Lattice QCD thermodynamics at μ=0 and $\mu\neq$0

Zoltán Fodor
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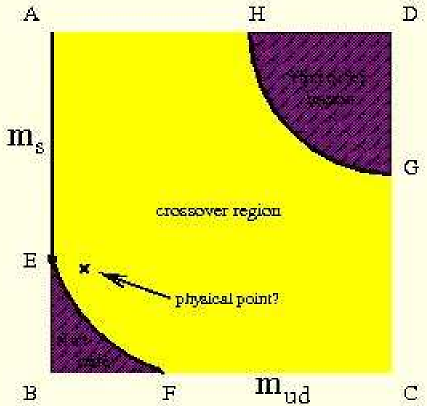
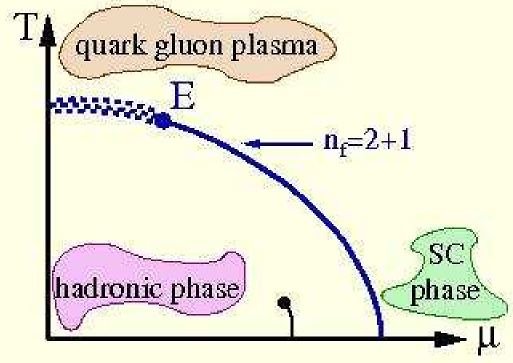
1. Standard picture of the phase diagram
2. Lattice formulation, fermion doubling, rooting
3. Lattice results at L_t=4 (μ>0)
4. Lattice results at L_t=4,6 (equation of state)
5. Lattice results at L_t=4,6,8,10 (nature and T_c)
6. Conclusions

Standard picture of the phase diagram and its uncertainties





physical quark masses: important for the nature of the transition

n_f=2+1 theory with m_q=0 or ∞ gives a first order transition

for intermediate quark masses we have an analytic cross over (no χPT)

F. Karsch et al., Nucl. Phys. Proc. 129 (2004) 614; Lattice'07 G. Endrodi, O. Philipsen

continuum limit is important for the order of the transition:

n_f=3 case (standard action, N_t=4): critical $m_{ps}\approx$300 MeV

with different discretization error (p4 action, N_t=4): critical $m_{ps}\approx$70 MeV

the physical pseudoscalar mass is just between these two values

what happens for physical quark masses, in the continuum, at what T_c?

N_t=4,6,8,10 lattices correspond to $a\approx$ 0.3 fm, 0.2 fm, 0.15 fm, 0.12 fm

CPU: $\approx N_t^{12}$ (thermodynamics): N_t=10 needs 50-times more than N_t=6

Partition function

$$Z = \int dU d\Psi d\bar{\Psi} e^{-S_E}$$

S_E is the Euclidean action

Parameters:

gauge coupling g

quark masses m_i $(i = 1..N_f)$

(Chemical potentials μ_i)

Volume (V) and temperature (T)

Finite T $\leftrightarrow$ finite temporal lattice extension

$$T = \frac{1}{N_t a}$$

Continuum limit: $a \to 0$

Renormalization: keep the physical spectrum constant

at finite T:

continuum limit $\Longleftrightarrow$ $N_t \to \infty$

CPU grows like N_t^{13}, thus N_t=10 instead of N_t=6 needs 50-100$\times$ more CPU


IBM


nVIDIA
DC-12V 0.13A

Overlap improving multi-parameter reweighting

Z. Fodor and S.D. Katz, Phys. Lett. B534 (2002) 87

$$Z(m,\mu,\beta) = \int \mathscr{D}U \exp[-S_g(\beta,U)] \det M(m,\mu,U) =$$

$$\int \mathscr{D}U \exp[-S_g(\beta_0,U)] \det M(m_0,\mu=0,U)$$

$$\left\{ \exp[-S_g(\beta,U) + S_g(\beta_0,U)] \frac{\det M(m,\mu,U)}{\det M(m_0,\mu=0,U)} \right\}$$

first line = measure, field configurations of the Monte-Carlo

curly bracket = can be measured on each configuration, weight

simultaneously changing several parameters: better overlap

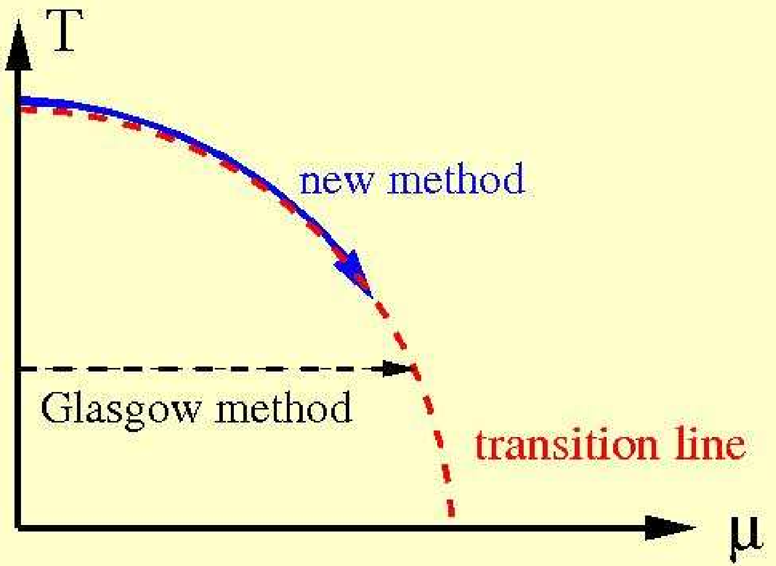
e.g. transition configurations are mapped to transition ones

expectation value of an observable O:

$$\langle 0 \rangle_{\beta,\mu,m} = \frac{\sum w(\beta,\mu,m) O(\mu,m)}{\sum w(\beta,\mu,m)}$$

observables to get the transition points at $\mu \neq 0$ (susceptibilities)

Comparison with the Glasgow method



one parameter reweighting	New method
single parameter (μ)	two parameters (μ and β)
purely hadronic configurations	transition configurations

$\mu \neq 0$ multi-parameter reweighting with Taylor expansion

C.R. Allton et al., Phys. Rev. D66 074507,'02, D68 014507,'03

$$Z(m,\mu,\beta) = \int \mathscr{D}U \exp[-S_g(\beta,U)] \det M(m,\mu,U) =$$

$$\int \mathscr{D}U \exp[-S_g(\beta_0,U)] \det M(m_0,\mu=0,U)$$

$$\left\{ \exp[-S_g(\beta,U) + S_g(\beta_0,U)] \frac{\det M(m,\mu,U)}{\det M(m_0,\mu=0,U)} \right\}$$

instead of evaulating determinants expand them in μ or $\exp(\mu)$:

$$\ln\left(\frac{\det M(\mu)}{\det M(0)}\right) = \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n \ln\det M(0)}{\partial \mu^n} \equiv \sum_{n=1}^{\infty} R_n \mu^n$$

faster than the complete evaluation of the determinants

only valid for somewhat smaller μ values than the full technique

QCD phase diagram from imaginary chemical potential

P.deForcrand, O.Philipsen, Nucl. Phys. B642 290,'02; B673 170, '03

M.D'Elia, M.P.Lombardo, Phys. Rev. D67 014505,'03

fermion determinant: real for imaginary chemical potential (μ_I)
$\Rightarrow$ no sign problem, no need for reweighting

directly obtain the (β_c,μ_I) transition line
analytically continue it to get the physical (β_c,μ) line

transition line (β_c, μ_I) is given by the susceptibility-peak

$$\chi = VN_t\langle(\mathcal{O} - \langle\mathcal{O}\rangle)^2\rangle, \qquad \partial\chi/\partial\beta = 0 \qquad \partial^2\chi/\partial\beta^2 < 0$$

on finite V the analytic $\chi(\mu_I, \beta)$ can be measured
using the implicitely given $\beta_c(\mu_I)$ one gets

$$\partial\beta_c/\partial\mu = -i\partial\beta_c/\partial\mu_I$$

Density of states (DOS) method

Constrained simulations:

Force some observable to have a given value

this way configurations with all values of the observable present

overlap problem not so serious

For any observable:

$$< O >= \int dx \, \langle O f(U) \rangle_x \, \rho(x) \Big/ \int dx \, \langle f(U) \rangle_x \, \rho(x)$$

ρ, the density of states is the constrained partition function for some observable ϕ

$$\rho(x) \equiv Z_\phi(x) = \int \mathcal{D}U \, g(U) \, \delta(\phi - x).$$

Possible choices for ϕ:

ϕ =PI (Bhanot et.al, '87; Karliner et.al,'88; Azooïti et.al,'90; Luo, '01; Takaishi, '04)

$\phi = \Theta$ (complex phase) (Gocksch, '88)

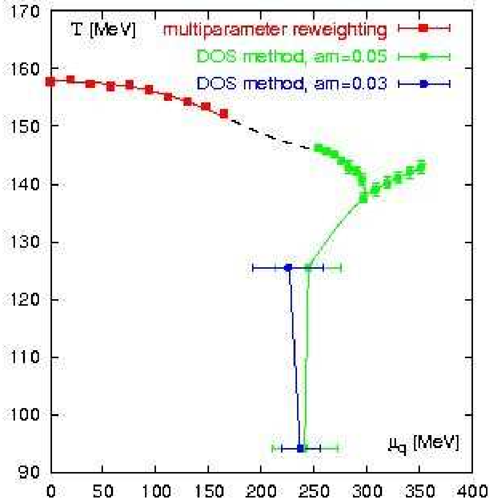
$\Phi = n_q$ (Ambjorn et. al., '02)

Our choice: $\phi = P$ $\quad g = |\mathrm{det}M| \exp\{-S_G\},$ $\quad f = \exp\{i\theta\}$

Results for QCD at large μ

Z. Fodor, S.D. Katz, C. Schmidt, JHEP 0703:121,2007 [hep-lat/0701022]

$N_f = 4$ staggered QCD on 6^4, $8 \cdot 6^3$ lattices



existence of a triple point around $\mu_q \approx$ 300 MeV and T $\lesssim$ 135 MeV

Note, L_t=6 lattices: smallest T is 73 MeV (if m_ρ fixes the scale)

Mass dependence checked:
small T transition point does not depend on pion mass

Equation of state from lattice simulations

energy density (ϵ) and pressure (p) from partition function:

$$\epsilon(T) = \frac{T^2}{V}\frac{\partial(\log Z)}{\partial T} \qquad p(T) = T\frac{\partial(\log Z)}{\partial V}.$$

T, V are varied by a, take derivative with respect of a

$$\frac{\epsilon - 3p}{T^4} = -\frac{L_t^3}{L_s^3} a \frac{d(\log Z)}{da}$$

the pressure ($p \propto \log[Z]$) along the LCP by the integral method:

$$\frac{p}{T^4} = L_t^4 \int d(\beta, m \cdot a) \left(\frac{\partial(\log Z)}{\partial \beta}, \frac{\partial(\log Z)}{\partial(m \cdot a)} \right)$$

Renormalization of the pressure

We want $p(T=0)=0$ and $\epsilon(T=0)=0 \rightarrow$
Simulations at both

$T>0 \ (N_t \ll N_s)$ and $T=0 \ (N_t \gtrsim N_s)$

are necessary and then subtraction:

$$\frac{p}{T^4} = \frac{p_T}{T^4} - \frac{p_0}{T^4}; \qquad \frac{\epsilon}{T^4} = \frac{\epsilon_T}{T^4} - \frac{\epsilon_0}{T^4}$$

numerical precision needed for the subtraction increases with N_t^4
$\rightarrow$ CPU costs grow faster ($\mathcal{O}(1/a^{13})$) than for $T=0$ simulations

Today

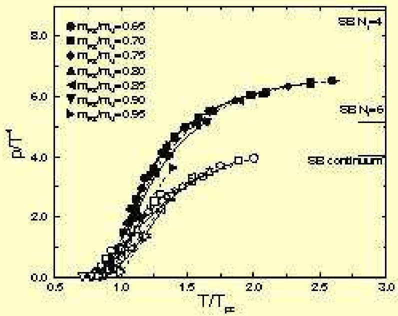
$N_t = 4$ is easy

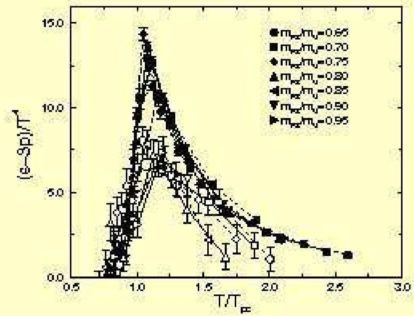
$N_t = 6$ is difficult

$N_t = 8$ is a challenge

Previous lattice results

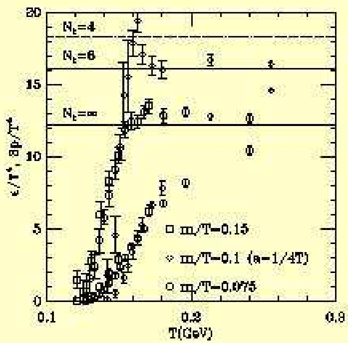
Wilson fermions: slower

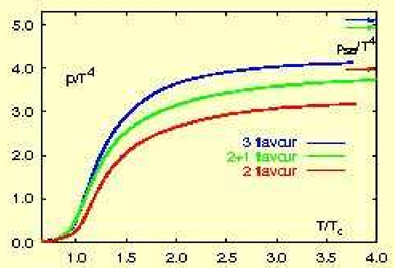




[Ali-Khan et al, '01]

Staggered fermions: faster



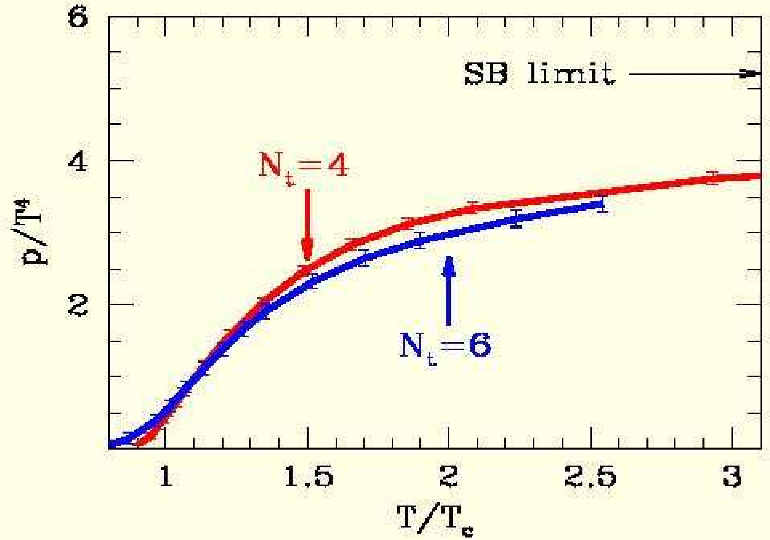


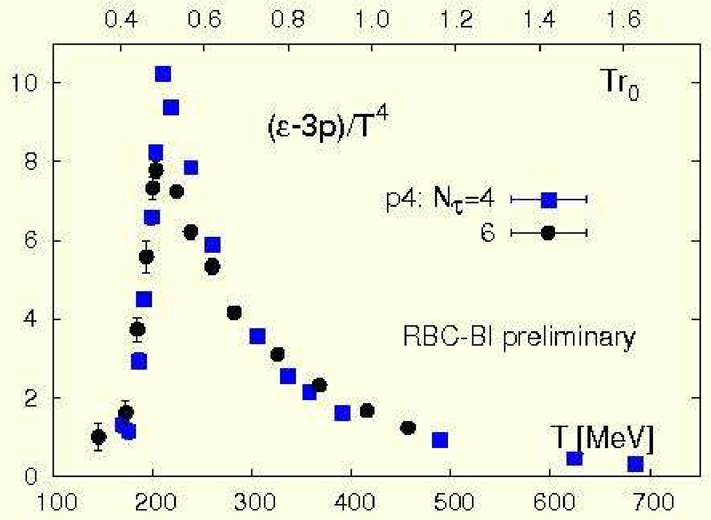
[Bernard et al, '96]   [Karsch, Laermann, Peikert, 2000]

Ongoing project: Bielefeld-Brookhaven-Columbia-Riken

Equation of state and scaling

Y.Aoki, Z.Fodor, S.D.Katz, K.K.Szabo, JHEP, 0601, 089, 2006.   F.Karsch, hep-ph/0701210



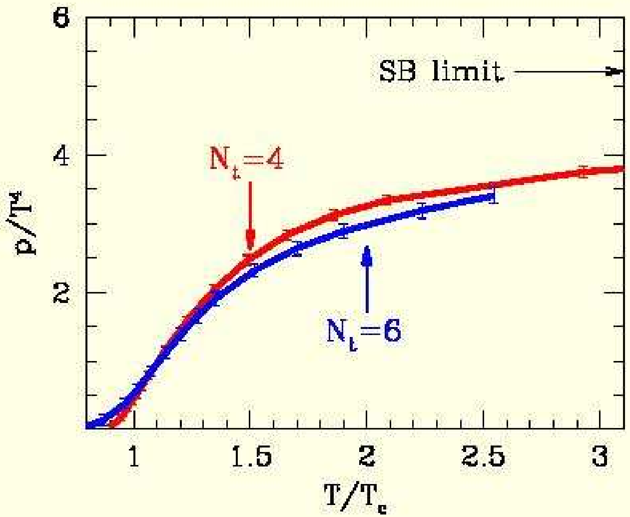


→ $N_t = 8$ is needed for final continuum-extrapolated result

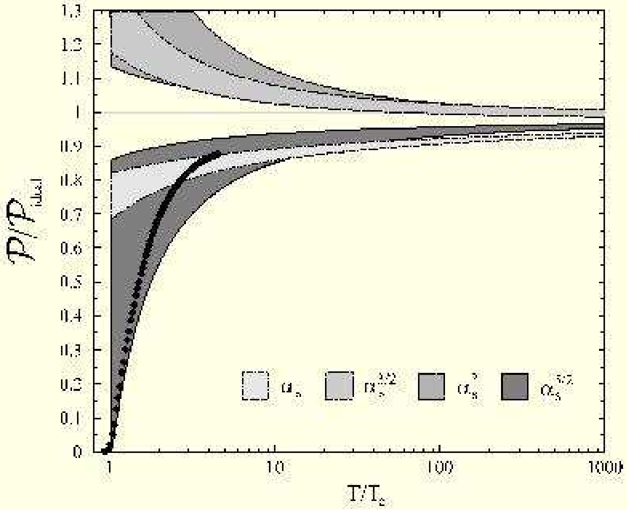
recent L_t=4,6 results also from the MILC collaboration: hep-lat/0611031

Link to perturbation theory: equation of state at large temperatures

lattice results for the EoS extend upto a few times T_c



perturbative series "converges" only at asymptotically high T



- the standard technique is the integral method:

$\bar{p}$=T/V·log(Z), but Z is difficult ⇒ $\bar{p}$ integral of (∂log(Z)/$\partial\beta$,∂log(Z)/∂m)
substract the T=0 term, the pressure is given by: $p(T)=\bar{p}(T)-\bar{p}(T=0)$

- back of an envelope estimate:

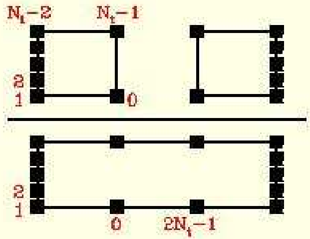
$T_c\approx$150–200 MeV, m_π=135 MeV and try to reach T=20·T_c for N_t=8 (a=0.0075 fm)
⇒$N_s > 4/m_\pi \approx 6/T_c$ = 6·20/T = 6·20·$N_t \approx$ 1000 ⇒ completely out of reach

a. substract successively: $p(T)=\bar{p}(T)-\bar{p}(T=0)=[\bar{p}(T)-\bar{p}(T/2)]+[\bar{p}(T/2)-\bar{p}(T/4)]+\ldots$

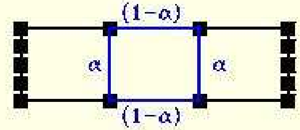
$\Longrightarrow$ for substractions at most twice as large lattices are needed

b. instead of the integral method calculate: $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\cdot\log[Z^2(N_t)/Z(2N_t)]$

$$\frac{Z^2(N_t)}{Z(2N_t)} =$$



$$\bar{Z}(\alpha) =$$



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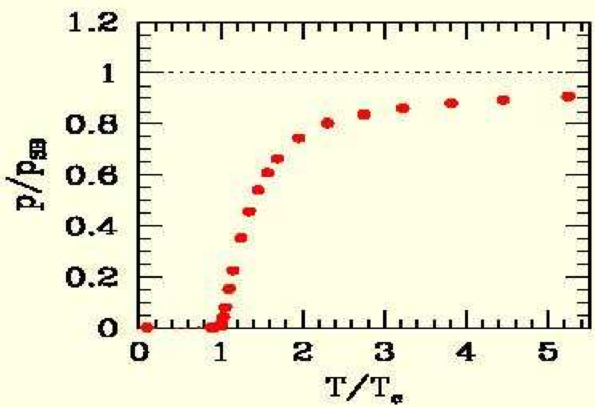
one gets directly $\bar{p}(T)-\bar{p}(T/2)=T/(2V)\int_0^1 d\log[\bar{Z}(\alpha)]/d\alpha\cdot d\alpha=T/(2V)\int_0^1 \langle S_{1b}-S_{2b}\rangle\alpha\cdot d\alpha$



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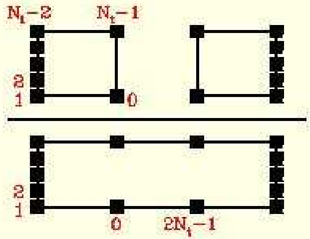

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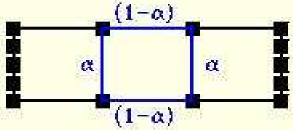

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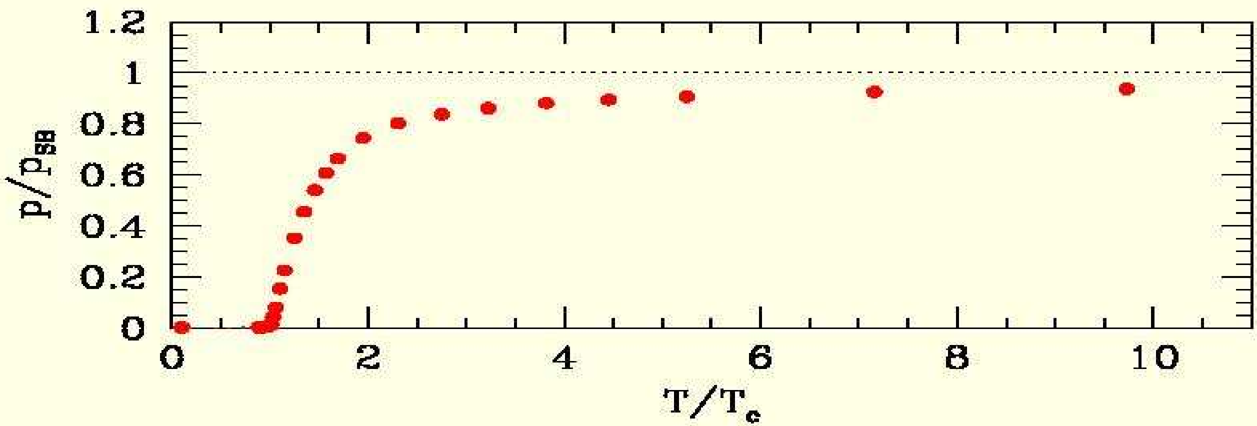

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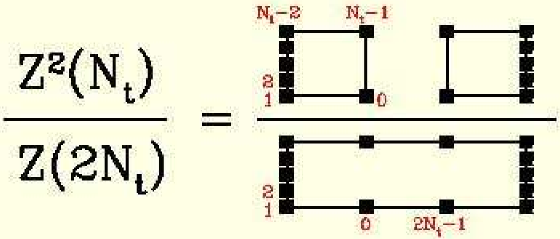

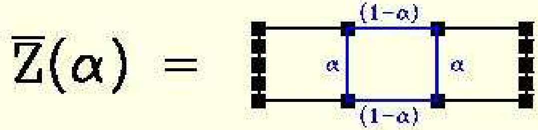

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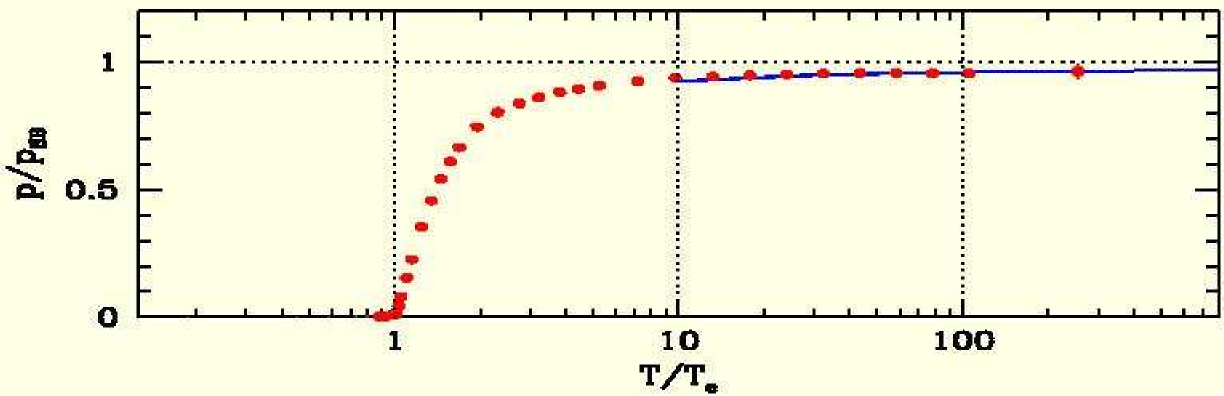

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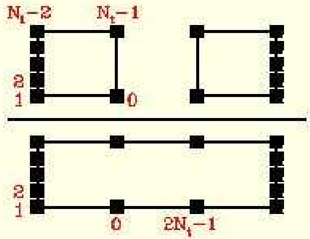
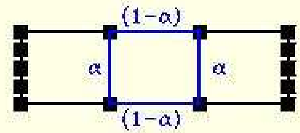
$\frac{Z^2(N_t)}{Z(2N_t)} =$ 

$\bar{Z}(\alpha) =$ 

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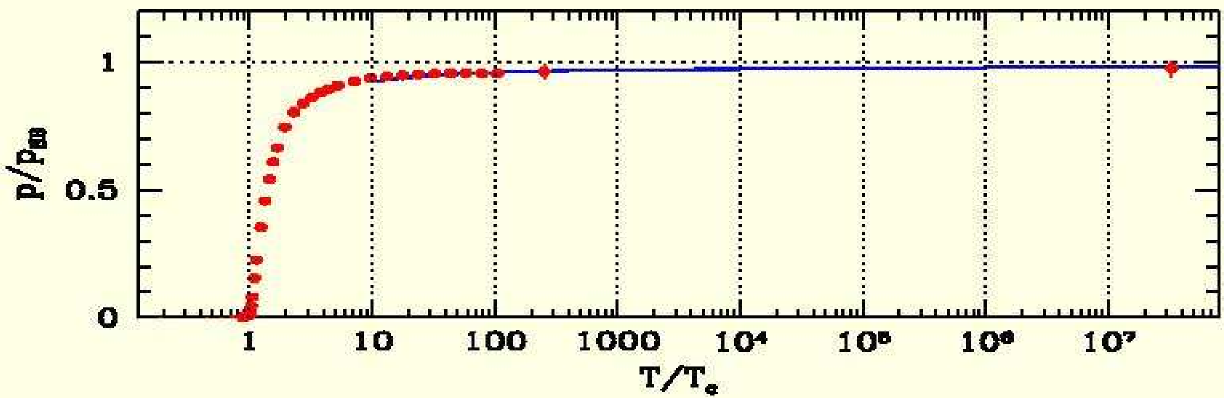

long awaited link between lattice thermodynamics and pert. theory is there

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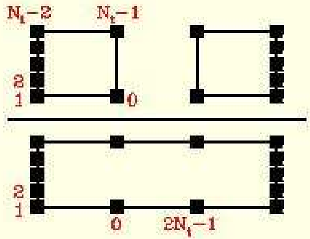

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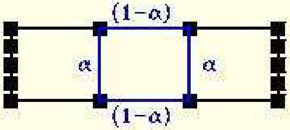

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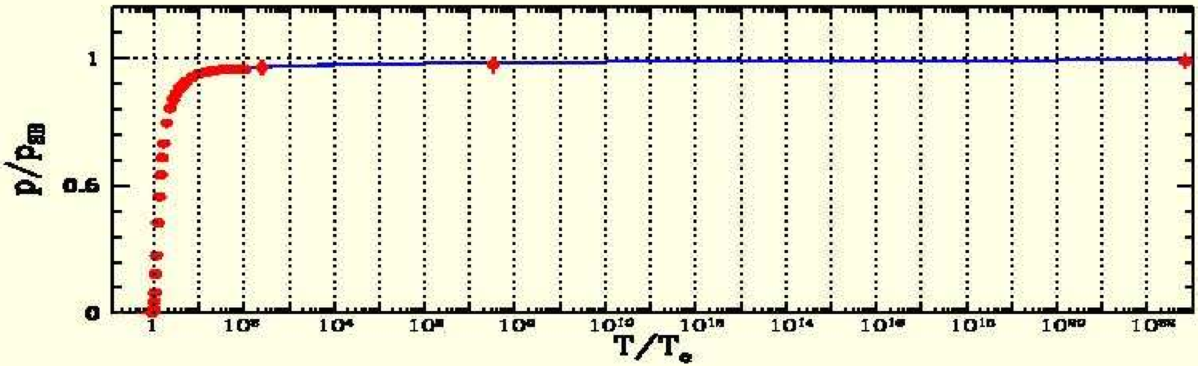

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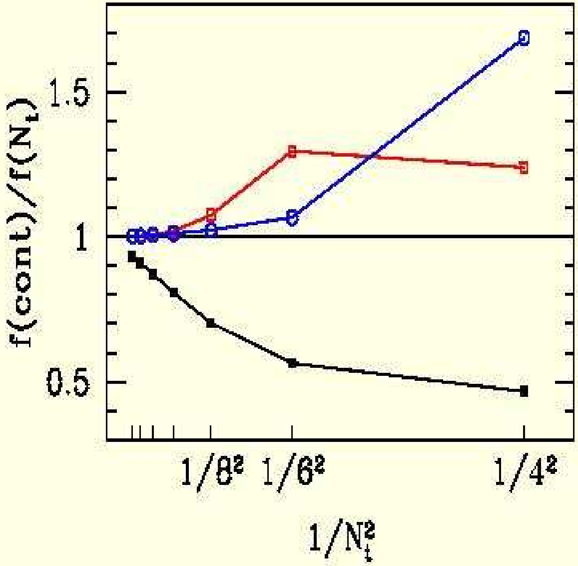
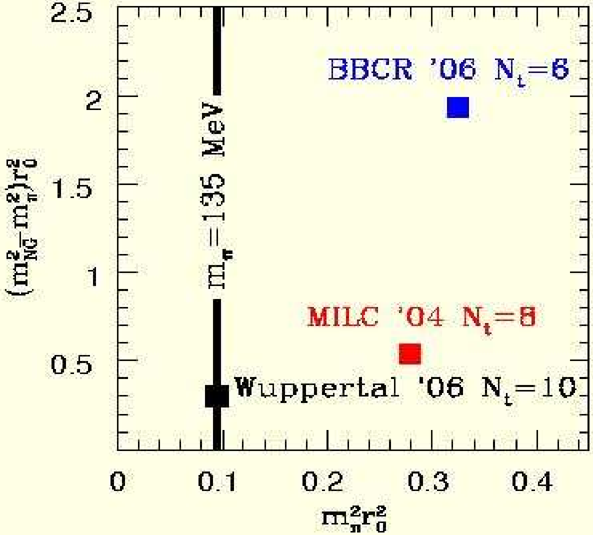
The nature of the QCD transition

Y.Aoki, G.Endrodi, Z.Fodor, S.D.Katz, K.K.Szabo, Nature, 443 (2006) 675 [hep-lat/0611014]

Symanzik improved gauge, stout improved n_f=2+1 staggered fermions
simulations along the line of constant physics: m_π=135 MeV, m_K=500 MeV



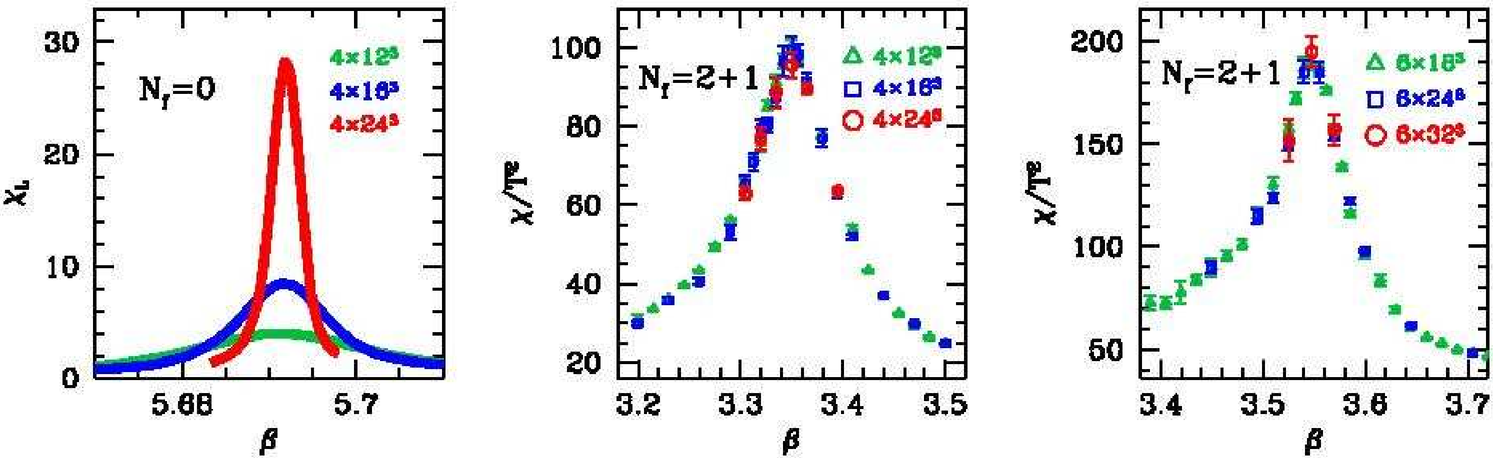


extrapolation from N_t and N_t+2 (standard action) ≈ as good as N_t with p4
N_t=8,10 gives ≈ ±1%, but a<0.15, 0.12 fm needed to set the scale (±1%)
thermodynamic quantities are obtained "more precisely" than the scale
(p4 independent config. is >10× more CPU ⇒ instead balance: a→0)

Finite size scaling of the chiral susceptibility: $\chi=(T/V)\partial^2\log Z/\partial m^2$

first order transition $\Longrightarrow$ peak width $\propto$ 1/V, peak height $\propto$ V

cross over $\Longrightarrow$ peak width $\approx$ constant, peak height $\approx$ constant



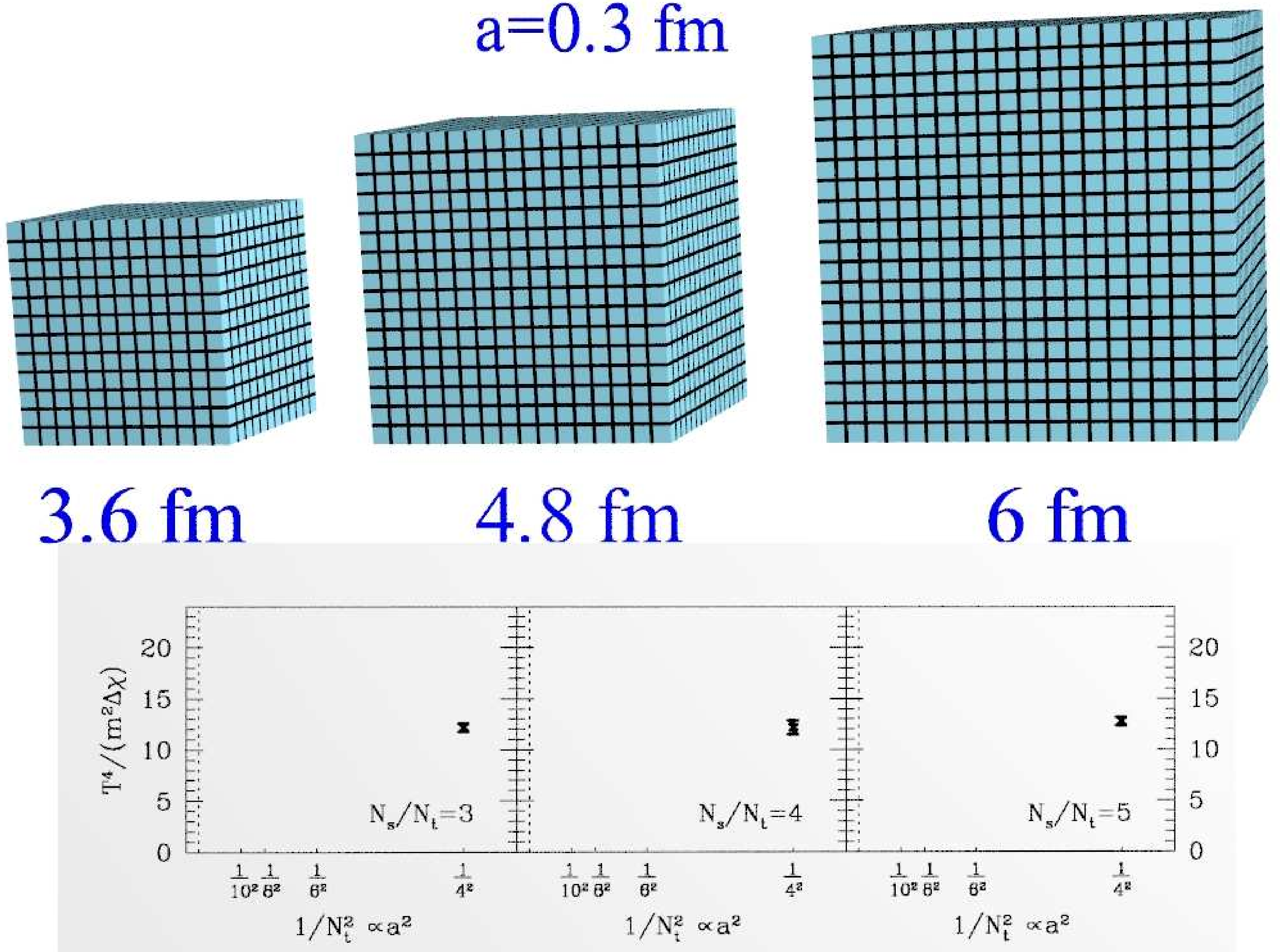
for aspect ratios 3–6 (an order of magnitude larger volumes)

volume independent scaling $\Longrightarrow$ cross-over

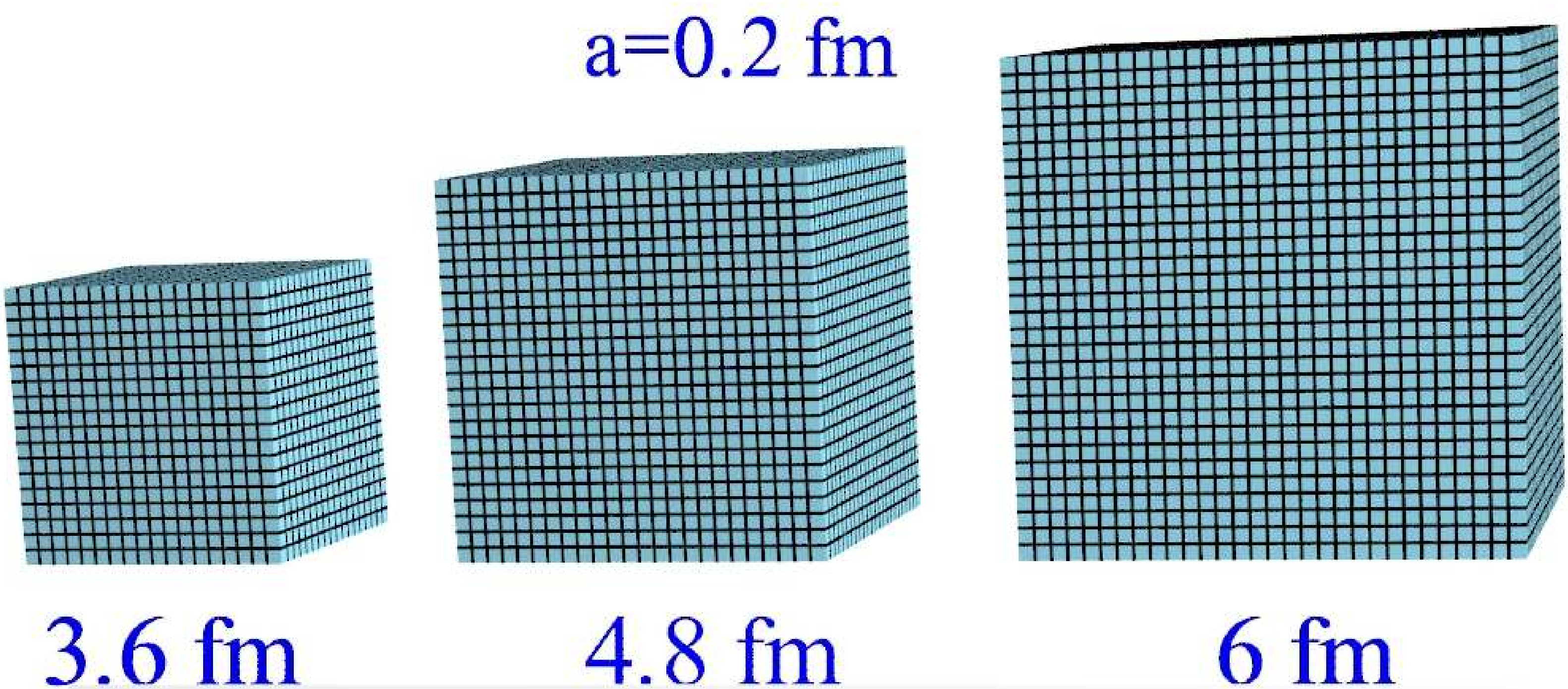
do we get the same result (cross-over) in the continuum limit?

one might have the unlucky case as we had in n_f=3 QCD:

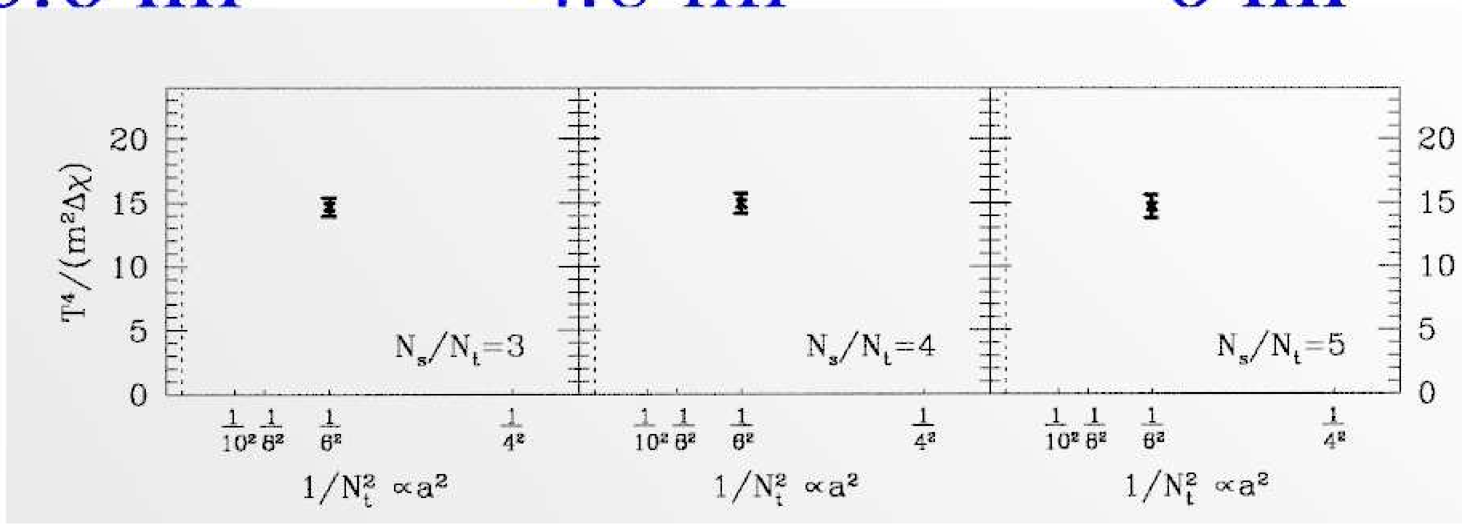
for physical m_π discretization errors changed the order



a=0.3 fm
3.6 fm
4.8 fm
6 fm
$T^4/(m^2\Delta\chi)$
0
5
10
15
20
$N_s/N_t=3$
$N_s/N_t=4$
$N_s/N_t=5$
$\frac{1}{10^2}$ $\frac{1}{8^2}$ $\frac{1}{6^2}$ $\frac{1}{4^2}$
$1/N_t^2 \propto a^2$

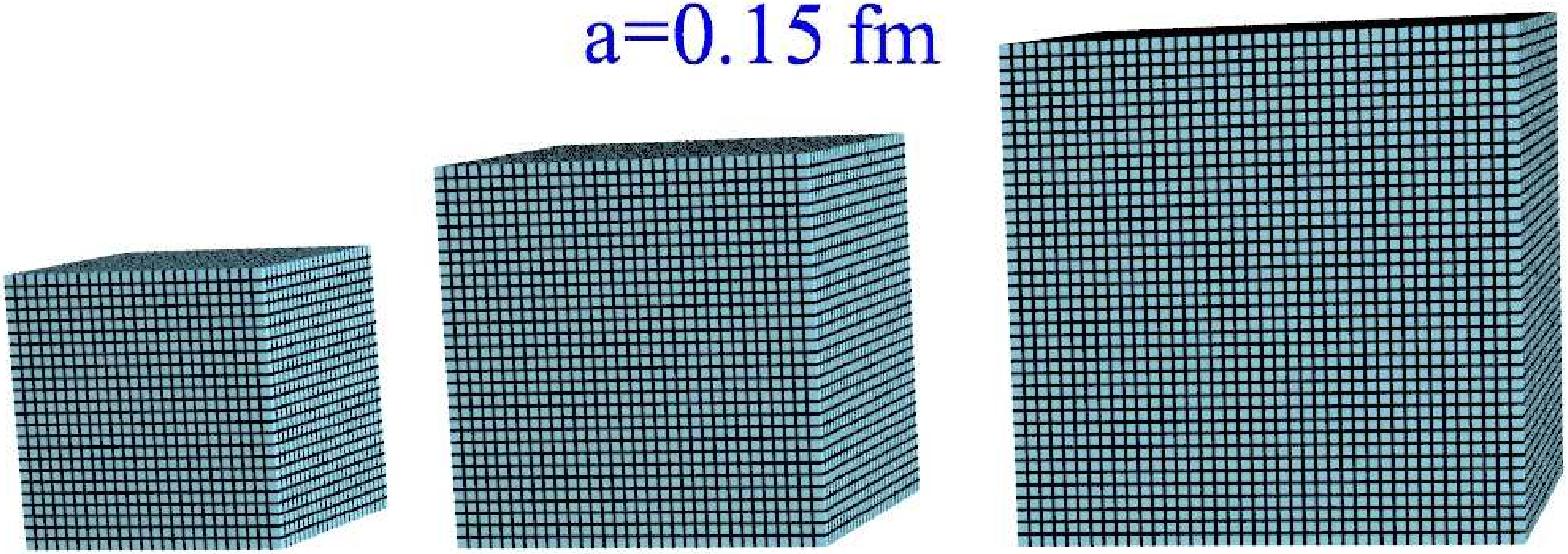


a=0.2 fm
3.6 fm
4.8 fm
6 fm

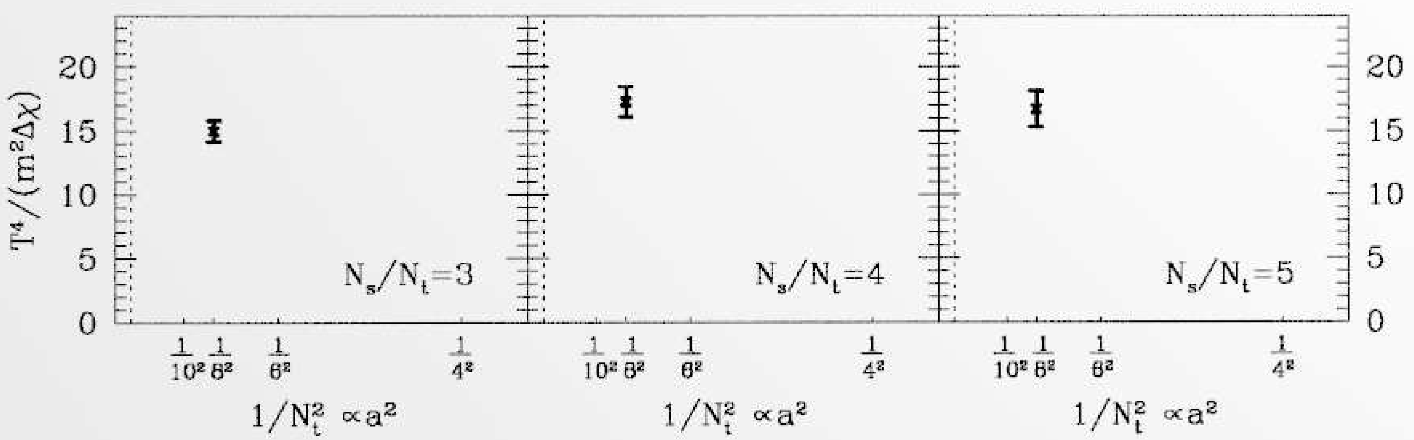


T⁴/(m²Δχ)
0
5
10
15
20
N_s/N_t=3
N_s/N_t=4
N_s/N_t=5
1/10²
1/8²
1/6²
1/4²
1/N_t² ∝a²

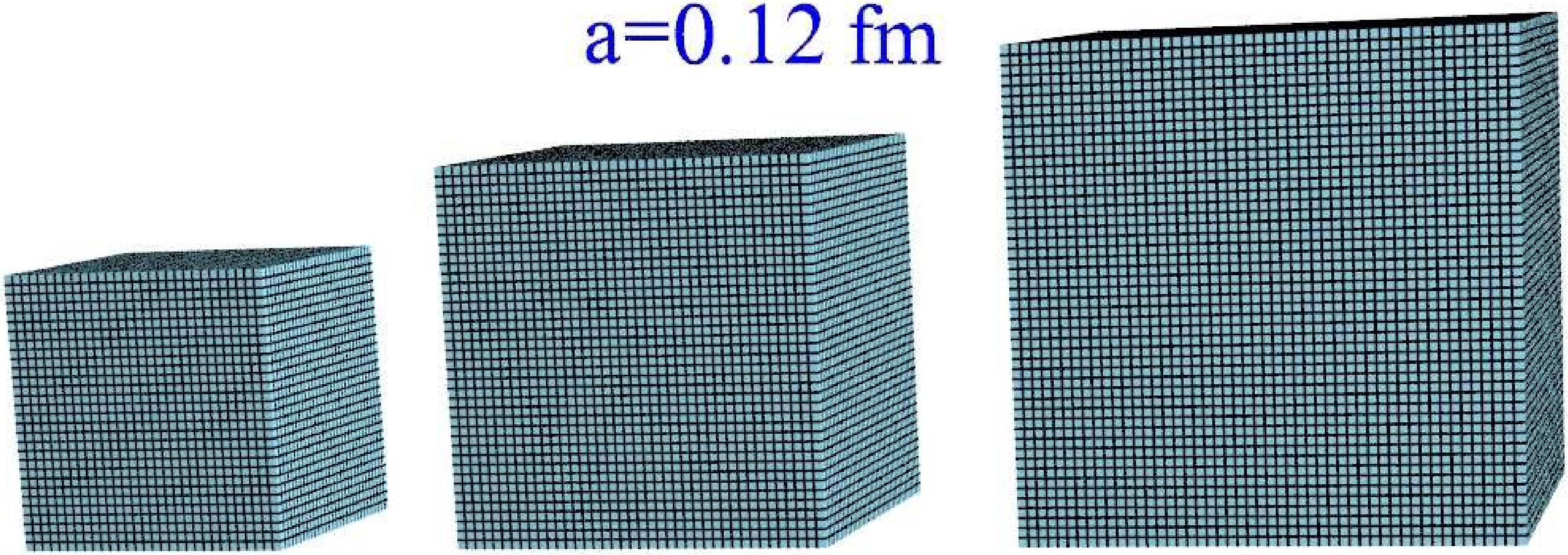
a=0.15 fm



3.6 fm 4.8 fm 6 fm



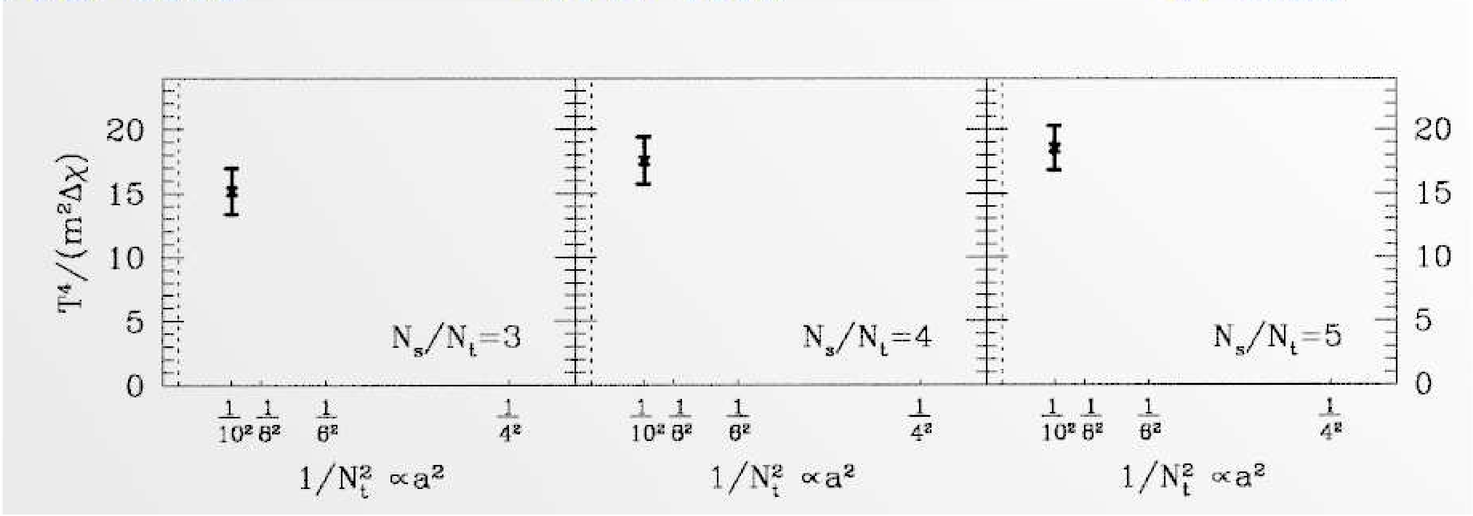
a=0.12 fm

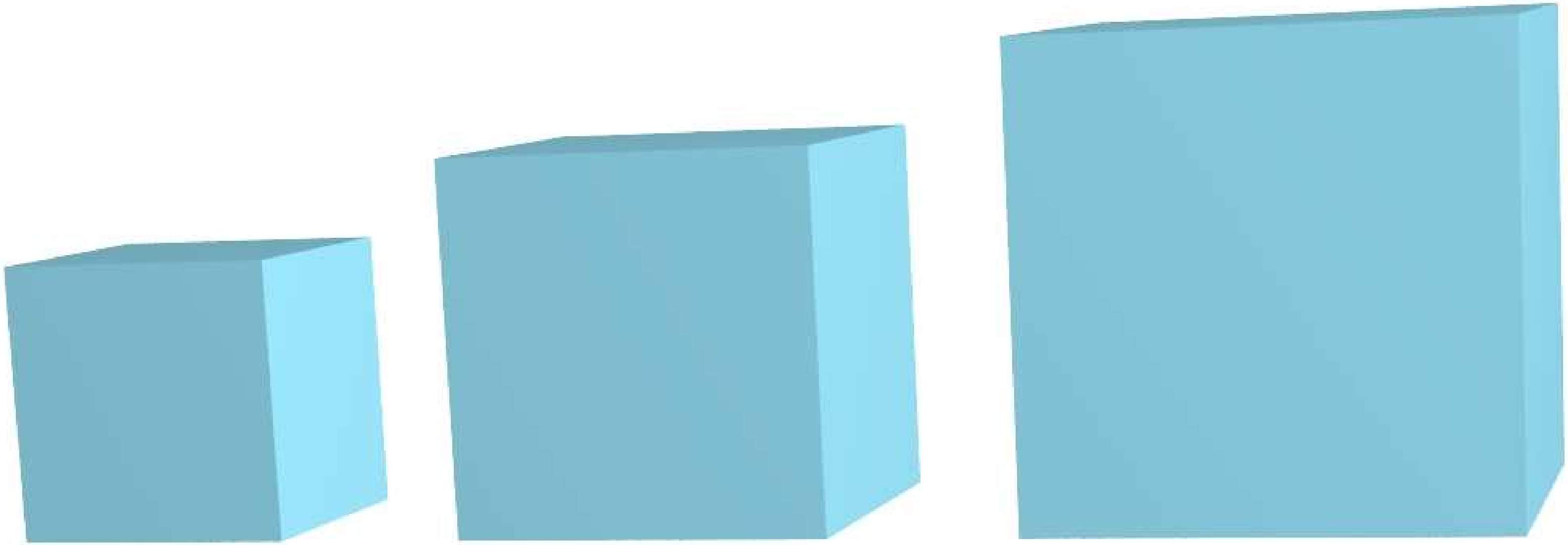


3.6 fm

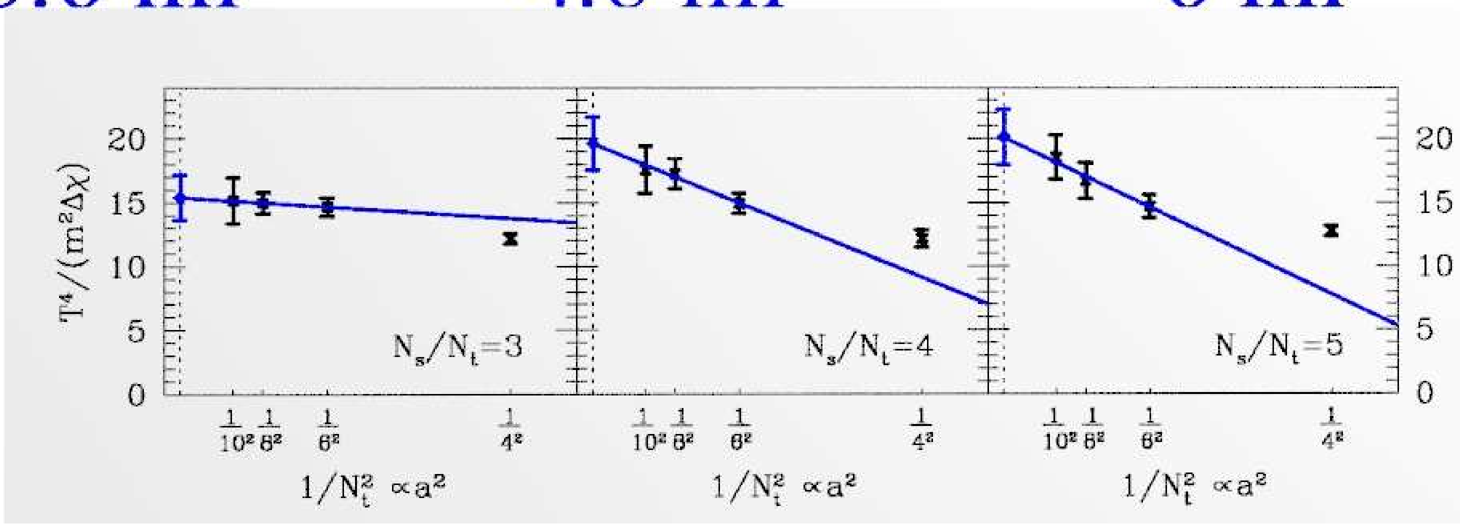
4.8 fm

6 fm

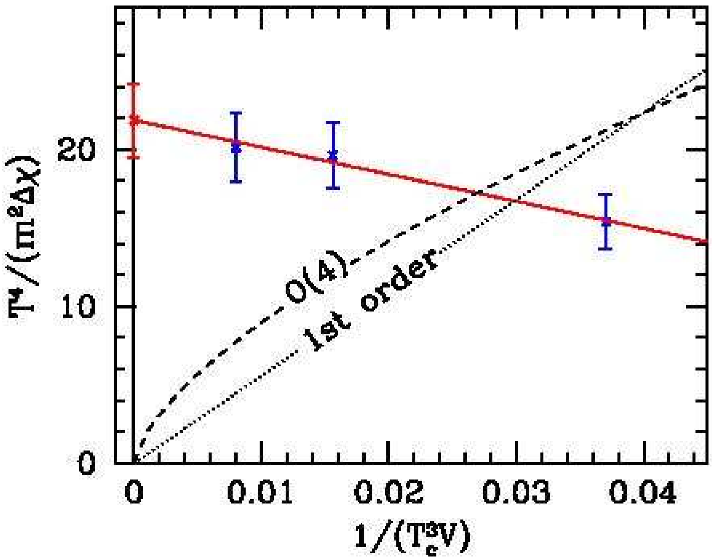




3.6 fm 4.8 fm 6 fm



• finite size study of continuum extrapolated $m^2\Delta\chi$ (N_t=4: off)



the result is consistent with an approximately constant behavior for a factor of 5 difference within the volume range

chance probability for 1/V is 10^{-19} for O(4) is $7 \cdot 10^{-13}$

continuum result with physical quark masses in staggered QCD:

the QCD transition at μ=0 is a cross-over

The transition temperature

Y. Aoki, Z. Fodor, S.D. Katz, K.K. Szabo, Phys. Lett. B. 643 (2006) 46 [hep-lat/0609068]

$T = 0$:

set the physical scale and locate the physical point

Three quantities are needed (m_π and m_K for the quark masses)

Several possibilities for the third quantity

- string tension (not existing in full QCD)
- static quark potential at intermediate distances ($r_0^2 \cdot dV/dr$=1.65)
- directly measurable quantities (e.g. f_K)

Further quantities are predictions (e.g. r_0, f_π, m_{K^*})

$T > 0$:

cross-over $\rightarrow$ different definitions give different T_c

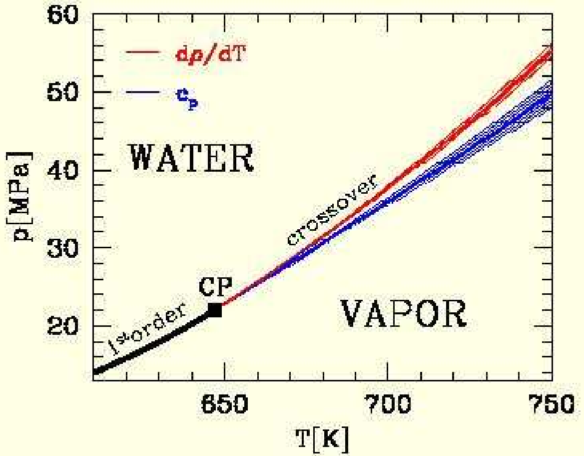
Possible choices:

- Chiral susceptibility
- Quark number susceptibility
- Polyakov-loop

T>0 Simulations

No well defined T_c

Example of water-steam transition



above the critical point c_p and $d\rho/dT$ give different T_cs.

Our choices in QCD

$\frac{m^2 \Delta\chi}{T^4} \rightarrow$ chiral transition

Quark number susceptibility
Polyakov loop $\rightarrow$ de-confinement transition

• an even more often experienced example

melting of ice shows singular behavior: ice $\longrightarrow$ water

melting of butter shows analytic behaviour (broad transition, cross-over)
natural fats are mixed triglycerids of fatty acids from C_4 to C_{24}
these are saturated or unsaturated of even carbon numbers

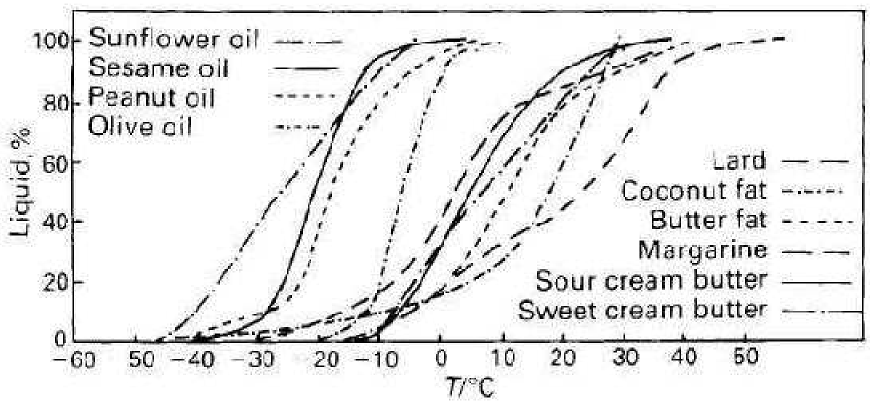
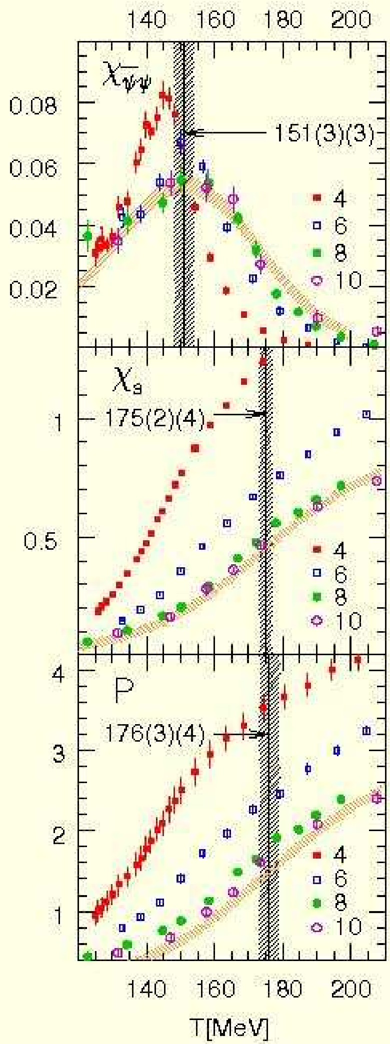


Fig. 4. Liquid proportions of various fats and oils

since in QCD we have an analytic cross-over
we will see very similar temperature dependence for all quantities
e.g. chiral condensate, strange quark susceptibility or Polyakov loop



Chiral susceptibility

T_c=151(3)(3) MeV

ΔT_c=28(5)(1) MeV

Quark number susceptibility

T_c=175(2)(4) MeV

ΔT_c=42(4)(1) MeV

Polyakov loop

T_c=176(2)(4) MeV

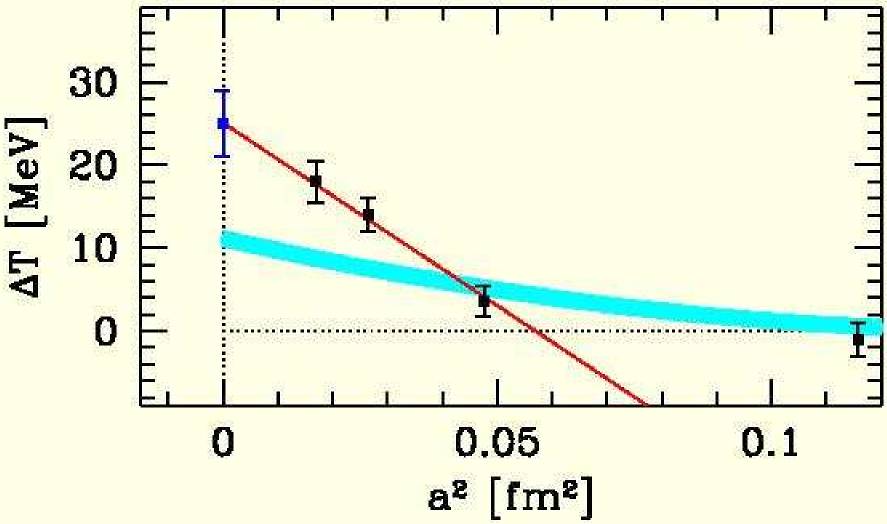
ΔT_c=38(5)(1) MeV

N_t=6,8,10 are in the a^2 scaling regime, N_t=8,10 are practically the same

⇒ 25(4) MeV difference between the chiral & the deconfinement transitions

normalization changes T_c (multiply a Gaussian by T^2 ⇒ peak shifts)
continuum: e.g. $\Delta\chi/T^2$ gives ≈10 MeV higher T_c than $m^2\Delta\chi/T^4$ (blue curve)

the difference can be seen only at small lattice spacings
C. De Tar hotQCD N_t=8 (asqtad): T_c from χ tends to be at smaller values



precise data at N_t=8 and 10 are needed to see the difference

• $T_c(\chi_{\bar\psi\psi})$ consistent with MILC '2004: $T_c = 169(12)(4)$ MeV

• BBCR collaboration: recent result [M. Cheng et.al, Phys. Rev. D74 (2006) 054507]
Transition temperature from $\chi_{\bar\psi\psi}$ and Polyakov loop, from both quantities

T_c=192(7)(4) MeV, $\Longrightarrow$ for $\chi_{\bar\psi\psi}$ contradicts our result (≈40 MeV)

Main differences to our work

no renormalization, χ/T^2 is used: explains only ≈ 10 MeV difference

only $N_t = 4$ & 6 (cutoff: a ≈0.3 fm & 0.2 fm or a^{-1}≈700 MeV & 1 GeV)

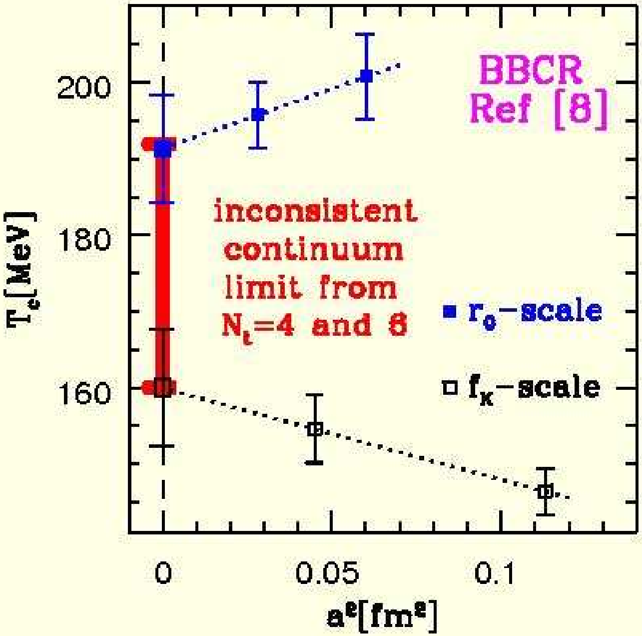
scale is set by r_0 instead of f_K (influences only the overall accuracy)

What is the reason for this discrepancy?

Their last concluding remark: it is desirable to
"obtain a reliable independent scale setting for the transition temperature from an observable not related to properties of the static potential".

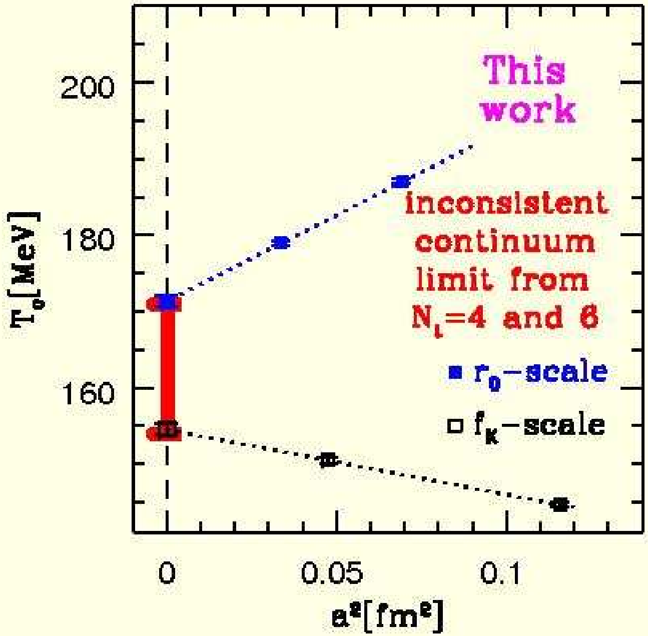
What if they used f_K to set the scale?

We repeated some of their $T = 0$ simulations to determine f_K



Alternatively:

We can use r_0 and only N_t=4,6

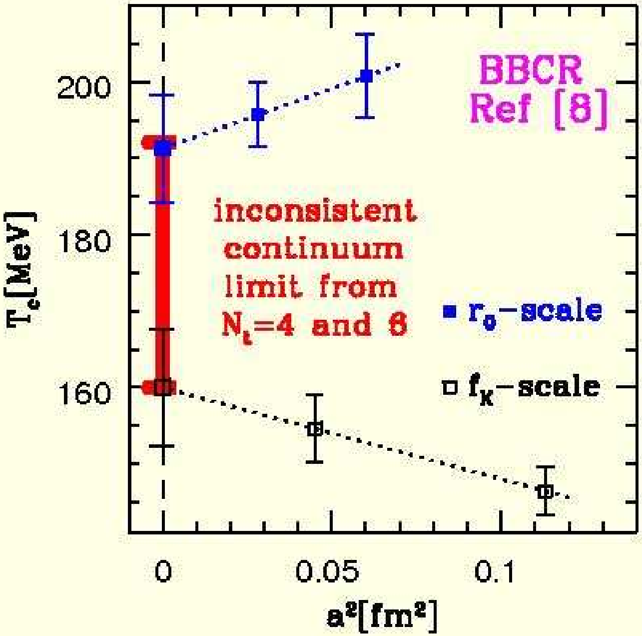


Continuum extrapolations from $N_t = 4, 6$ are inconsistent!

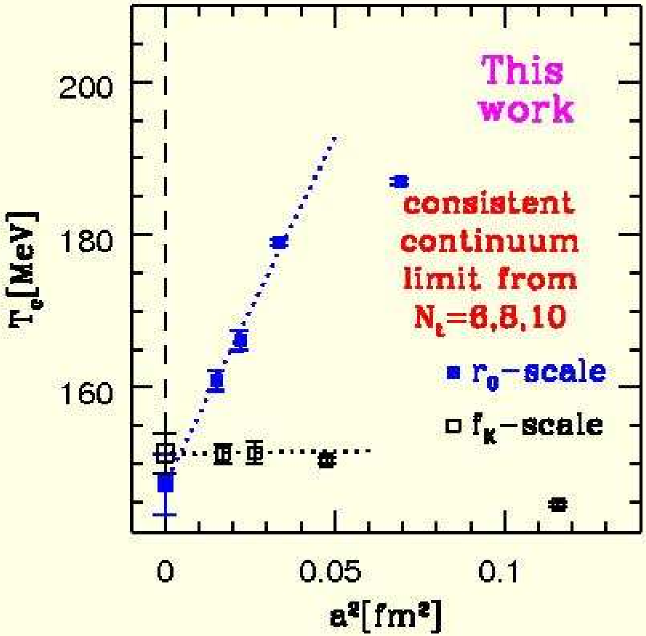
not surprising: eg. asqtad at $N_t \approx 10$ has $\approx$10% scale difference between r_1 & f_K
Lüscher (Dublin) & DelDebbio et al: a=.06fm $\approx$20% difference between r_0 & m_{K^*}

What if they used f_K to set the scale?

We repeated some of their $T = 0$ simulations to determine f_K



Alternatively:
We can use r_0 and only N_t=4,6,8,10



Continuum extrapolations from $N_t = 6, 8, 10$ are consistent!

Conclusion: continuum limit from N_t=4,6 isn't safe ($a \approx$0.3, 0.2 fm or 0.7, 1GeV)

Conclusions

lattice thermodynamics: important (already/soon full) results

nature of the transition: analytic transition (cross-over)

T_c discrepancies between groups: resolve it in the continuum

equation of state: still needs a continuum extrapolation

$\mu>0$ results are quite far from the continuum limit (N_t=4)

"all" results are within the staggered formalism (non-locality)

$\Longrightarrow$ closer to the continuum + non-staggered fermions