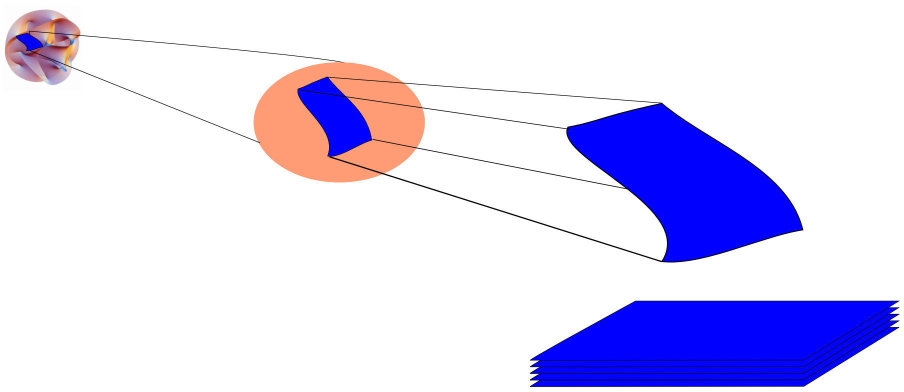
Tools for String Phenomenology

Sakura Schäfer-Nameki

King's College, London



CERN Winter School, February 4-8, 2013

4 Lectures on String Pheno

Lecture 1: Why?
SUSY GUT settings in String Theory

Lecture 2: How?
Higgs Bundles as the Tool for String Phenomenology

Lecture 3: What exactly?
F-theory as a UV completion of Higgs bundles

Lecture 4: Seriously?!
Complete F-theory models and implications

Lecture 3: What exactly?

F-theory: from Higgs bundles to Geometry

Spectral Covers:

Hayashi, Kawano, Tatar, Watari, Donagi, Wijnholt, Marsano, Saulina, SSN, Weigand...

F-theory:

Vafa, Morrison, Katz, Bershadsky, Sadov, Kachru, ...

F-theory and geometry of elliptic CY4:

Esole, Yau, Marsano, SSN, Weigand, Grimm, Mayrhofer, Kuntzler, Krause, Lawrie,....

F-theory via M-theory:

Vafa, Morrison, Grimm, Hayashi, Cvetic, Klevers, ...

Finally, we need to talk about F-theory

F-theory = non-perturbative Type IIB vacua

- Coupling: complex field $\tau = C_0 + ie^{-\phi}$
- S-duality of Type IIB = $SL_2\mathbb{Z}$ action, e.g. $\tau \to -1/\tau$
- [Vafa] Geometrize τ consistent with $SL_2\mathbb{Z}$
  $\Rightarrow$ Tag on to geometry a T^2 or elliptic curve, where τ= complex structure of curve
- 4 dim: compactify on T^2 fibered Calabi-Yau fourfold Y
  $y^2 = x^3 + fx + g$

$$\begin{array}{ccc} T^2_\tau & \longrightarrow & Y_4 \\ & & \downarrow \\ & & B_3 \end{array}$$

Various ways to reach F-theory

- Non-perturbative IIB theory
- F-theory on K3-fibered CY4 is dual to heterotic on elliptic CY3
- Duality to M-theory: useful approach to learn about effective theory

$$M/S^1_A \times S^1_B \xrightarrow{R_A \to 0} IIA/S^1_B \xrightarrow{R_B \to 0} IIB$$

$$R_A, R_B \to 0\,, \qquad g_s = R_A/R_B = \text{fixed}$$

More generally: F-theory from M-theory on $\mathbb{E}_\tau$

$$\text{Elliptic curve} \qquad \mathbb{E}_\tau \sim S^1_A \times S^1_B : \qquad \begin{cases} \text{Im}(\tau) = g_s = \text{fixed} \\ \text{Vol}(\mathbb{E}_\tau) \to 0 \end{cases}$$

Gauge degrees of freedom/7-branes in F-theory

Gauge degrees of freedom (like in Type IIB) arise from 7-branes:

7-branes in IIB sources F_9: $z =$ direction perpendicular to 7-brane

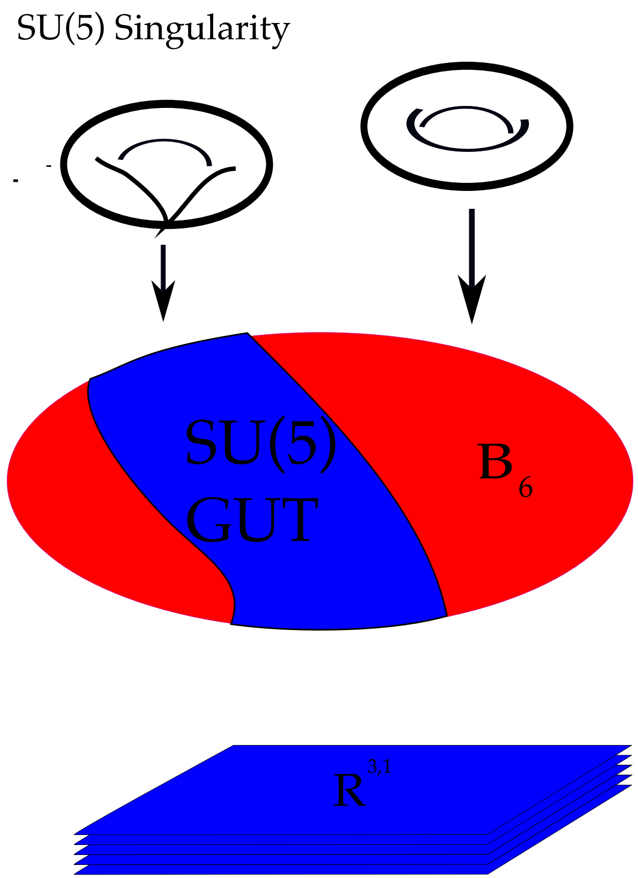
$$d \star F_9 = \delta(z - z_0) \quad \Rightarrow \quad \oint_{S^1} dC_0 = 1$$

which has solution locally

$$\tau(z) = \tau(z_0) + \frac{1}{2\pi i} \log(z - z_0) + \cdots$$

- Monodromy: $\tau \to \tau + 1$
- (p, q) 7-branes generalize this to $SL_2\mathbb{Z}$ monodromies
- τ diverges at location of 7-brane
  $\tau =$complex structure of elliptic curve
  $\Rightarrow$ Location of 7-branes are loci where fiber is singular

Gauge degrees of freedom/7-branes in F-theory

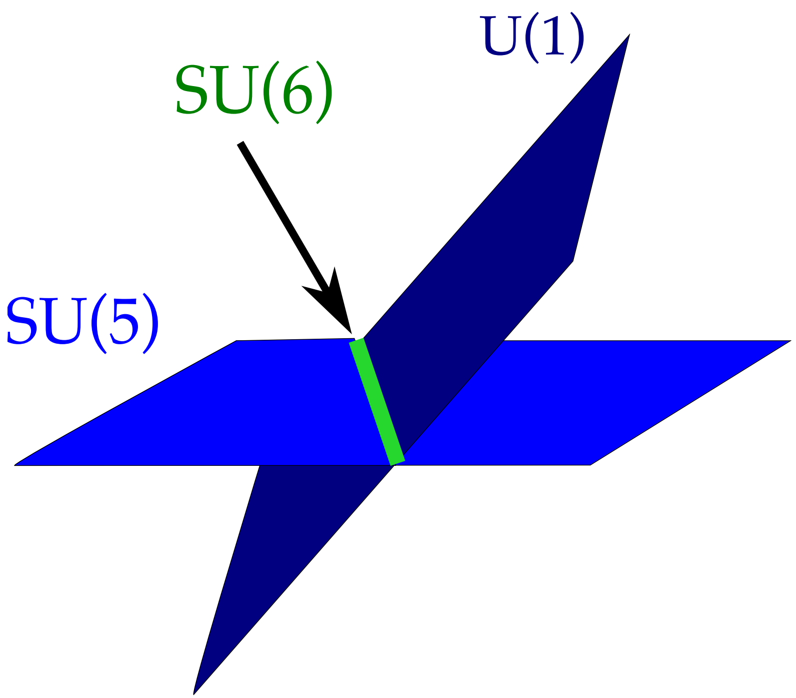


F-theory: realizes branes in terms of geometric singularities

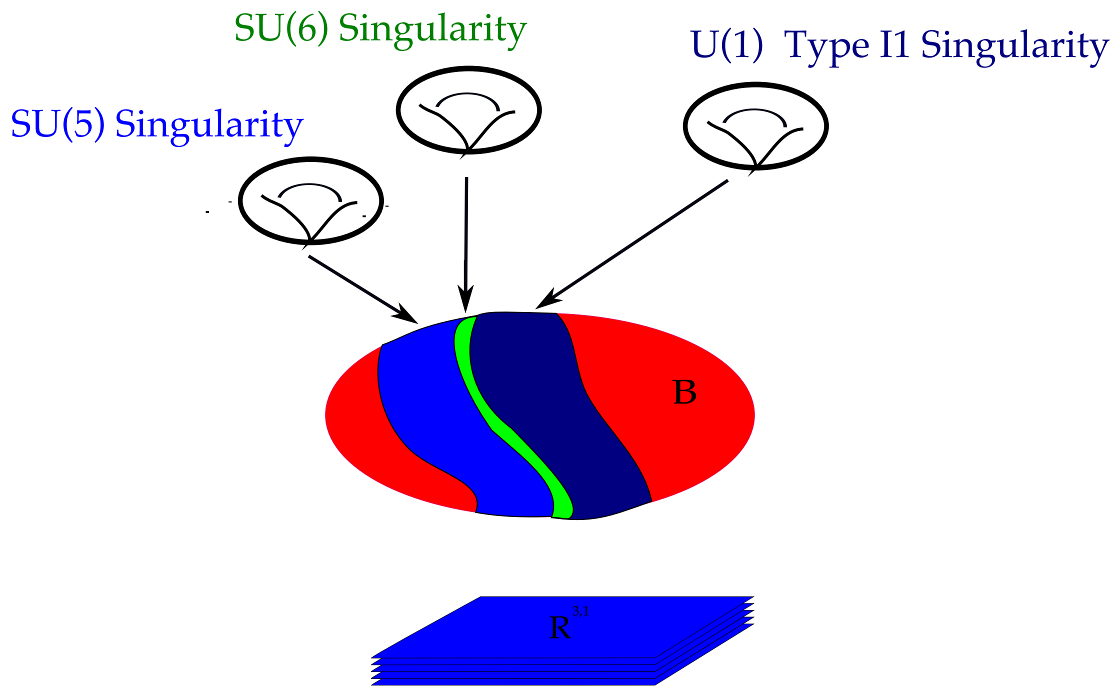
8d $\mathcal{N} = 1$ SYM with gauge groups: $SU(n)$, $SO(2n)$, $E_{6,7,8}$

Matter fields geometrically

7-branes inside B_6 wrapping surfaces, which intersect over a curve Σ:

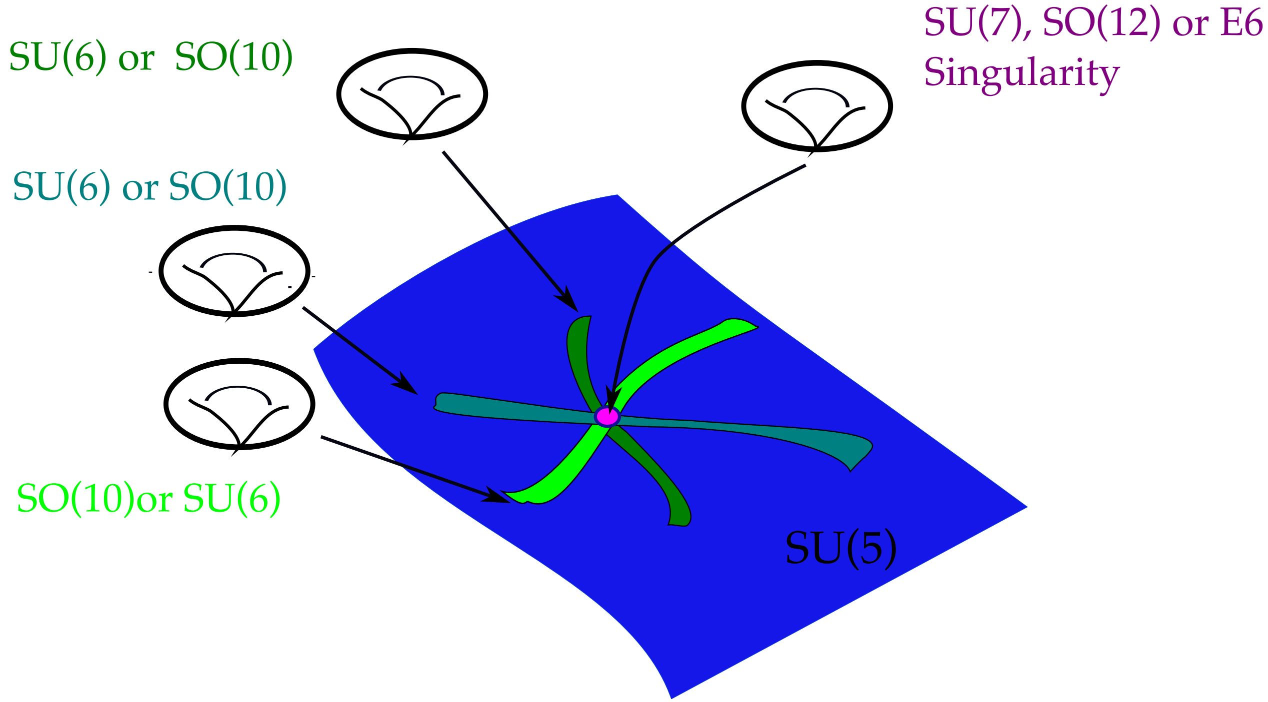


$\Rightarrow$



$\Rightarrow$ Bifundamental matter is localized along curves Σ

Yukawa couplings from Triple-Intersections



Yukawa couplings from triple intersection of matter curves:

$$G_p \to SU(5) \times U(1)_1 \times U(1)_2$$

Such as

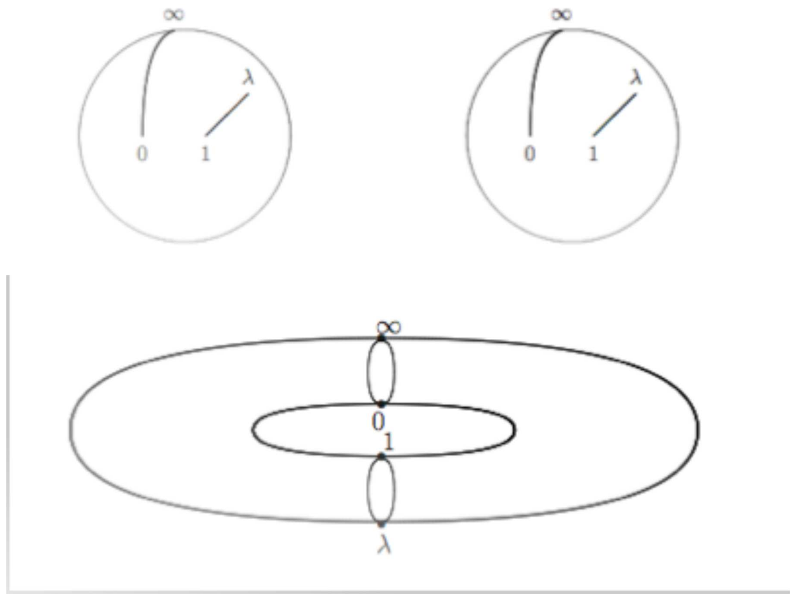
$$SO(12): \ \bar{\mathbf{5}}_H \times \bar{\mathbf{5}}_M \times \mathbf{10}_M \quad E_6: \ \mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_H \quad SU(7): \ \mathbf{5} \times \bar{\mathbf{5}} \times \mathbf{1}$$

Elliptic curves

Classic theory of elliptic curves over $\mathbb{C}$:

Weierstrass form: $y^2 = x^3 + fx + g$

Geometrically: gives a branched covering over y-plane, with cuts connecting the 3 roots, wlog 0, 1, λ of $x^3 + fx + g$ and $\infty \Rightarrow$ gives a torus:



Singular elliptic curves

Singular loci arise when two branchpoints collide, ”cycle shrinks to 0 size”, i.e. roots of

$$x^3 + fx + g = (x-a)(x-b)(x-c)$$

collide, i.e. when $(a-b) = 0$ or $(a-c) = 0$ or $(b-c) = 0$. In terms of f, g:

$$a + b + c = 0\,, \qquad g = -abc\,, \qquad f = ab + ac + bc$$

Then $(a-b)^{(}a-c)^{(}b-c)^{=}0$ can be rewritten as

$$\Delta = 4f^3 + 27g^2 = 0$$

Singular Elliptic Fibrations

For elliptic fibrations over curves, Kodaira classified all the possible fiber singularities. Generally understood to hold for fibers over higher dimensional spaces in codimension 1.
Fibers characterized by an intersection graph of the resolution $\mathbb{P}^1$s:

ON ANALYTIC SURFACES: II 565

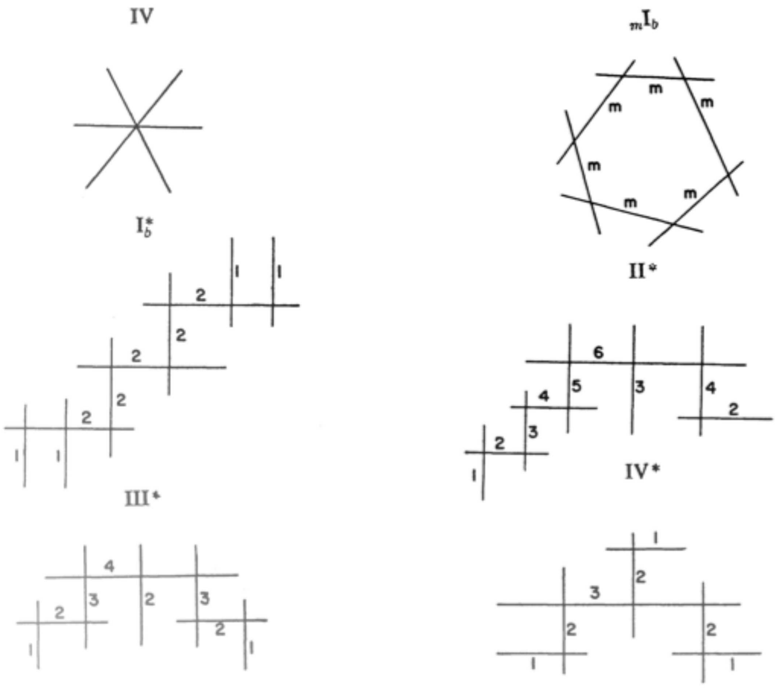


FIGURE 1. Each line represents $\Theta_{\rho s}$; the integer attached to the line gives $n_{\rho s}$.

Elliptic fibrations

Consider fibrations: $\mathbb{E}_\tau \to B_3$, given by a Weierstrass form

$$y^2 = x^3 + fx + g$$

- f and g are functions on the base B_3.
- Let z be a local coordinate on B_3
  $\Rightarrow z = 0$ gives a divisor S in B_3 (surface), aka S_{GUT}
- How to characterize that fiber above S is singular? Let:

$$f = \sum_i f_i z^i \,, \qquad g = \sum_i g_i z^i$$

Singular along $z = 0$: $\quad \Delta = 4f^3 + 27g^2 = O(z^n)$

- $\Delta = 0$ as an equation in B_3 yields a divisor
  $\Rightarrow$ singularity in codimension 1 (one equation in B_3)

	$\mathrm{ord}_S(f)$	$\mathrm{ord}_S(g)$	$\mathrm{ord}_S(\Delta)$	singularity	local gauge group factor
I_0	≥ 0	≥ 0	0	none	–
I_1	0	0	1	none	–
I_2	0	0	2	A_1	$SU(2)$
I_m, $m \geq 1$	0	0	m	A_{m-1}	$Sp([\frac{m}{2}])$ or $SU(m)$
II	≥ 1	1	2	none	–
III	1	≥ 2	3	A_1	$SU(2)$
IV	≥ 2	2	4	A_2	$Sp(1)$ or $SU(3)$
I_0^*	≥ 2	≥ 3	6	D_4	G_2 or $SO(7)$ or $SO(8)$
I_m^*, $m \geq 1$	2	3	$m+6$	D_{m+4}	$SO(2m+7)$ or $SO(2m+8)$
IV^*	≥ 3	4	8	E_6	F_4 or E_6
III^*	3	≥ 5	9	E_7	E_7
II^*	≥ 4	5	10	E_8	E_8
non-minimal	≥ 4	≥ 6	≥ 12	non-canonical	–

Kodaira's classification of singular fibers and gauge groups

The powerhouse for F-theory singularities: Tate form

How do we characterize a specific type of singularity?

Starting with Weierstrass form with specific Kodaira singular fiber

$$y^2 = x^3 + fx + g$$

Can transform globally (almost always) into (generalized) Tate form

[Bershadskya et al],[Katz, Morrison, SSN, Sully]

$$y^2 = x^3 + a_1 xy + a_2 x^2 + a_3 y + a_4 x + a_6$$

Fiber type encoded in

$$a_n = z^{i_n} b_n \,, \qquad b_n = O(1)$$

Tate Algorithm and Tate Forms

Tate Algorithm determines singularity type of an elliptic curve/fibration, and finds a minimal form of it, i.e. ″canonical form for a singular elliptic fibration″. Starting point for Tate algorithm: $f = \sum_i f_i z^i$ and $g = \sum_i g_i z^i$

$$\Rightarrow \qquad \Delta = 4f^3 + 27g^2 = \left(4f_0^3 + 27g_0^2\right) + \left(12 f_1 f_0^2 + 54 g_0 g_1\right) z + O\left(z^2\right)$$

- If z does not divide $\Delta \longrightarrow$ smooth I_0 fiber
- If $z|\Delta$: then there exists u_0 such that

$$f_0 = -\frac{1}{3}u_0^2 + O(z)\,, \qquad g_0 = \frac{2}{27}u_0^3 + O(z)$$

Shifting $(x, y) \mapsto (x + \frac{1}{3}u_0, y)$ the equation becomes

$$y^2 = x^3 + u_0 x^2 + (f_1 z + f_2 z^2 + \cdots)x + (g_1 + \frac{1}{3}u_0 f_1)z + (g_2 + \frac{1}{3}u_0 f_2)z^2 + \cdots$$

which is the Tate form for an I_1 singularity. (U(1))

Tate Algorithm and Tate Forms

- If $z^2|\Delta$

$$\Delta = 4u_0^3(g_1 + \frac{1}{3}u_0 f_1)z + O(z^2)\,, \qquad \text{where } (g_1 + \frac{1}{3}u_0 f_1) = 0 + O(z^2)$$

  so we can set it to zero to leading order and obtain the Tate form for I_2 (SU(2))

$$y^2 = x^3 + u_0 x^2 + (f_1 z + f_2 z^2 + \cdots)x + (g_2 + \frac{1}{3}u_0 f_2)z^2 + \cdots$$

- If $z^3|\Delta$ then

$$\Delta = u_0^2\left(4u_0\left((g_2 + \frac{1}{3}u_0 f_2)\right) - f_1^2\right)z^2 + O\left(z^3\right)$$

  Then there exists s_0, μ such that

$$u_0 = \frac{1}{4}\mu s_0^2 + O(z)\,, \qquad f_1 = \frac{1}{2}\mu s_0 t_1 + O(z)\,.$$

  - If $\mu|_S \neq 0$, then u_0 has a square root and the resulting singularity can be globally put into I_3^s Tate form ($SU(3)$).
  - If μ has zeros on S then there is no global change of coordinates to bring it into Tate form for I_3^{ns} ($Sp(1)$) $\Rightarrow$ generalized Tate form.

- Inductively for all $z^n|\Delta$

Tate Algorithm and Tate Forms

Can transform Weierstrass globally into (generalized) Tate form
except for $SU(m)$, $6 \leq m \leq 9$, $Sp(n)$, $n = 3, 4,$, $SO(l)$, $l = 13, 14$

[Katz, Morrison, SS-N, Sully]

$$y^2 = x^3 + a_1 xy + a_2 x^2 + a_3 y + a_4 x + a_6$$

Fiber type: $a_n = z^{i_n} b_n$, with $b_n = O(1)$ are given in Tate table:

NB: for outliers we know examples where there is no Tate form, e.g. E_6 deformed to A_5

$$y^2 - \frac{9}{4} t^2 xy + z^2 y = x^3$$

which is already I_5 Tate form. Singularity type is I_6, so following the Tate alg, require a non-holomorphic coordinate change.

Type	Group	a_1	a_2	a_3	a_4	a_6	Δ
I_1	—	0	0	1	1	1	1
I_2	$SU(2)$	0	0	1	1	2	2
I_3^{ns}	$Sp(1)$	0	0	2	2	3	3
I_3^{s}	$SU(3)$	0	1	1	2	3	3
I_{2n}^{ns}	$Sp(n)$	0	0	n	n	$2n$	$2n$
I_{2n}^{s}	$SU(2n)$	0	1	n	n	$2n$	$2n$
I_{2n+1}^{ns}	$Sp(n)$	0	0	$n+1$	$n+1$	$2n+1$	$2n+1$
I_{2n+1}^{s}	$SU(2n+1)$	0	1	n	$n+1$	$2n+1$	$2n+1$
III	$SU(2)$	1	1	1	1	2	3
IV^{ns}	$Sp(1)$	1	1	1	2	2	4
IV^{s}	$SU(3)$	1	1	1	2	3	4
$I_0^{*\,ns}$	G_2	1	1	2	2	3	6
$I_0^{*\,ss}$	$SO(7)$	1	1	2	2	4	6
$I_0^{*\,s}$	$SO(8)^*$	1	1	2	2	4	6
$I_1^{*\,ns}$	$SO(9)$	1	1	2	3	4	7
$I_1^{*\,s}$	$SO(10)$	1	1	2	3	5	7
$I_2^{*\,ns}$	$SO(11)$	1	1	3	3	5	8
$I_2^{*\,s}$	$SO(12)^*$	1	1	3	3	5	8
$I_{2n-3}^{*\,ns}$	$SO(4n+1)$	1	1	n	$n+1$	$2n$	$2n+3$
$I_{2n-3}^{*\,s}$	$SO(4n+2)$	1	1	n	$n+1$	$2n+1$	$2n+3$
$I_{2n-2}^{*\,ns}$	$SO(4n+3)$	1	1	$n+1$	$n+1$	$2n+1$	$2n+4$
$I_{2n-2}^{*\,s}$	$SO(4n+4)^*$	1	1	$n+1$	$n+1$	$2n+1$	$2n+4$
$IV^{*\,ns}$	F_4	1	2	2	3	4	8
$IV^{*\,s}$	E_6	1	2	2	3	5	8
III^*	E_7	1	2	3	3	5	9
II^*	E_8	1	2	3	4	5	10
non-min	—	1	2	3	4	6	12

Tate Form for $SU(5)$

Tate form for an $SU(5)$ singularity

$$P_{Tate}: \qquad y^2 = x^3 + b_1 xy + b_2 z x^2 + b_3 z^2 y + b_4 z^3 x + b_6 z^5$$

More precisely:
P_{Tate} is an equation for a hypersurface in $X_5 = \mathbb{P}^2(\mathcal{O} \oplus K_B^{-2} \oplus K_B^{-3})$ or $\mathbb{P}^{1,2,3}$.

Adjunction formula relates canonical bundle of X and Y and normal bundle $N_{Y|X}$

$$K_Y = K_X|_Y \otimes N_{Y|X}$$

But P_{Tate} is like a local coordinate near Y, so that $N_{Y|X} = K_X^{-1}|_Y$, whereby

$$K_Y = K_X|_Y \otimes K_X^{-1}|_Y = \mathcal{O}\ .$$

Tate Form for $SU(5)$

$$y^2 = x^3 + b_1 xy + b_2 z x^2 + b_3 z^2 y + b_4 z^3 x + b_6 z^5$$

and

$$\Delta = z^5 \delta_5 + z^6 \delta_6 + O(z^7)$$

- b_n are sections of bundles over base B_3
- b_n will in fact encode Higgs bundle data (in a second)
- Given that $\dim_{\mathbb{C}} B_3 = 3$, we can consider higher codimension singularities

$$\begin{aligned}
\text{codim } 1: &\quad z = 0 \\
\text{codim } 2: &\quad z = b_1 = 0 \\
&\quad z = (b_1(b_1 b_6 - b_3 b_4) + b_2 b_3^2 = 0 \\
\text{codim } 3: &\quad z = b_1 = b_2 = 0 \\
&\quad z = b_1 = b_3 = 0
\end{aligned}$$

Higher codimension singularities

Or, what's the meaning of δ_5 and δ_6?

$\Rightarrow$ codimension 2 and 3 singularities (in the base)

- Heuristics 1:

  At $z = \delta_5 = 0$ the $SU(5)$ 7-branes intersects flavor brane $\delta_5 = 0$

  $\Rightarrow$ Bifundamental Matter

  At $z = \delta_5 = \delta_6 = 0$ two flavor branes intersect with the $SU(5)$ 7-branes

  $\Rightarrow$ Yukawas

- Heuristics 2:

  Setting $\delta_5 = z = 0$ or $\delta_5 = \delta_6 = z = 0$ yields a higher rank singularity

  $\Rightarrow \Delta = O(z^6)$ or $O(z^7)$

  $\Rightarrow G \rightarrow SU(5)$ generating bifundamental matter/Yukawas

Flawed: no Kodaira classification in higher codimension

$\Rightarrow$ Precise picture from Resolution of singularities

Resolution of the $SU(5)$ singularity

[Esole,Yau], [Marsano, SSN], [Lawrie, SSN]

$SU(5)$ Tate model:

$$y^2 = x^3 + b_6 z^5 + b_4 z^3 x + b_3 z^2 y + b_2 z x^2 + b_1 x y$$

Discriminant has vanishing order at S_{GUT}

$$\Delta \sim z^5 D(b_m, z)$$

Eq is singular along $x = y = z = 0$.
Why? Tangent space degenerates: $\partial_x = \partial_y = \partial_z = 0$ at this locus.

- $x = y = z = 0$ singular locus, i.e. derivatives all vanish
  Blowup: Introduce new $\mathbb{P}^2$ defined by $\zeta_1 = 0$, and new projective coordinates $[x, y, \zeta_0]$

$$x \to x\zeta_1, \quad y \to y\zeta_1, \quad z \to \zeta_0\zeta_1$$

$\Rightarrow \zeta_1 = 0$ is exceptional divisor

- $x = y = \zeta_1 = 0$ singular locus, i.e. derivatives all vanish
  Blowup: Introduce new $\mathbb{P}^2$ defined by $\zeta_2 = 0$, and new projective coordinates $[x, y, \zeta_1]$

$$x \to x\zeta_2, \quad y \to y\zeta_2, \quad \zeta_1 \to \zeta_1\zeta_2$$

$\Rightarrow \zeta_2 = 0$ is exceptional divisor

Smooth in codim 1

$$y(y + b_1 x + b_3 \zeta_1 \zeta_0^2) = \zeta_1 \zeta_2 \left(b_6 \zeta_1^2 \zeta_0^5 + b_2 x^2 \zeta_0 + b_4 \zeta_1 x \zeta_0^3 + \zeta_2 x^3 \right)$$

Read: cf. conifold

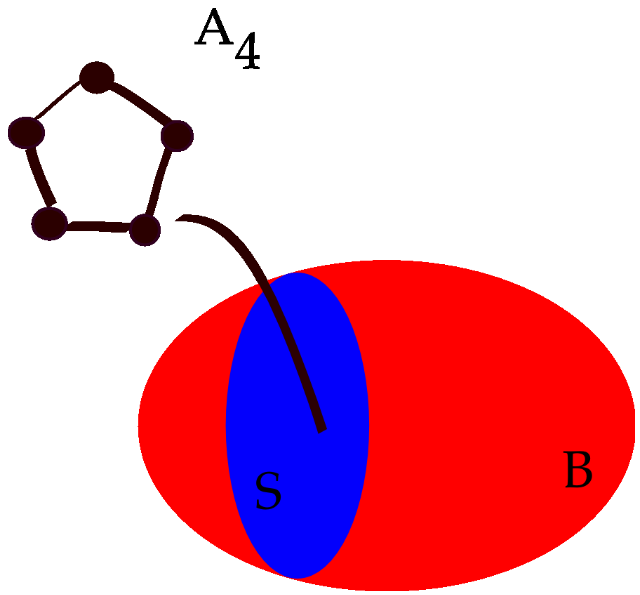
$$y\tilde{y} = \zeta_1 \zeta_2 Z$$

More generally, e.g. $SU(2k+1)$

$$y\tilde{y} = \zeta_1 \cdots \zeta_{k-1} Z$$

Structure of Resolved CY4: Intuitive stuff

Resolving an $SU(5)$ singularity, we expect 4 new $\mathbb{P}^1$s, which can be fibered over S_{GUT} and give rise to 4 new divisors in the resolved geometry:



$$y\tilde{y} = \zeta_1 \zeta_2 Z$$

$\Rightarrow$ 4 exceptional divisors

$\Rightarrow$ $\mathcal{D}_{-\alpha_i}$ "Cartan Divisors"

$\Rightarrow$ Irreducible components of $\zeta_i = 0$

$\zeta_i = y = 0$ and $\zeta_i = \tilde{y} = 0$

$\Rightarrow$ $\mathcal{D}_{-\alpha_i} \cdot \mathcal{D}_{-\alpha_j}$

= Cartan matrix of $\widehat{A_4}$

Higher codimension

$$y(y + b_1 x + b_3\zeta_1\zeta_0^2) = \zeta_1\zeta_2 \left(b_6\zeta_1^2\zeta_0^5 + b_2 x^2\zeta_0 + b_4\zeta_1 x\zeta_0^3 + \zeta_2 x^3\right)$$

Geometry is still singular in higher codimension e.g. in codimension 2

$$y = \zeta_1 = \zeta_2 = b_1 = 0$$

$\Rightarrow$ Small resolutions (see conifold). E.g.

$$y \to y\delta_1\,, \qquad \zeta_1 \to \zeta_1\delta_1$$

where $\delta_1 = 0$ describes a $\mathbb{P}^1$

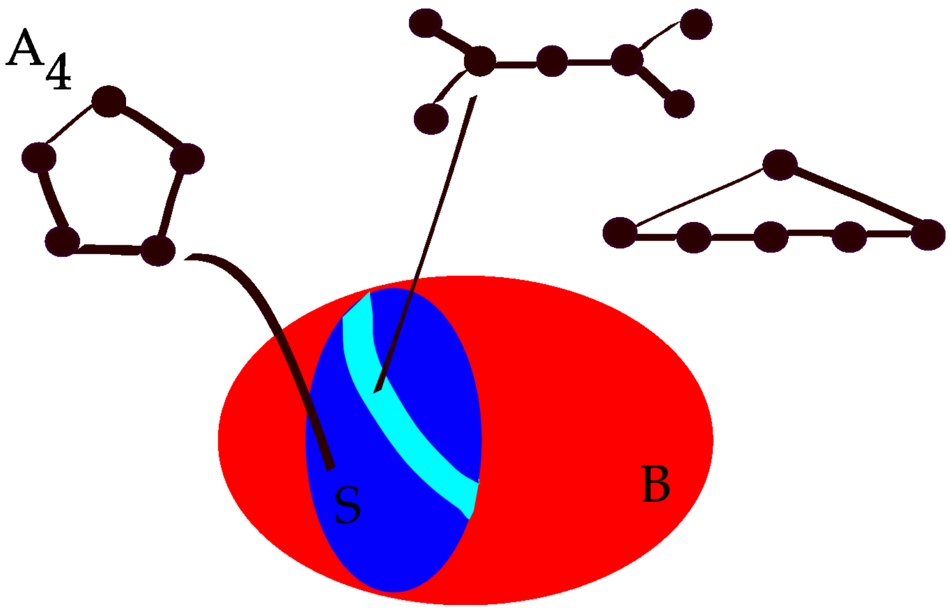
$$\delta_1 = 0 : \qquad [y, \zeta_1]$$

Smooth geometry:

$$y\left(\delta_1\left(b_3\zeta_1\zeta_0^2 + \delta_2 y\right) + b_1 x\right) = \zeta_1\zeta_2\left(b_2 x^2\zeta_0 + \delta_1\zeta_1\zeta_0^3\left(b_6\delta_1\zeta_1\zeta_0^2 + b_4 x\right) + \delta_2\zeta_2 x^3\right)$$

What's the structure of the fibers in higher codim, e.g. along $b_1 = 0$?

Matter Fibers: naive expectation



Fibers are generically not of ADE type (Kodaira) along higher codim

$$\Delta = \delta_5 z^5 + \delta_6 z^6 + O(z^7)$$

Structure of resolved CY4: Not so intuitive Stuff

[Esole,Yau], [Marsano, SSN], [Lawrie, SSN]

- codimension 2, aka matter: Naively expect $SO(10)$ or $SU(6)$ fibers
  ⇒ This is confirmed [Marsano, SSN], [Lawrie, SSN] although initially claimed otherwise [Esole,Yau]

- codimension 3, aka Yukawas: Naively expect $SO(12)$ or E_6 fibers
  ⇒ However, fibers are not Kodaira for E_6
  ⇒ Top Yukawas seem to be in trouble.

In higher codim: no mathematical classification of singular fibers

Constructive proof of fiber types obtained in [Lawrie, SSN]

Structure of resolved CY4: Not so intuitive Stuff, Unriddled

[Marsano, SSN]

$$y\left(\delta_1\left(b_3\zeta_1\zeta_0^2+\delta_2 y\right)+b_1 x\right)=\zeta_1\zeta_2\left(b_2x^2\zeta_0+\delta_1\zeta_1\zeta_0^3\left(b_6\delta_1\zeta_1\zeta_0^2+b_4x\right)+\delta_2\zeta_2x^3\right)$$

Codimension 2:

Cartan divisors split along $b_1=0$ (SO(10)) into surfaces S

$$\begin{aligned} D_{-\alpha_2} &\stackrel{b_1\cdot}{\longrightarrow} S_{(0,0,-1,0,1)}+S_{(0,1,-1,1,-1)} \\ D_{-\alpha_4} &\longrightarrow S_{(0,1,0,0,-1)}+S_{(0,1,-1,1,-1)}+D_{-\alpha_1}\cdot[b_1]\,, \end{aligned}$$

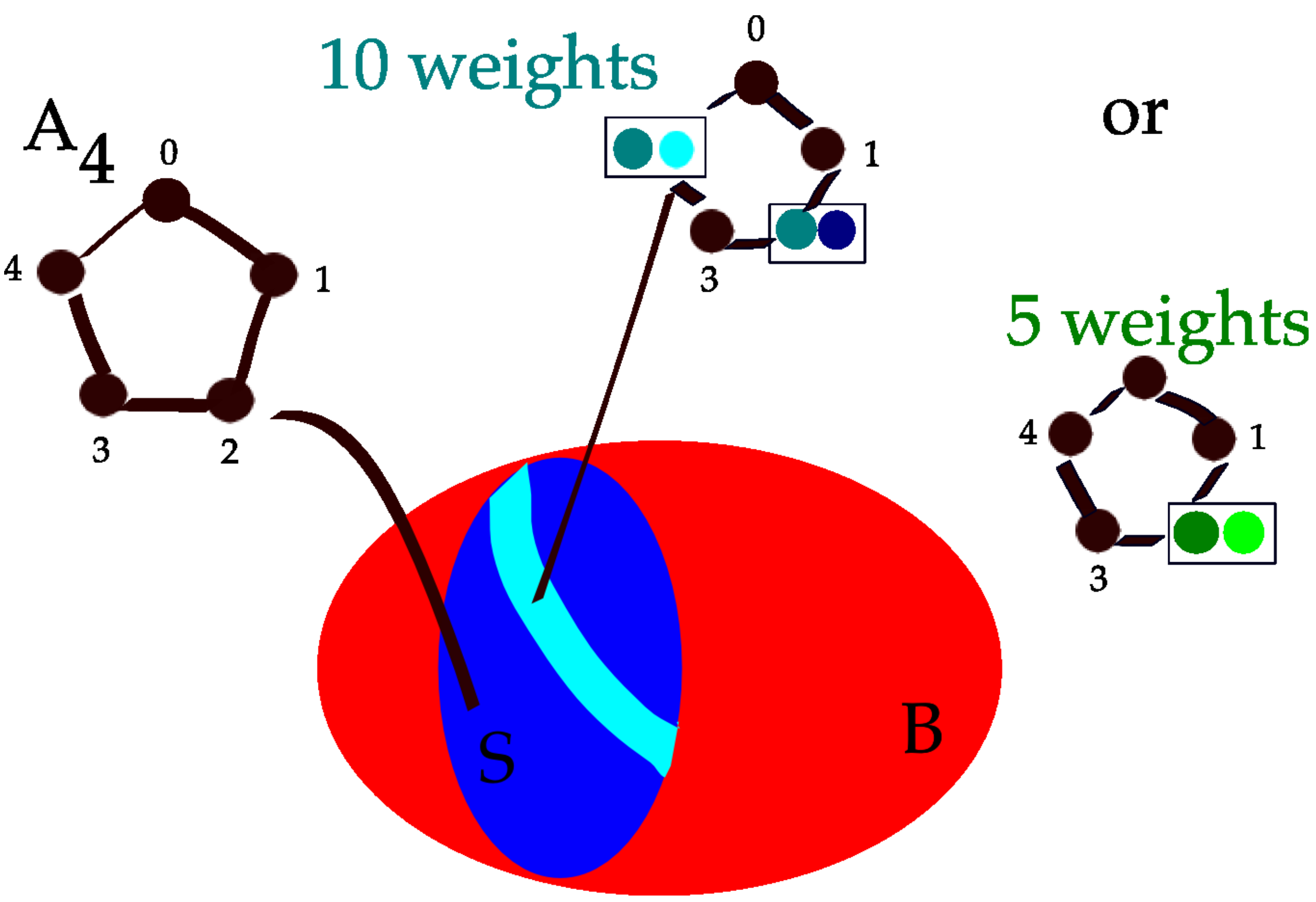
Cartan divisors split along $P_{\mathbf{5}}=0$ (SU(6))

$$D_{-\alpha_3} \longrightarrow S_{(0,0,1,-1,0)}+S_{(0,0,0,-1,1)}$$

⇒ Fibers in higher codim are Kodaira with correct multiplicities

⇒ splitting into weights of the corresponding representation

Matter Fibers



Along codim 2 enhancement: fibers split into weights of SU(5) representations

Structure of resolved CY4: Not so intuitive Stuff, Unriddled

[Marsano, SSN]

Codimension 3:

At E_6 loci $b_1 = b_2 = z = 0$ one of the matter surfaces splits further

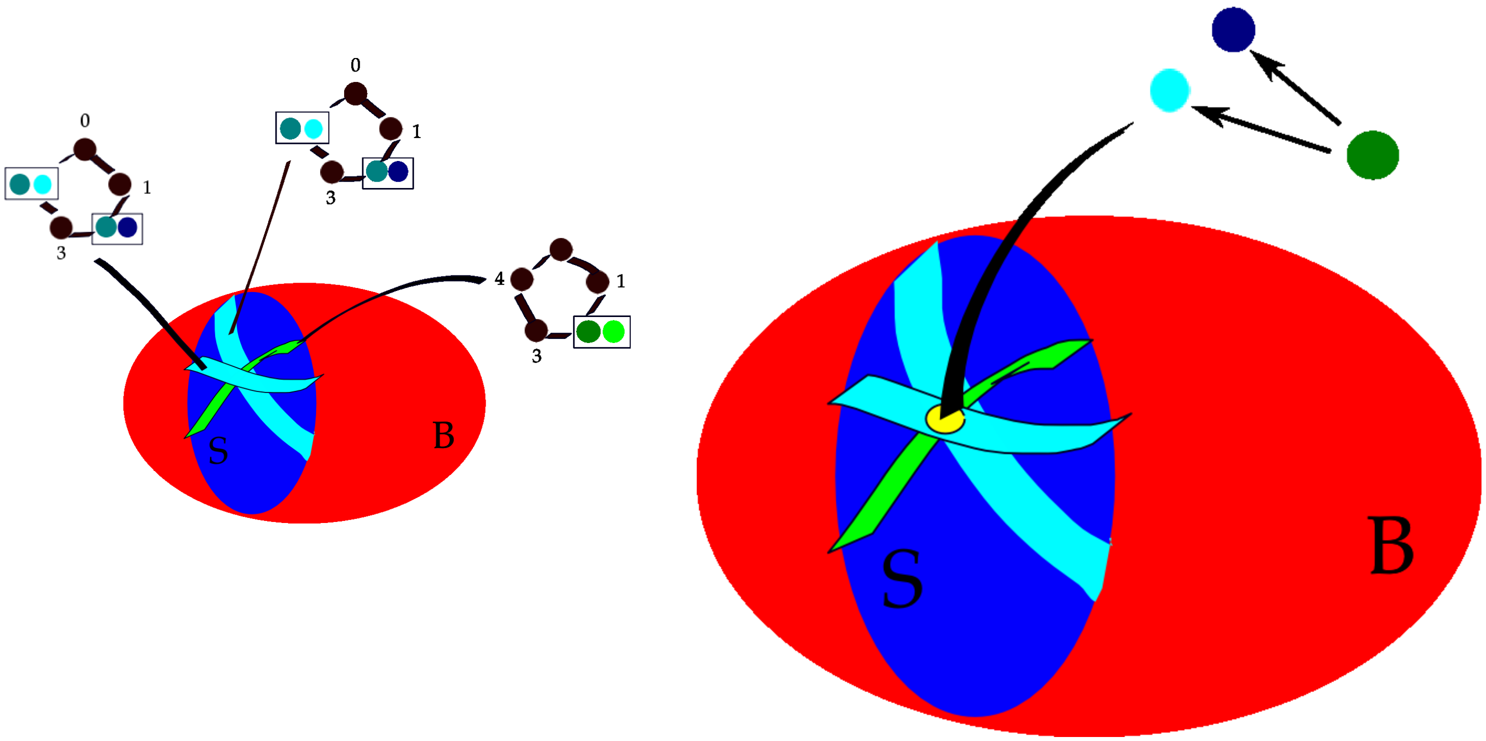
$$S_{(0,1,0,0,-1)} \longrightarrow \Sigma_{(0,1,-1,1,-1)} + \Sigma_{(0,0,1,-1,0)}$$

$\Rightarrow$ In this case: NOT E_6 Kodaira fiber

$\Rightarrow$ But Splitting guarantees existence of $\mathbf{10} \times \mathbf{10} \times \bar{\mathbf{5}}$ top Yukawa:

$\bar{\mathbf{5}}$ curve becomes reducible and splits into two **10** curves.

Yukawas



Codim 3: curves corresponding to weights split consistent with Yukawas

General resolved ADE singularity

[Lawrie, SSN]

For A_{2k} along $z = 0$ in a fourfold: After k blowups

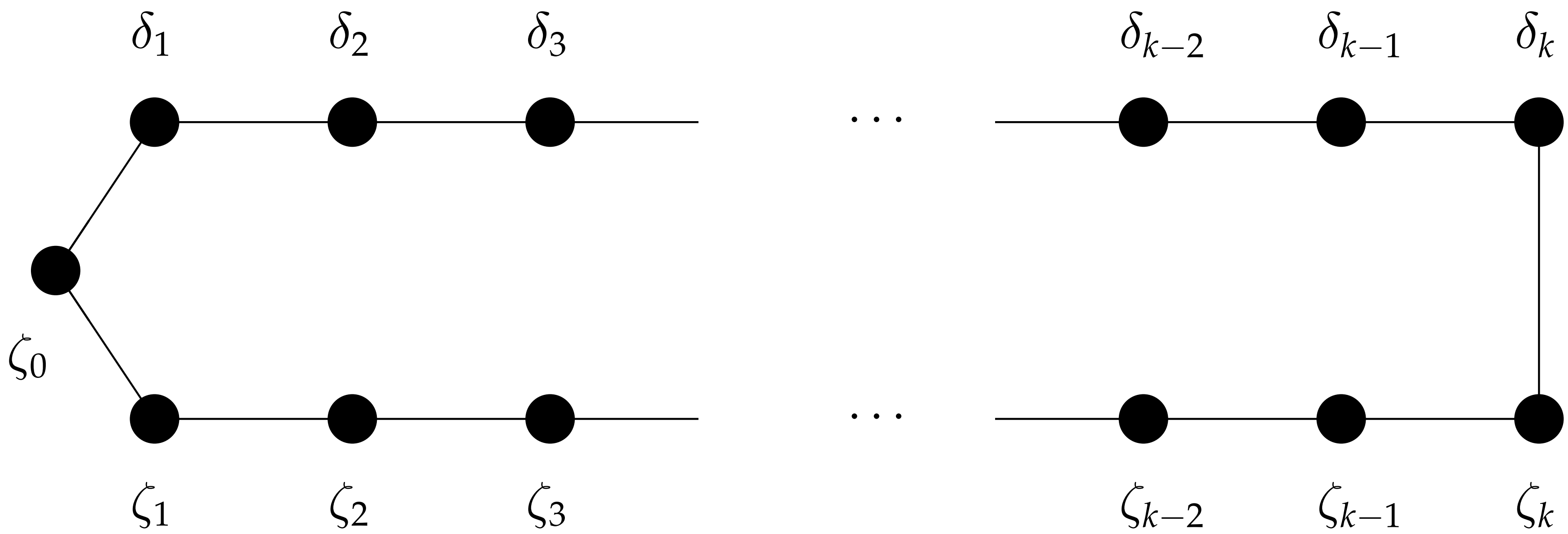
$$y\left(y + b_1 x + b_3 \zeta_0^k B(\zeta) C(\zeta)\right)$$
$$= \zeta_1 \cdots \zeta_{k-1} \left[\zeta_0 \zeta_k b_2 x^2 + x^3 A(\zeta) \zeta_k^k + b_4 x \zeta_0^k C(\zeta) - b_6 \zeta_0^{2k} B(\zeta) C(\zeta^2)\right]$$

- Exceptional divisors $\zeta_i = 0$, $i = 1, \cdots k - 1$ are reducible
  $\Rightarrow$ Resolved in codim 1, yielding affine A_{2k} intersection graph
- Network of small resolutions:
  $\Rightarrow$ smooth in codim 2 and 3.

Structure of Fibers for Tate Forms

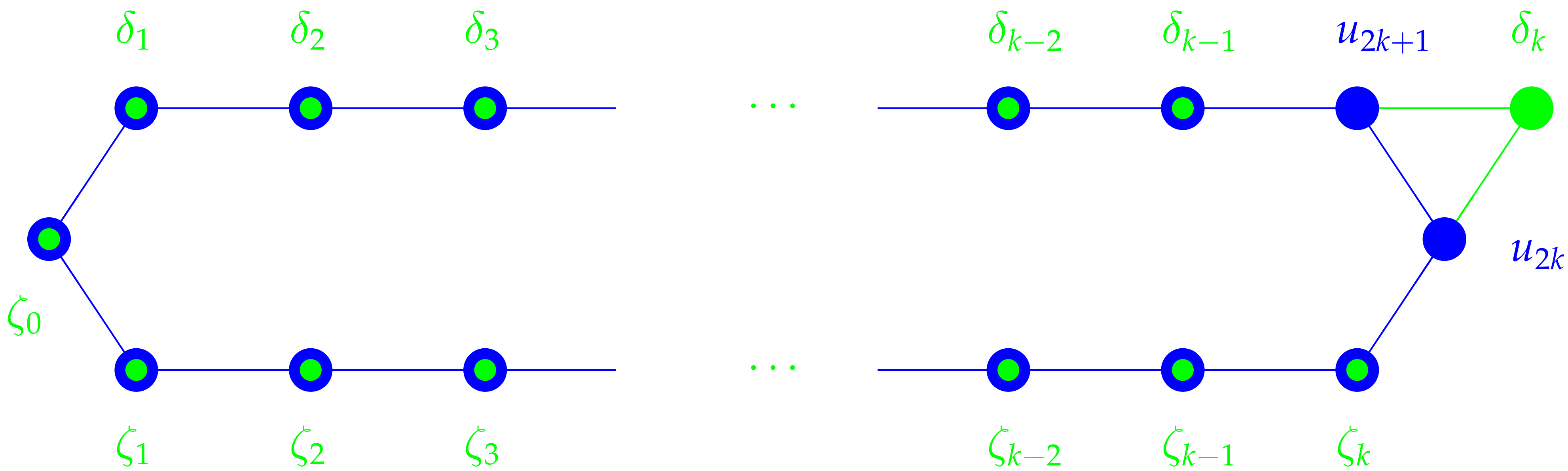
[Lawrie, SSN]

Consider for instance A_{2k} Tate Form. After resolution, the fiber in codimension 1 yields the affine A_{2k} Dynkin diagram:



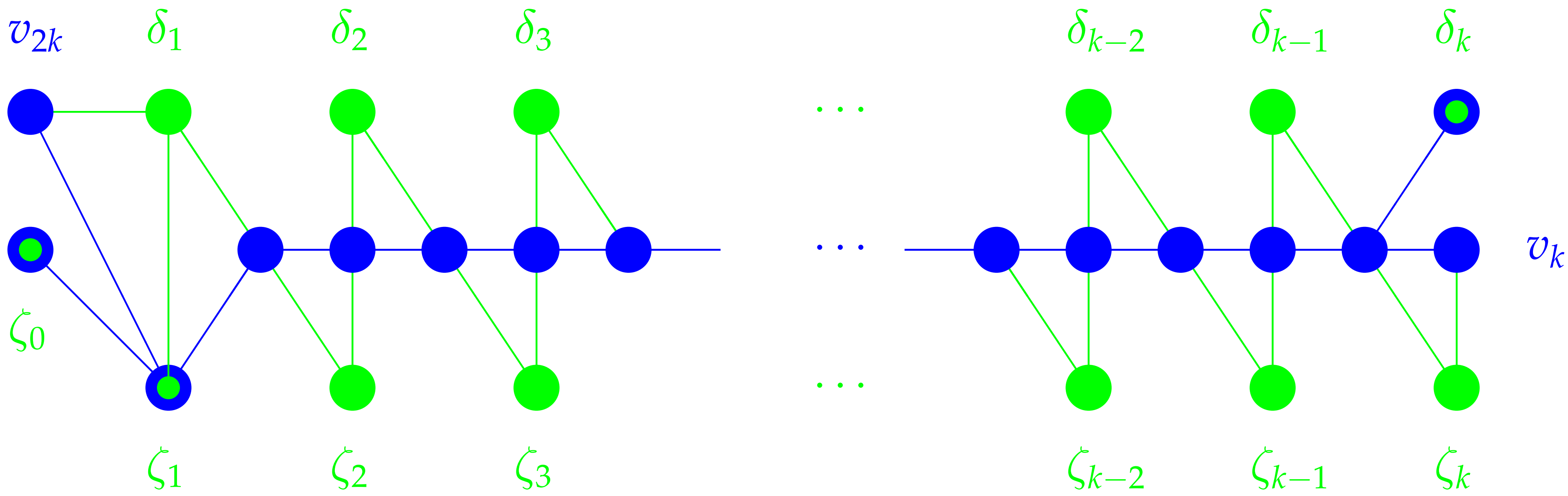
Splitting of fiber along "A_{2k+1}" Locus

New fiber components carry charges associated to **2k** fundamental matter.



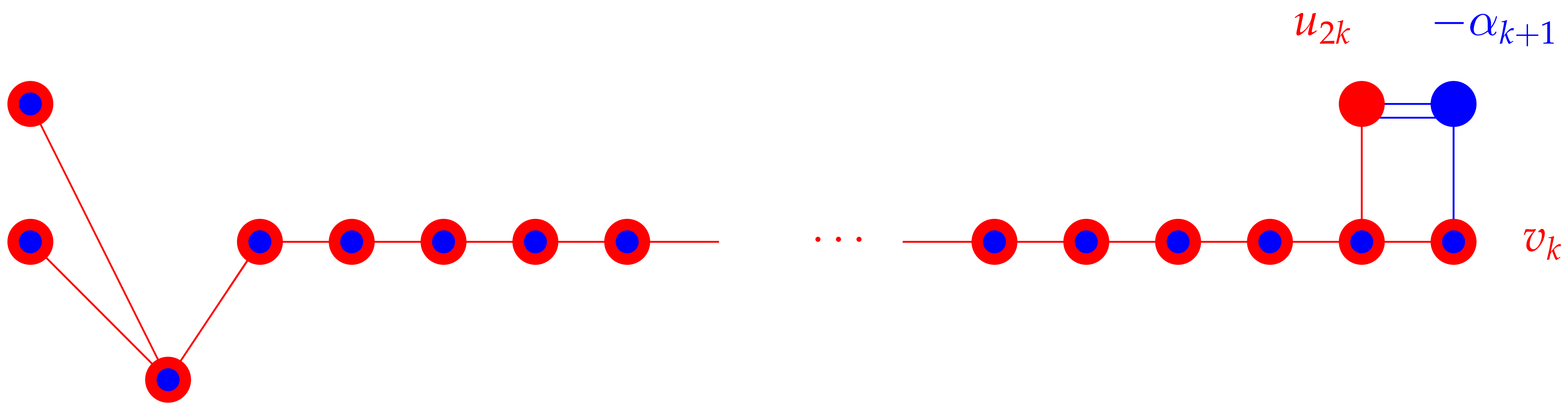
Splitting of Fiber along "D_{2k+1}" Locus

New fiber components carry charges associated to $\Lambda^2\mathbf{2k}$ matter, however additional degrees of freedom from surface components in fiber.



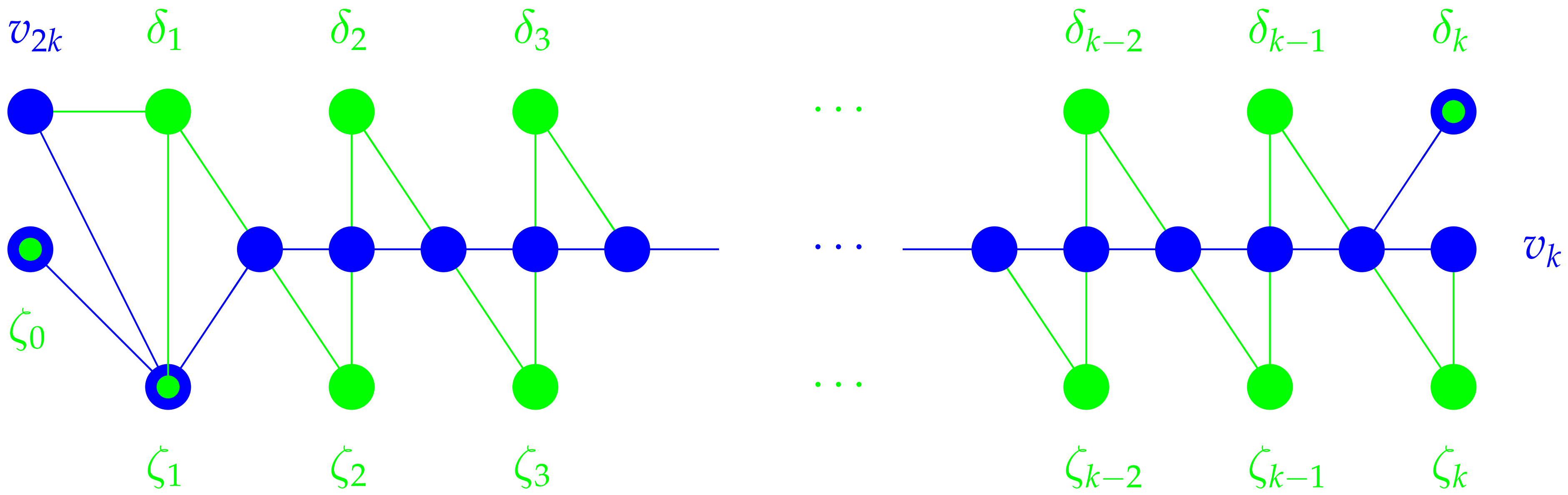
Splitting along "D_{2k+2}" Yukawa locus

Matter in $\Lambda^2\mathbf{2k}$ splits to generate $\overline{\mathbf{2k}} \otimes \overline{\mathbf{2k}} \otimes \Lambda^2\mathbf{2k}$ Yukawa coupling.



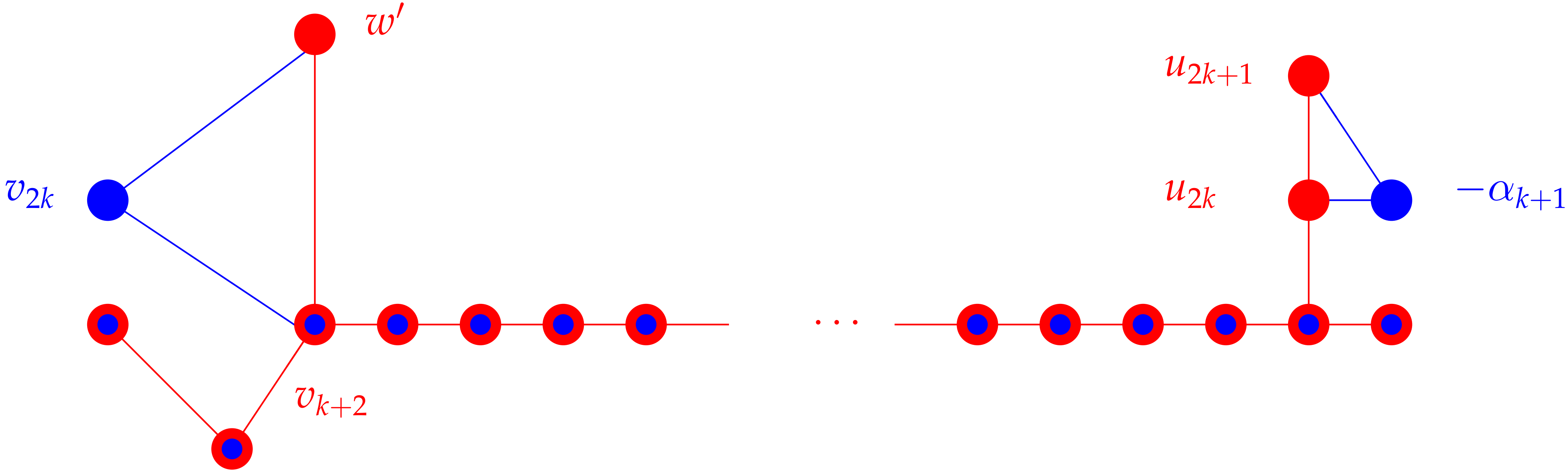
Splitting of Fiber along "D_{2k+1}" Locus

New fiber components carry charges associated to $\Lambda^2\mathbf{2k}$ matter.



Splitting along "E" type locus

Matter in $\Lambda^2\mathbf{2k}$ splits to generate $\overline{\Lambda^2\mathbf{2k}} \otimes \overline{\Lambda^2\mathbf{2k}} \otimes \Lambda^4\mathbf{2k}$ Yukawa coupling. Definitely not Kodaira fiber. Group theoretic interpretation, and what additional dofs in gauge theory?



Summary: higher codimension singularity resolution

- in higher codimension, fibers not necessarily Kodaira type
  $\Rightarrow$ Non-Minimal Singularities, what additional degrees of freedom?
- splitting of the roots is precisely of the type, that generates new curve classes, associated to $SU(5)$ weights
- In codim 3, the splitting is furthermore such that Yukawas are generated
- Classification of fiber in all codim for ADE Tate forms, generalizing Kodaira classification in codim 1 [Lawrie, SSN]

Tate Forms and Spectral Covers

[Marsano, Saulina,SS-N], [Marsano, SS-N], [Kuentzler, SS-N]

We've discussed local models for 7-branes wrapped on $S \times \mathbb{R}^{1,3}$
$\Rightarrow$ What's connection to Tate forms of CY4? How to see $\mathcal{C}$?

Local limit of (resolved) CY4 $\tilde{Y}_4$ limits to $\mathcal{C}$

$$y^2 = x^3 + b_5 xy + b_4 z x^2 + b_3 z^2 y + b_2 z^3 x + b_0 z^5$$

Consider the divisor in Y_4 (and in resolution $\tilde{Y}_4$)

$$\mathcal{C}_{spectral}: \qquad b_5 xy + b_4 z x^2 + b_3 z^2 y + b_2 z^3 x + b_0 z^5 = 0$$

Limiting behavior close to $x = y = z = 0$: let $t = y/x$ and take limit $t, z \to 0$ with $z/t =$ fixed

$$\mathcal{C}_{spectral} \quad \to \quad t^5 \mathcal{C}$$

where $b_n|_S$ as coefficients. All the matter and Yukawa points translate precisely into the higher codimension loci in the CY4.

Additional benefits from resolution: G-flux

G-flux encodes gauge field via ($\omega_i = (1,1)$ forms)

$$G_4 = dC_3 = F_i \wedge \omega_i$$

$\Rightarrow$ Key to get chirality

Four-form $G_4 \in H^{2,2}(Y_4)$, with one leg in fiber and satisfy

$$G \wedge J = 0\,, \qquad G + \frac{1}{2} c_2(Y_4) \in H^4(Y_4, \mathbb{Z})$$

Proper quantization requires c_2. $(2,2)$ forms are dual to surfaces: construct G-fluxes from exceptional divisors of resolution

[Marsano, Saulina, SSN], [Grimm, Weigand], [Kuntzler, SSN], [Collinucci, Savelli]

Lecture 4:
Putting all this to use in model building

EXTRA SLIDES: *G*-flux

Local Model and Spectral Covers

The holomorphic data of the local models of interest comprise:

- $SU(N)$ Higgs bundle on surface S
  - Higgs bundle breaks E_8 to commutant: $SU(5)$, $SO(10)$, E_6
  - Specialization to spectral data of Higgs bundle
    $\Rightarrow$ "spectral models"

$$p_{Higgs}: \qquad \mathcal{C}_{Higgs} \longrightarrow S$$

- Non-abelian vector bundle V on S with $c_1(V) = 0$
  - Line bundle $\mathcal{N}_{Higgs} \to \mathcal{C}_{Higgs}$ determining $V = p_{Higgs,*}\mathcal{N}_{Higgs}$
  - $c_1(\mathcal{N}_{Higgs}) = \gamma_{Higgs} + \frac{r_{Higgs}}{2}$ so that
    $p_{Higgs,*}\gamma_{Higgs} = 0$ and r= ramification divisor of p_{Higgs}
    $\Rightarrow$ spectral cover flux

Example: Local $SU(5)$ from Spectral Cover

[Donagi,Wijnholt], [Marsano, Saulina, SS-N]

[Hayashi, Kawano, Tatar, Watari]

Higgsing characterized by spectral data of ϕ (eigenvalues λ_i)

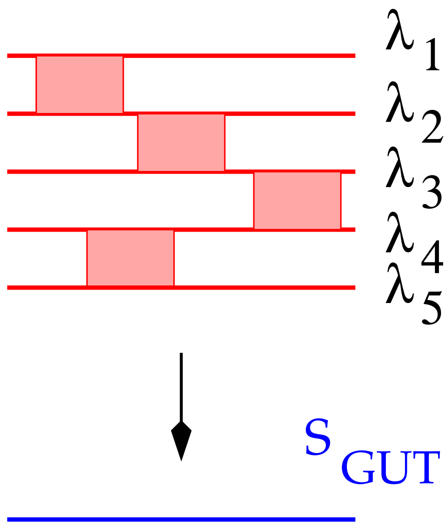
$$E_8 \xrightarrow{\langle\phi\rangle} SU(5)_{\mathrm{GUT}} \times U(1)^4$$

E_8 gauge theory with ϕ and A varying over S encoded in spectral cover $\mathcal{C}_{Higgs}$

$$\det(s-\phi) = \beta_0 s^5 + \beta_2 s^3 + \beta_3 s^2 + \beta_4 s + \beta_5 = 0$$

where $\beta_n = \beta_n(\lambda_i)$ and spectral cover flux
$\Rightarrow$ Local engineering of complete MSSM

[Marsano, Saulina, SS-N]



Spectral cover flux for $SU(N)$

Realize Spectral Cover in $\mathbb{P}^1$-bundle $\pi : Z = \mathbb{P}(\mathcal{O} \oplus K_S) \to S$.
Let $\sigma =$ hyperplane class of $\mathbb{P}^1$. Consider curves

$$\mathcal{C}_{Higgs} \cdot \pi^* \Sigma \qquad \text{and} \qquad \mathcal{C}_{Higgs} \cdot \sigma$$

Suitable linear combintations of these are both properly quantized and satisfy $c_1 = 0$. For example

$$\gamma = (N\sigma - \pi^*(\Sigma_N)) \cdot \mathcal{C}_{Higgs}$$

where $\Sigma_N =$curve at $s = 0$ in $\mathcal{C}_{Higgs}$.

$\Rightarrow$ $\mathcal{C}_{Higgs}$ can be used to construct suitable fluxes

NB: There can be quantization subtleties for N even, e.g. $SO(10)$.

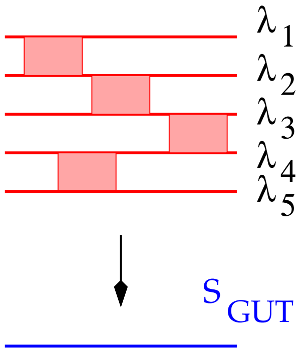
$U(1)$ symmetries and Factored Spectral Cover

[Tatar, Tsuchiya, Watari], [Marsano, Saulina, SS-N]

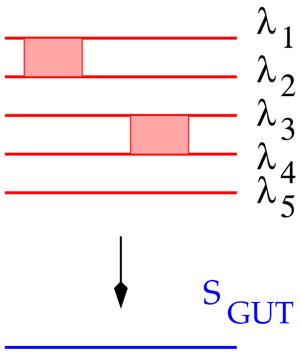
Phenomenologically: require $U(1)$s. Realization in spectral cover:

Independent gauged $U(1)$ symmetries are encoded in # factors of $\mathcal{C}_{Higgs}$

$U(1)$ gauge bosons are elements in Cartan subalgebra:



- G = transitive subgroup of S_5:
  only invariant combination is $\sum_{i=1}^{5} \lambda_i = 0$
  $\Rightarrow$ no gauged $U(1)$
- λ_i in reducible representation of G:
  $\mathcal{C}_{10}$ factors into N components $\Rightarrow (N-1)$ gauged $U(1)$s
  $\Rightarrow \mathcal{C}_{Higgs} = \prod_i \mathcal{C}_i$



Global version: Spectral Divisor

[Marsano, Saulina,SS-N], [Marsano, SS-N], [Kuentlzer, SS-N]

A few subtleties:

Firstly: There is in fact a whole family of divisors $\mathcal{C}_{spectral}$. Single out the one that

- Reduces to spectral divisor $\mathcal{C}_{Higgs}$ in the vicinity of S
- Additional $U(1)$: Factored $\mathcal{C}_{spectral} = \mathcal{C}^{(m)}_{spectral}\mathcal{C}^{(n)}_{spectral}$

Global version: Spectral Divisor

[Marsano, Saulina,SS-N], [Marsano, SS-N], [Kuentlzer, SS-N]

Secondly: For general $SU(N)$ covers, the Tate form is not the right place to start, as it will not give rise to the Higgs bundle spectral covers, e.g. for E_6

$$E_6 : \qquad \beta_6 s^3 + \beta_4 s - \beta_3 = 0$$

Define spectral form, which is

G	Spectral form	of singularity
E_7	y^2	$= x^3 + b_4 z^3 x + b_6 z^5$
E_6	$y^2 + b_3 z^2 y$	$= x^3 + b_4 z^3 x + b_6 z^5$
$SO(10)$	$y^2 + b_3 z^2 y$	$= x^3 + b_2 z x^2 + b_4 z^3 x + b_6 z^5$
$SU(5)$	$y^2 + b_1 xy + b_3 z^2 y$	$= x^3 + b_2 z x^2 + b_4 z^3 x + b_6 z^5$
$SO(11)$	y^2	$= x^3 + b_2 z x^2 + b_4 z^3 x + b_6 z^5$

As usual: Construction checked to be consistent with het/F.

Putting the Spectral Divisor to use: Global G-flux

(2, 2) forms in CY4 is dual to a surfaces

$\Rightarrow$ construct G-flux from holomorphic surfaces D_{G_4}

Recall: spectral flux from curves dual to $\gamma \in H^{1,1}(\mathcal{C}_{Higgs})$. Likewise

$$D_{G_4} \quad \text{dual to} \quad G_4 \in H^{2,2}(\mathcal{C}_{spectral})$$

More precisely: construct D_{G_4} from $\mathcal{C}_{spectral}$ from

- Line bundle $\mathcal{N}_{spectral} \longrightarrow \mathcal{C}_{spectral}$
- Generalization of ramification divisor $\mathcal{L}_r$ to map $p_{spectral} : \mathcal{C}_{spectral} \longrightarrow B_3$

to construct

$$G_4 = \iota_{spectral,*} \left(c_1(\mathcal{N}_{spectral}) - \frac{c_1(\mathcal{L}_r)}{2} \right)$$

Spectral Divisor and G-flux

So – like in the spectral flux case – consider surfaces

$$\mathcal{S}_D = \mathcal{C}_{spectral} \cdot D \,, \qquad D = \text{ divisors in } B_3$$

and $\mathcal{S}_\sigma$ defined as containing in the local limit $\mathcal{C}_{Higgs} \cdot \sigma$.

Again, suitable linear combinations of these give correctly quantized global G-flux

$$G = \frac{1}{2}(2n+1)\left(N\mathcal{S}_\sigma - \mathcal{S}_{p^*(\Sigma_N)}\right)$$

- Direct generalization of local spectral cover fluxes
- Can get a brute force $U(1)$ from factored $\mathcal{C}_{spectral}$

$$G = \left(n\mathcal{C}^{(m)}_{spectral} - m\mathcal{C}^{(n)}_{spectral}\right) \cdot D - G_0 \quad \sim \quad \omega \wedge F \qquad \text{for } F = U(1) \text{ gauge flux}$$

$G_0 =$ surface, ensures that G is $\perp$ to horizontal and vertical divisors in Y_4.

Checks: ”Brute force” G-flux from holomorphic surfaces

From the resolved geometry, the proposal using spectral divisor can be directly checked.

$(2,2)$ forms in CY4 is dual to holomorphic surfaces
$\Rightarrow$ G-flux from holomorphic surfaces, orthogonal to vertical and horizontal divisors, i.e. miss surfaces that contain fiber or sit in base
$\Rightarrow$ G-flux from D_{G_4} satifying

$$D_{G_4} \cdot \sigma_{global} \cdot D_1 = 0 \,, \qquad D_{G_4} \cdot D_1 \cdot D_2 = 0$$

D_i= pullbacks of divisors in B

In resolved CY4:
G can be properly quantized from computation of $c_2(CY4)$ and D_{G_4} construction from exceptional divisors of resolution.
$\Rightarrow$ reproduce the construction via spectral divisor.