'}$
 $\xrightarrow{K_{\gamma}} X_{\gamma} + (1 - X_{\gamma})^{-1} (X_{2\gamma+\gamma'} + X_{\gamma+\gamma'})$
 $= X_{\gamma} + X_{\gamma+\gamma'}$

(This one occurs e.g. in $SU(2)$ $N_f=1$...)

Ex If $\langle \gamma, \gamma' \rangle = 2$ then

$$K_{\gamma'} K_{\gamma} = \left(\prod_{n=1}^{\infty} K_{n\gamma + (n-1)\gamma'} \right) K_{\gamma+\gamma'}^{-2} \left(\prod_{n=0}^{\infty} K_{(n-1)\gamma + n\gamma'} \right) \quad [\text{KS}]$$

This is just what we need for $N=2$ SYM, $G = SU(2)$...

$$\underbrace{K_{\gamma'} K_{\gamma}}_{\substack{\uparrow \quad \uparrow \\ \text{dyon} \quad \text{dyon} \\ \text{strong coupling}}} = \underbrace{\left(\prod_{n=1}^{\infty} K_{n\gamma + (n-1)\gamma'} \right)}_{\text{dyons}} \underbrace{K_{\gamma+\gamma'}^{-2}}_{\text{W boson}} \underbrace{\left(\prod_{n=0}^{\infty} K_{(n-1)\gamma + n\gamma'} \right)}_{\text{dyons}} \quad [\text{Denef}]$$

weak coupling

e.g. if $\langle \gamma, \gamma' \rangle = 3$ then

$$\mathcal{Z}_{\gamma, \gamma'} = \mathcal{Z}_{\gamma} \left[\prod_{\substack{m, n > 0 \\ m/n \nearrow}} \mathcal{K}_{n\gamma + m\gamma'}^{c_{n,m}} \right] \mathcal{Z}_{\gamma'}$$

where the coefficients $c_{n,m}$ are complicated, supported on a dense set of $\frac{m}{n}$, and grow exponentially with the charges!

In particular, $(c_{n,m})_{n \geq 1} = 3, -6, 18, -84, 465, \dots$

This does occur, already in $\mathcal{N}=2$ SYM $G = SU(3)$!

[Gaiotto, Longhi, Marniero, Moore, Neitzke]

So far: know how (in principle) to use:

1) IR physics (encoded in functions \mathcal{Z}_{γ})

2) $\Omega(\gamma)$'s at one part of moduli space

to determine $\Omega(\gamma)$ everywhere in moduli space, using KS formula.

Next: ① explain why KS is true

② recipe for computing $\Omega(\gamma)$ directly (geometric) in "theories of class S"

By now \exists many approaches to "proving" KS formulae.

(Garoff-Moore-Neitzke, Cecotti-Vafa, Dimofte-Gaiotto-Schulman,
Manschot-Prohac-Sen, ...)

I'll describe an approach via **SUSY line defects**.

Ex In $U(1)$ gauge theory, Wilson line operator is

$$L = \exp\left[i \int_p A\right]. \quad p = \text{path in } \mathbb{R}^{3,1}$$

SUSY version of this: fix $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$

$$L_\vartheta = \exp\left[i \int_p A + e^{-i\vartheta} \Psi ds + e^{i\vartheta} \bar{\Psi} ds\right]$$

If p is a straight path, L_ϑ preserves $\frac{1}{2}$ of the SUSY.

Indeed, if p is timelike then L_ϑ preserves the operators we called $R_\vartheta, \bar{R}_\vartheta$ in prev talk — just like a BPS particle with $\vartheta = -\arg(Z)$.

So: Define a **SUSY line defect** with phase ϑ to be a line defect that preserves $R_\vartheta, \bar{R}_\vartheta$.

e.g. nonabelian Wilson, 't Hooft, Wilson-'t Hooft...
but also \exists in non-Lagrangian field theories!

Inserting L_ϑ then is like inserting a very heavy BPS particle at rest, as an external probe, with phase ϑ .

In the presence of L_g the Hilbert space is modified: $\mathcal{H} \rightsquigarrow \mathcal{H}_{L_g}$

BPS bound becomes $M \geq \text{Re}(e^{i\theta} Z)$

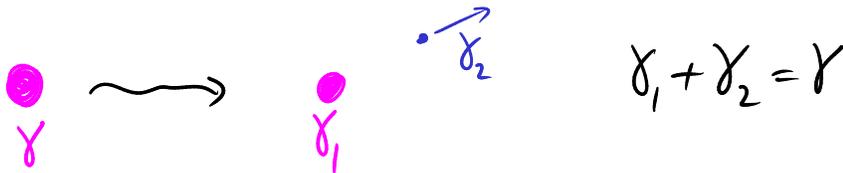
A "framed BPS state" is a state in \mathcal{H}_{L_g} saturating this bound (annihilated by R_g, \bar{R}_g).



They are counted by framed index $\bar{\Omega}(L_g, \gamma) = \text{Tr}_{\mathcal{H}_{L_g}} (-1)^{2J_3}$

As with $\Omega(\gamma)$, can ask how $\bar{\Omega}(L_g, \gamma)$ behaves under deformation of parameters. Framed BPS states can decay by emitting an ordinary ("vanilla") BPS state.

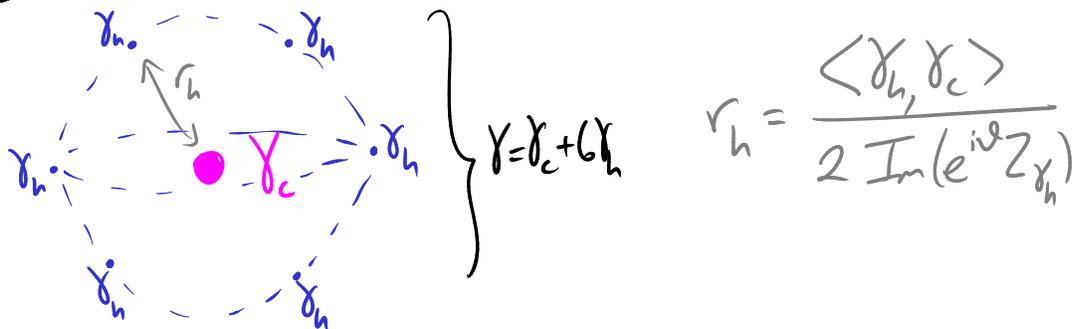
e.g.



This can happen when the 2 constituents have the same phase, i.e.

$$\theta = -\arg(Z_{\gamma_2})$$

Moreover, in this process one can readily calculate how $\bar{\Omega}(L_g, \gamma)$ jumps: the states which appear/disappear in the spectrum have a nice semiclassical picture [Denef/Denef-Moore]. e.g.



Quantize: Each "half-particle" can be in one of $\Omega(\gamma_h) \cdot \langle \gamma_h, \gamma_c \rangle$ states.

These states generate a Fock space (bosonic/fermionic depends on sign of $\Omega(x_h)$) of "halo states".

As $\mathcal{J} \rightarrow -\arg Z_{x_h}$, $r_h \rightarrow \infty$ and these states disappear from $\mathcal{H}_{L_g}^1$.

Effect on $\overline{\Omega}(L_g, \gamma)$ can best be summarized by:

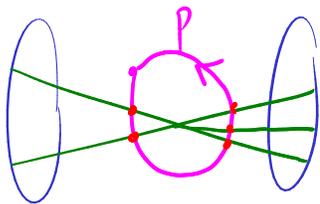
Define
$$F_{L_g} = \sum_{\gamma \in \Gamma} \overline{\Omega}(L_g, \gamma) X_\gamma$$

Then, when \mathcal{J} crosses $-\arg Z_{x_h}$, F_{L_g} is transformed by

$$X_\gamma \longrightarrow X_\gamma (1 - X_{x_h})^{\pm \Omega(x_h) \langle \gamma, x_h \rangle} \quad \text{i.e. } K_{x_h}^{\pm \Omega(x_h)} !$$

But: When we travel around any closed loop in parameter space

$$\mathcal{B} \times S^1$$



the generating funcn. $F(L_g, X_\gamma)$ must return to its original value!

So, $\prod_{\text{pnc}} K_\gamma^{\pm \Omega(\gamma)}$ preserves $F(L_g, X_\gamma)$.

If the theory has "enough" line defects L_g

(so that any X_γ is a linear combination of the L_g for various L)

then this proves $\prod_{\text{pnc}} K_\gamma^{\pm \Omega(\gamma)} = \underline{1}$ which was the KS formula.

Ex Argyres-Douglas theory AD_3 : $\text{rank}(\Gamma) = 2$ $X_{\delta_1} = X$ $X_{\delta_2} = Y$

5 natural line operators:
 $F(L_1) = X$
 $F(L_2) = X + XY$
 $F(L_3) =$
 $F(L_4) =$
 $F(L_5) = \dots$

An alternative approach: take the $\mathcal{N}=2$ theory on $\mathbb{R}^3 \times S^1$

What's it like in the IR?

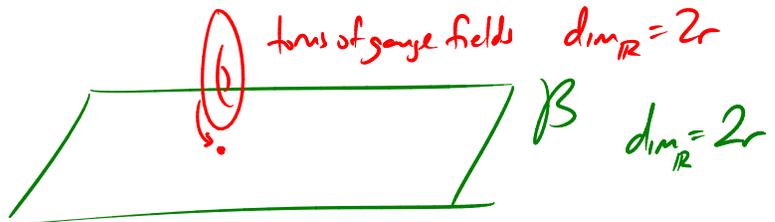
First approx: take IR action in 4d, dim reduce on $S^1_{\mathbb{R}}$.

$U(1)^r$ gauge theory
 $\mathcal{N}=2$

Dualize gauge fields \rightsquigarrow scalars.
 Left with a σ -model in 3d.

Target space:

hyperkähler



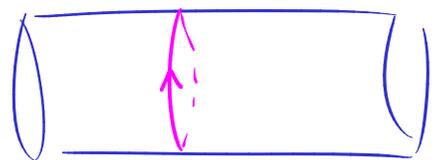
Corrected by instantons: BPS particles of the 4d theory going around $S^1_{\mathbb{R}}$.

$$\sim e^{-R/2l}$$

\rightsquigarrow quantum-corrected HK metric
 which must be smooth (no phase transition)

Requiring this metric to be smooth actually = KS formula!

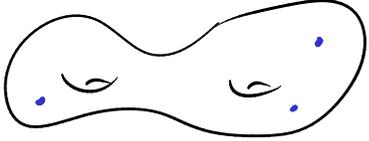
(Connection to line operators: consider the $\langle L_g \rangle$
 as a function on the HK moduli space.
 $SUSY \Rightarrow$ holomorphic in \mathbb{C} str labeled by $\zeta = e^{i\theta}$)



Today: BPS spectra via surface defects
 in "theories of class S"

The answer will look like s.t. that would come from string theory —
 but derived purely within gauge theory

Theories of class S:

Riemann surface C 

$K \geq 2$ (A_{K-1})

Marked points z_ℓ ($1 \leq \ell \leq n$)

Parameters $m_\ell^{(i)} \in \mathbb{C}$ $1 \leq \ell \leq n, 1 \leq i \leq K$ $\sum_{i=1}^K m_\ell^{(i)} = 0$
 (or more general "defect data")

\rightarrow $\mathcal{N}=2$ SUSY QFT in $d=4$
 [Witten, Gaiotto, Garofalo]

Tuple $\vec{\phi} = (\phi_1, \dots, \phi_K)$

ϕ_r is a meromorphic r -differential on C
 (i.e. $\phi_r(z) = f_r(z) dz^{\otimes r}$)

with poles only at the defects, residues controlled by $m_\ell^{(i)}$

\rightarrow point of the Coulomb branch

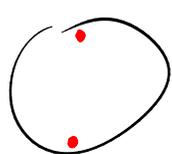
IR physics: governed by "Seiberg-Witten curve"

$$\Sigma = \left\{ (\lambda, z) \in T^*C : \lambda^K + \sum_{r=2}^K \lambda^{K-r} \phi_r(z) = 0 \right\}$$



"Seiberg-Witten differential" λ

Ex $\mathcal{N}=2$ SYM, $G = SU(2)$:
 $K=2$



$C = \mathbb{CP}^1$

$$\phi_2(z) = \left(\frac{\Lambda}{z} + u + \Lambda z \right) \left(\frac{dz}{z} \right)^2$$

$$\Sigma = \left\{ \lambda^2 + \phi_2(z) = 0 \right\} \quad \lambda = y dz$$

$$\boxed{y^2 + \left(\frac{\Lambda}{z^3} + \frac{u}{z^2} + \Lambda z \right) = 0}$$

Closed path P on C defects
up to homotopy
phase ϑ

[Dunkel-Morrison-Okada]

line defect, phase ϑ

Point z on C

surface defect S_z

Path P from z_1 to z_2
phase ϑ

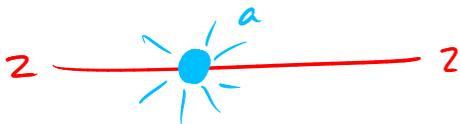
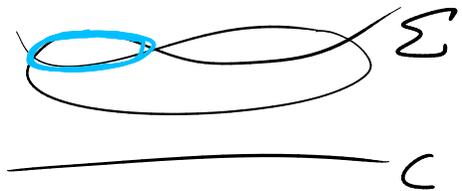
interface, phase ϑ

S_{z_1} S_{z_2}

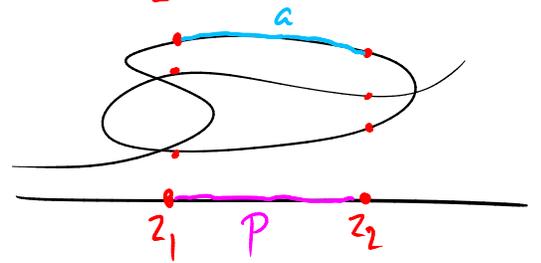
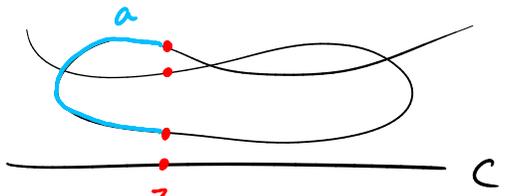
Now we may consider various kinds of BPS states. In pth:



4d BPS states:
charge $\gamma \in H_1(\Sigma, \mathbb{Z})$
count $\Omega(\gamma)$



charge represented by
open path
count $\mu(a)$

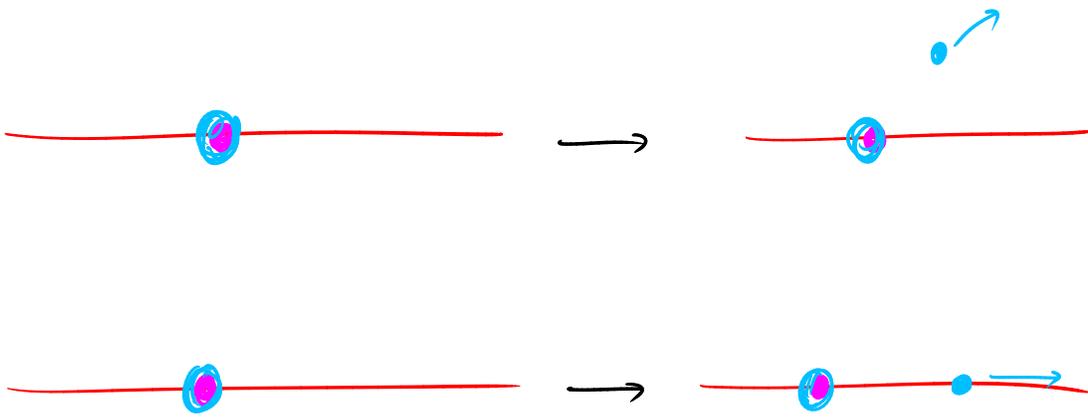


Now, how to compute the BPS spectra?

Begin with the $\Omega(a, L_p, g)$.

Use wall-crossing ideas as in last talk, now for coupled 2d/4d system.

Various kinds of wall-crossing can occur:



Using methods from last time for this more complicated setup,
plus 2 more principles:

$$\begin{aligned}
 & \mathcal{H}_{\text{BPS}} \left(\begin{array}{c} S_{z_1} \quad S_{z_2} \quad S_{z_3} \\ \text{---} \cdot \text{---} \cdot \text{---} \\ L_{p_1} \quad L_{p_2} \end{array} \right) \\
 &= \mathcal{H}_{\text{BPS}} \left(\begin{array}{c} z_1 \quad z_2 \\ \text{---} \cdot \text{---} \\ L_{p_1} \end{array} \right) \otimes \mathcal{H}_{\text{BPS}} \left(\begin{array}{c} z_2 \quad z_3 \\ \text{---} \cdot \text{---} \\ L_{p_2} \end{array} \right)
 \end{aligned}$$

$$\mathcal{H}_{\text{BPS}} \left(\begin{array}{c} z \quad z \\ \text{---} \cdot \text{---} \\ L_{\text{triv}} \end{array} \right) = \{ \text{groundstates of } S_z \}$$

Can determine all framed BPS degeneracies!

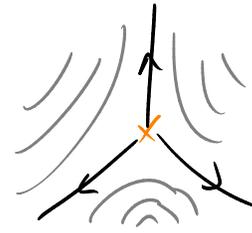
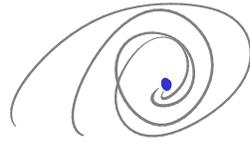
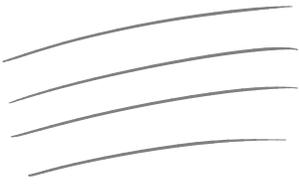
The answer is given in terms of a new geometric gadget, "spectral network" on C .

$$W(\varphi_2, \dots, \varphi_k, \mathcal{J}).$$

Begin with the special case $K=2$. So have a curve C and a mer quad. d.f.f. ϕ_2 on C .
Assume ϕ_2 has only simple zeroes.

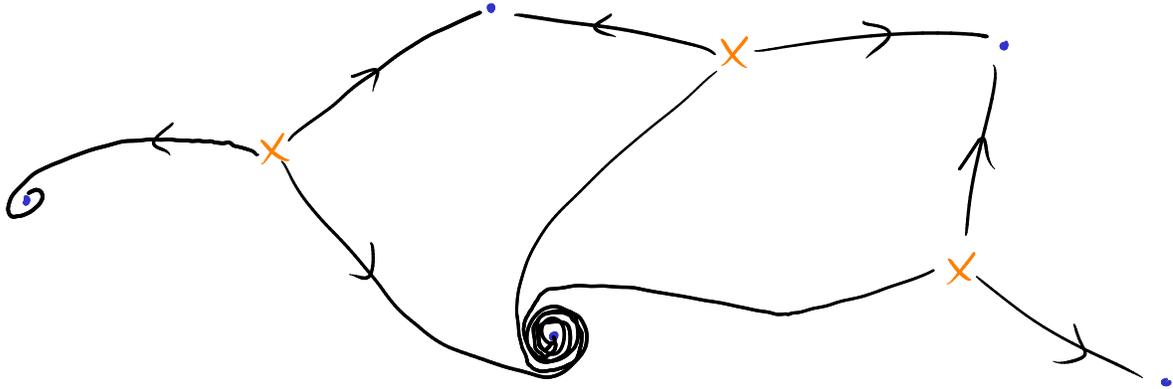
Given $\mathcal{J} \in \mathbb{R}/2\pi\mathbb{Z}$ define \mathcal{J} -trajectories of ϕ_2 to be paths on C
along which $e^{-i\mathcal{J}} \sqrt{\phi_2}$ is real. They make a foliation $F(\phi_2, \mathcal{J})$.

$F(\phi_2, \mathcal{D})$ is singular at defects and at zeros of ϕ_2 .



Focus on the 3 critical traj. emerging from each zero.

Let $W(\mathcal{D}, \phi_2)$ be the union of these.



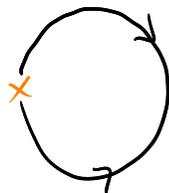
$W(\mathcal{D}, \phi_2)$ is the locus of marg. stab. for $\bar{\Omega}(a, L_{p, \mathcal{D}})$.

Then: The $\bar{\Omega}(a, L_{p, \mathcal{D}})$ are determined by the \cap between P and $W(\mathcal{D}, \phi_2)$.

- When \mathcal{D} varies, $W(\mathcal{D}, \phi_2)$ can jump
 $\Rightarrow \bar{\Omega}(a, L_{p, \mathcal{D}})$ jumps —
 interpretation: decaying by emitting 4d BPS states
 \Rightarrow jumps of $W(\mathcal{D}, \phi_2) = 4d$ BPS states



$\Omega(X) = 1$
hyper.



$\Omega(X) = -2$
vectormultiplet

[Kleban-Loch-May-Vafa-Warner]

Now for $K \geq 2$:

Label the sheets of Σ by $i=1, \dots, K$

→ get K locally defined 1-forms $\lambda^{(i)}$ on C .

Define an *ij-trajectory with phase \mathcal{I}* to be a path along which

$$e^{i\mathcal{I}}(\lambda^{(i)} - \lambda^{(j)}) \text{ is real, positive.}$$

Each branch point of $\Sigma \rightarrow C$ of type (ij) has 3 outgoing *ij*/*ji*-trajectories.

