Color Instabilities in the Quark-Gluon Plasma

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- What are the plasma instabilities?
- Why do the instabilities occur?
- Why the instabilities are important?

Instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$

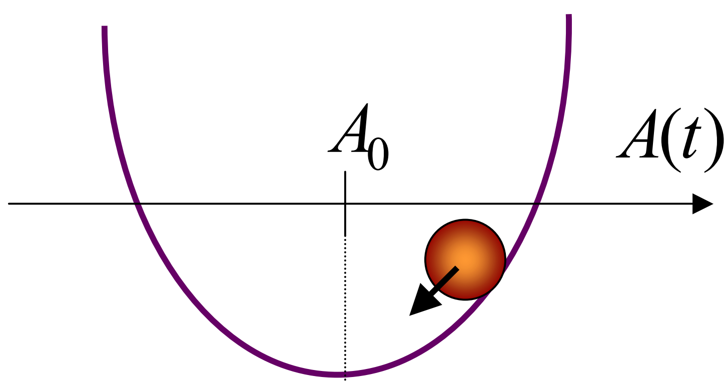
fluctuation

Instability

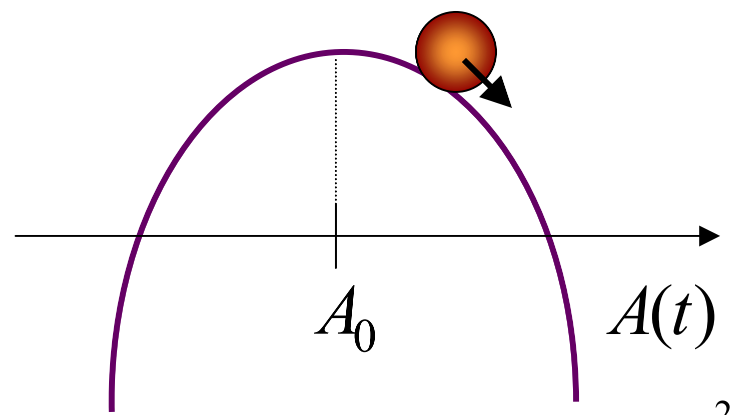
$$\delta A(t) \propto e^{\gamma t}$$

$$\gamma > 0$$

stable configuration



unstable configuration



Terminology

Plasma instabilities – interplay of particles and classical fields

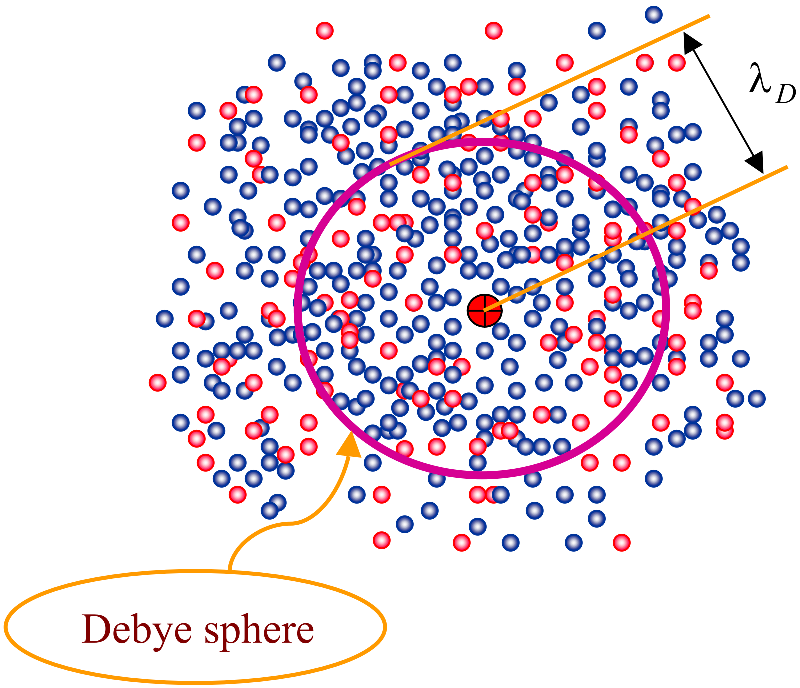
Quantum Field Theory – no particles, no classical fields

$$p_{\text{hard}} \sim T$$

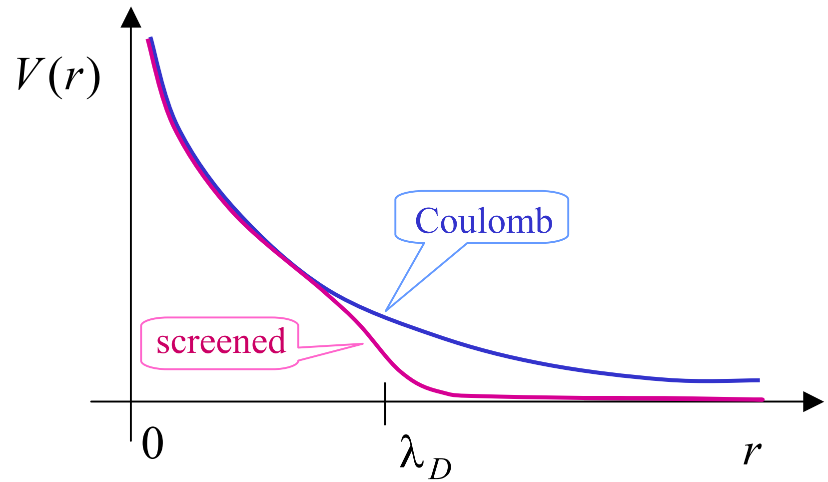
- particles – hard excitations, hard modes
- classical fields – highly populated soft excitations, soft modes

$$\sim 1/g^2 \qquad p_{\text{soft}} \sim gT$$

Plasma manifests collective behavior



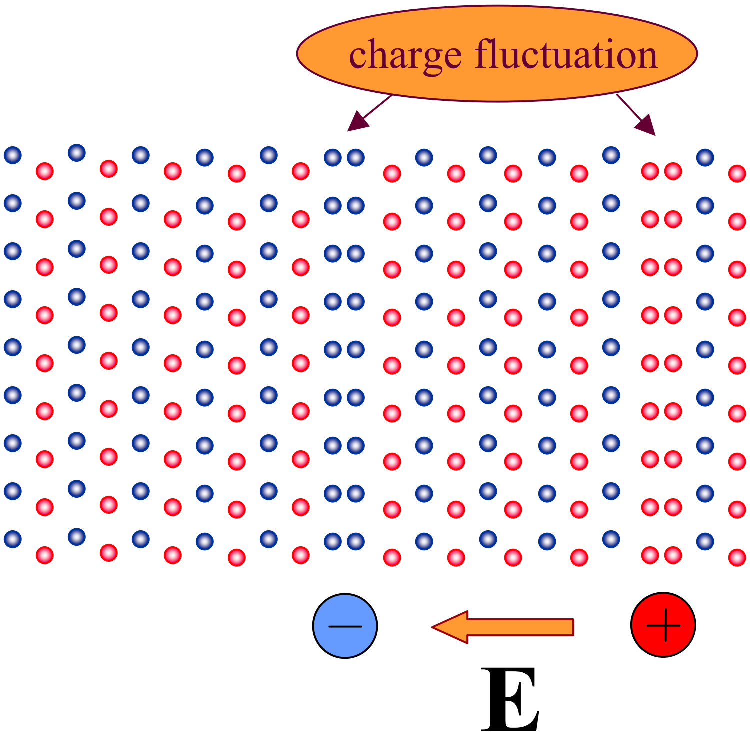
$$\lambda_D = \frac{1}{m_D} \sim \frac{1}{gT}$$ screening length $$V(r) \sim \frac{e^{-\frac{r}{\lambda_D}}}{r}$$



$$V_D = \frac{4}{3}\pi\lambda_D^3 \sim \frac{1}{g^3 T^3}, \qquad n \sim T^3, \qquad n\,V_D \sim \frac{1}{g^3} >> 1 \quad \text{if} \quad g << 1$$

In a weakly coupled plasma, there are many particles in a Debye sphere!

Plasma oscillations



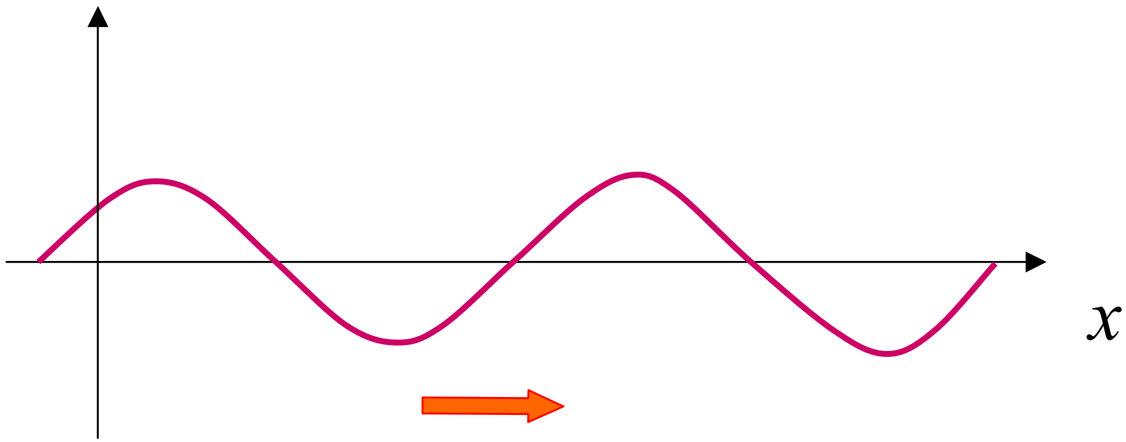
$$\mathbf{E}(t,\mathbf{r}) = \mathbf{E_0}\cos(\omega(\mathbf{k})\,t - \mathbf{k}\cdot\mathbf{r} + \varphi)$$

$$\omega(\mathbf{k}) \underset{\mathbf{k}\to 0}{\approx} \omega_{\mathbf{0}} \sim gT$$

plasma frequency

Landau damping

$$E^x(t,x) = E_0 \cos(\omega_0 t - kx)$$



$$v_\varphi = \frac{\omega_0}{k}$$

Resonance energy transfer from electric field to particles with $v = v_\varphi$

Plasma instabilities

- **instabilities in configuration space – hydrodynamic instabilities**

- **instabilities in momentum space – kinetic instabilities**

instabilities due to non-equilibrium momentum distribution

$$f(\mathbf{p}) \text{ is not } \sim \exp\left(-\frac{E}{T}\right)$$

Kinetic instabilities

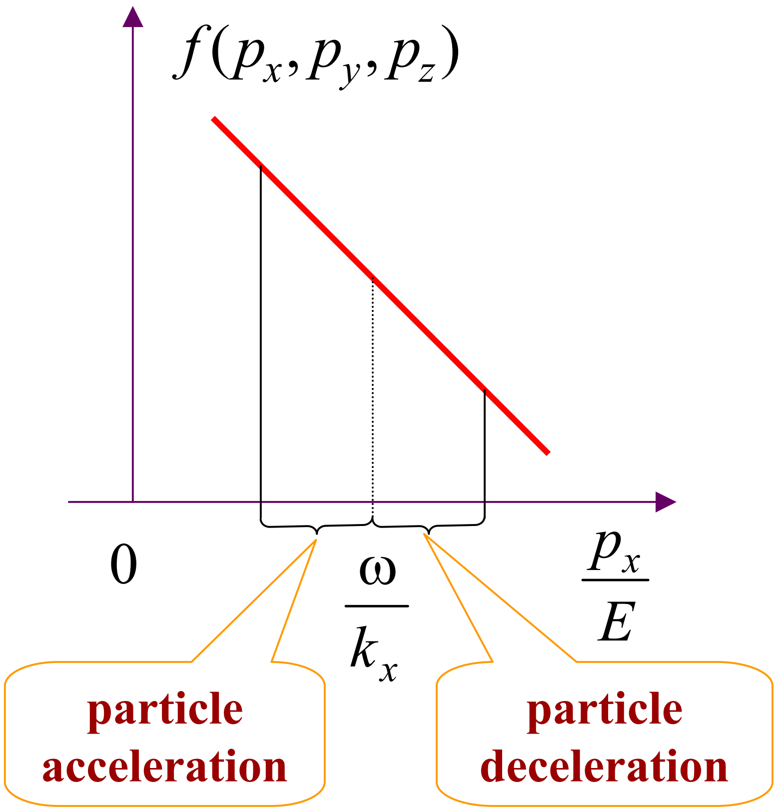
- **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}, \ \delta\rho \sim e^{-i(\omega t - \mathbf{kr})}$

- **transverse modes** – $\mathbf{k} \perp \mathbf{E}, \ \delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{kr})}$

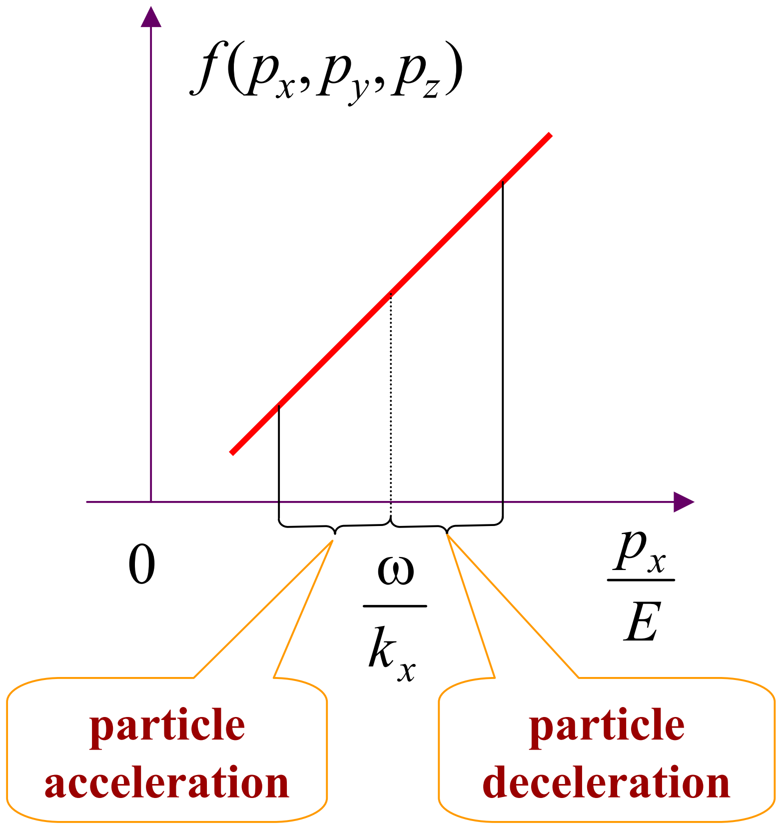
E – electric field, **k** – wave vector, ρ – charge density, **j** - current

Logitudinal modes

Electric field decays - damping



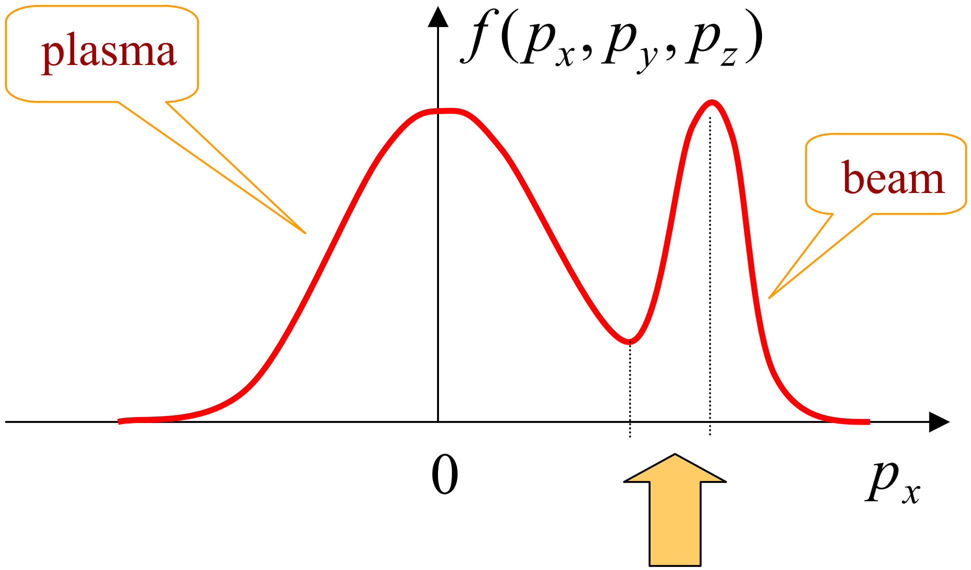
Electric field grows - instability



$\frac{\omega}{k_x}$ - phase velocity of the electric field wave, $\frac{p_x}{E}$ - particle's velocity

Logitudinal modes

unstable configuration

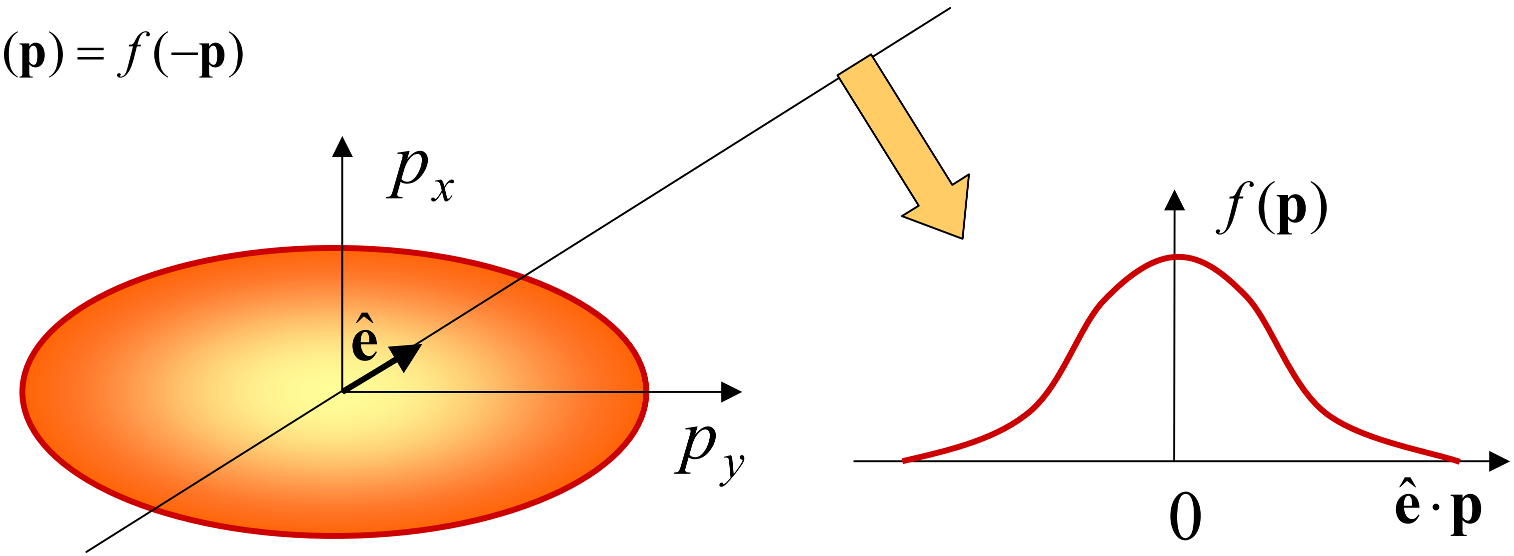


Energy is transferred from particles to fields

Transverse modes

Unstable modes occur due to anisotropy of the momentum distribution

$$f(\mathbf{p}) = f(-\mathbf{p})$$

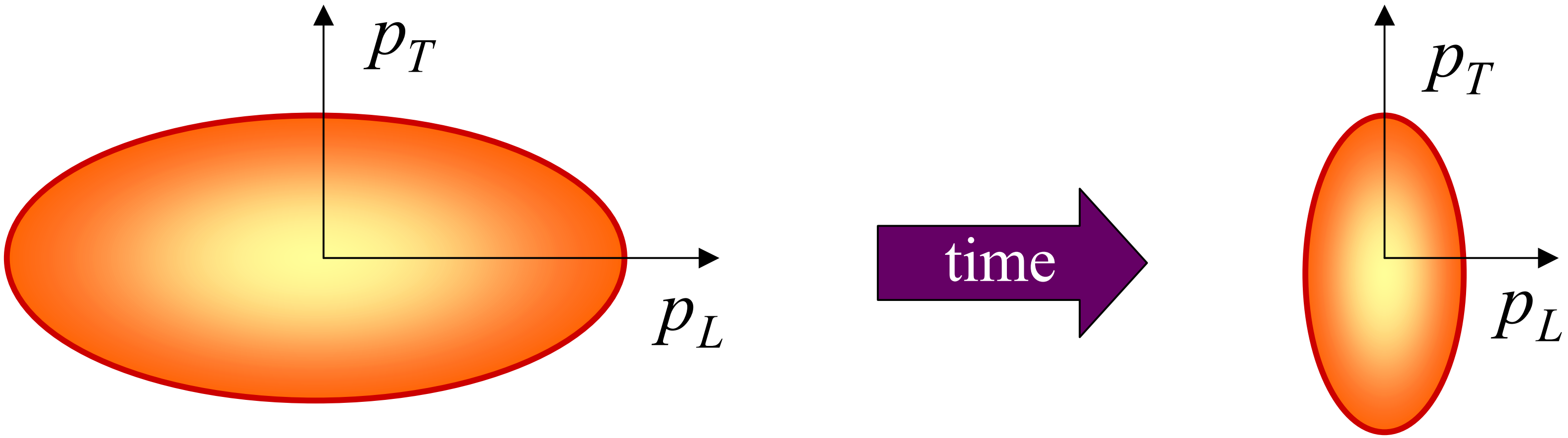


Momentum distribution distribution can monotonously decrease in every direction

Transverse modes are relevant for relativistic nuclear collisions!

Momentum Space Anisotropy in Nuclear Collisions

Parton momentum distribution is initially strongly anisotropic



CM after 1-st collisions

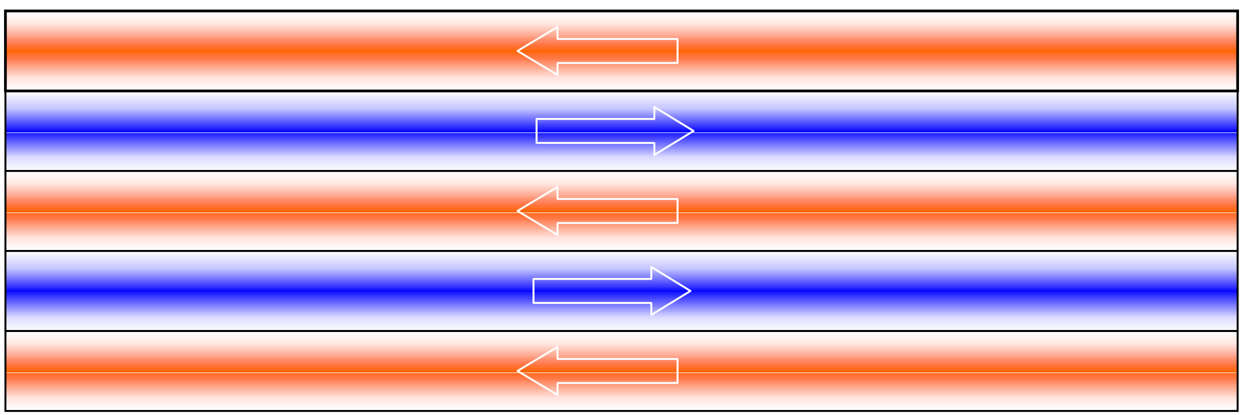
local rest frame

Seeds of instability

$\left\langle j_a^\mu(x) \right\rangle = 0$ **but current fluctuations are finite**

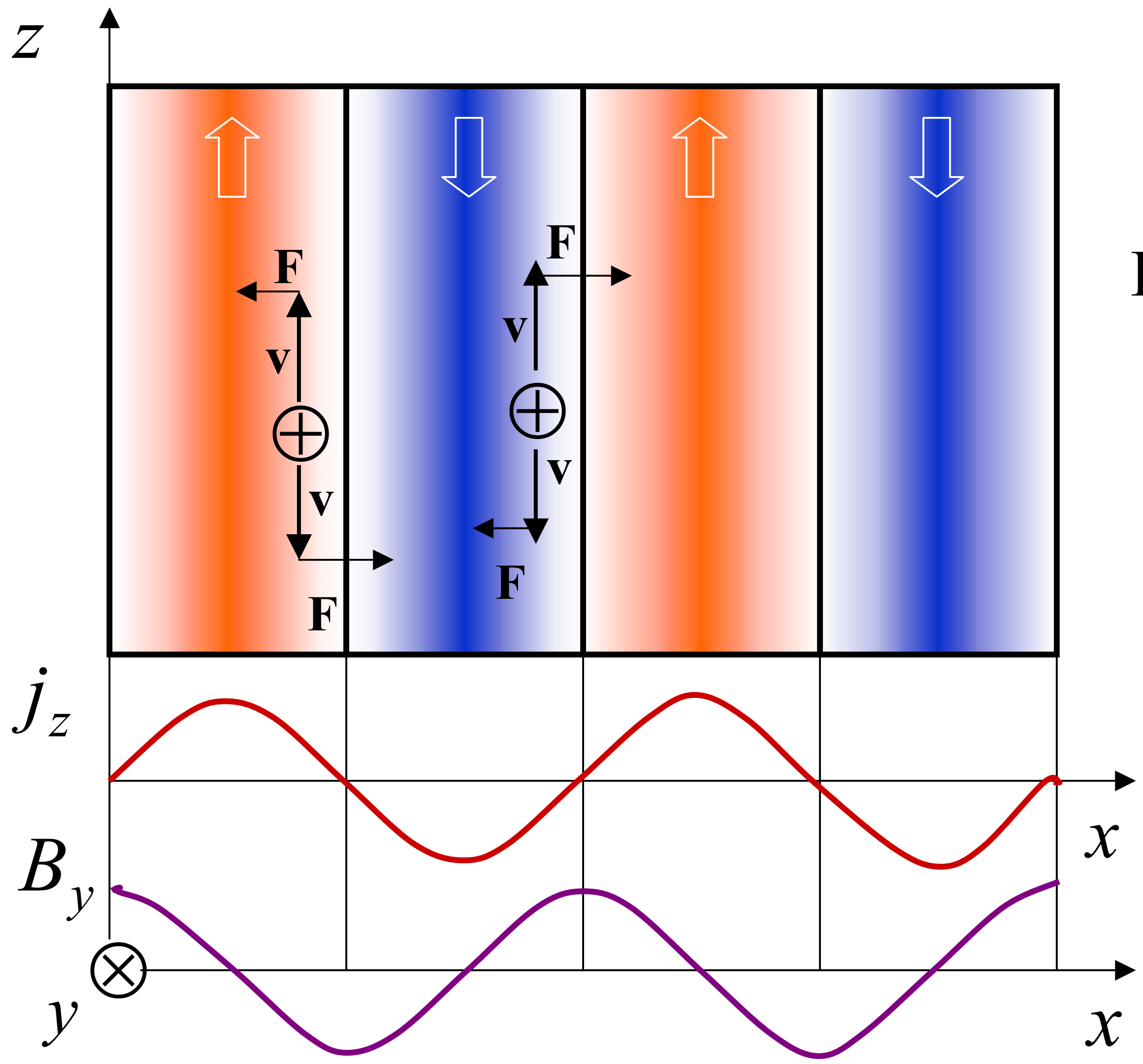
$$\left\langle j_a^\mu(x_1)\, j_b^\nu(x_2) \right\rangle = \frac{1}{8}\delta^{ab} \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p^2} f(\mathbf{p})\delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0$$

$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus

Mechanism of filamentation



Lorentz force

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

Ampere's law

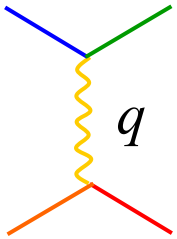
$$\nabla \times \mathbf{B} = \mathbf{j}$$

Instabilities vs. collisions

Time scale of parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

$$t_{\text{soft}} \sim \frac{1}{g^2 \ln(1/g)\, T}$$



hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Time scale of collective phenomena

$$t_{\text{collec}} \sim \frac{1}{g\,T}$$

$$g^2 << 1 \quad \Rightarrow \quad t_{\text{hard}} >> t_{\text{soft}} >> t_{\text{collec}}$$

The instabilities are fast!

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with Im$\omega > 0$ $\Rightarrow$ $A^\mu(x) \sim e^{\mathrm{Im}\,\omega t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. **How to get it?**

Transport theory – distribution functions

Distribution functions of quarks $Q(p,x)$ and antiquarks $\overline{Q}(p,x)$

are gauge dependent $N_c \times N_c$ matrices

The gauge transformation:

$$Q(p,x) \to U(x)\, Q(p,x)\, U^{-1}(x)$$

Distribution function of gluons $G(p,x)$ is $(N_c^2 - 1) \times (N_c^2 - 1)$ matrix

Transport theory – transport equations

fundamental:

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu Q\} = C \qquad \text{quarks}$$

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu \overline{Q}\} = \overline{C} \qquad \text{antiquarks}$$

adjoint:

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{\mathcal{F}_{\mu\nu},(x)\partial_p^\nu G\} = C_g \qquad \text{gluons}$$

free streaming — mean-field force — collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, ...], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]$$

mean-field generation

collisionless limit: $C = \overline{C} = C_g = 0$

Transport theory - linearization

fluctuation

$$Q(p,x) = Q_0(p) + \delta Q(p,x)$$

stationary colorless state $Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| >> |\delta Q(p,x)|, \quad |\partial_p^\mu Q_0(p)| >> |\partial_p^\mu \delta Q(p,x)|$$

Linearized transport equations

$$p_\mu D^\mu \delta Q(p,x) - g p^\mu F_{\mu\nu}(x) \partial_p^\nu Q_0(p) = 0$$

$$p_\mu D^\mu \delta \overline{Q}(p,x) + g p^\mu F_{\mu\nu}(x) \partial_p^\nu \overline{Q}_0(p) = 0$$

$$p_\mu \mathcal{D}^\mu \delta G(p,x) - g p^\mu \mathcal{F}_{\mu\nu}(x) \partial_p^\nu QG_0(p) = 0$$

Transport theory – polarization tensor

$$\delta Q(p,x) = g\int d^4x'\,\Delta_p(x-x')\,p^\mu F_{\mu\nu}(x)\partial_p^\nu Q_0(p)$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$j^\mu[\delta Q, \delta\overline{Q}, \delta G]$$

$$j^\mu(k) = \Pi^{\mu\nu}(k)A_\nu(k)$$

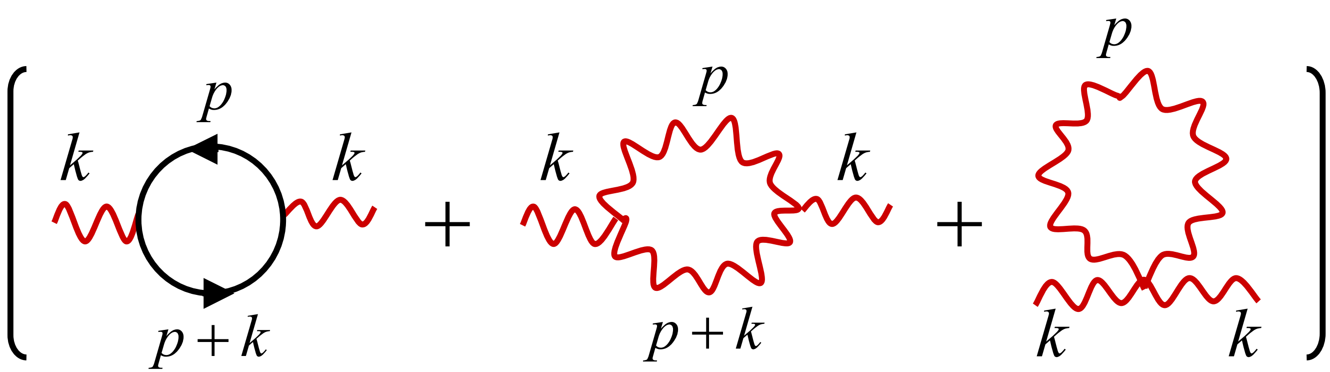
$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \overline{n}(\mathbf{p}) + 2n_g(\mathbf{p})$$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\left[g^{\mu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\right]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu\Pi^{\mu\nu}(k) = 0$$

Diagrammatic Hard Loop approach

$\Pi^{\mu\nu}(k) =$



Hard loop approximation: $k^\mu << p^\mu$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p^\mu}{E} \left[g^{\mu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+} \right] \frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

St. M. & M. Thoma, Phys. Rev. C **62**, 036011 (2000)

Chromo-hydrodynamic approach for short time scales

Collisionless transport equation of quark distribution function $Q(p,x)$

$$p_\mu D^\mu Q(p,x) - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu Q(p,x)\} = 0$$

$$\int dP$$

Taking into account antiquarks and gluons is straightforward

$$dP \equiv \frac{d^4 p}{(2\pi)^3} 2\Theta(p_0)\,\delta(p^2)$$

Covariant continuity

$$D_\mu n^\mu(x) = 0$$

$$n^\mu(x) \equiv \int dP\, p^\mu Q(p,x)$$

C. Manuel & St. M., Phys. Rev. **D74**, 105003 (2006)

Chromo-hydrodynamic approach cont.

$$p_\mu D^\mu Q(p,x) - \frac{g}{2} p^\mu \{F_{\mu\nu}, \partial_p^\nu Q(p,x)\} = 0$$

$$\int dP\, p^\mu \;\boxed{}$$

$$D_\mu T^{\mu\nu}(x) - \frac{g}{2}\{F_{\mu\nu}, n^\mu(x)\} = 0$$

$$T^{\mu\nu}(x) \equiv \int dP\, p^\mu p^\nu Q(p,x)$$

$$p^2 = 0$$

$$T^\mu_\mu(x) = 0$$

Chromo-hydrodynamic equations

$$D_\mu n^\mu(x) = 0$$

$$D_\mu T^{\mu\nu}(x) - \frac{g}{2}\{F^{\mu\nu}, n_\mu(x)\} = 0$$

Postulated form of $n^\mu(x)$ and $T^{\mu\nu}(x)$:

isotropy in the local rest frame

$$n^\mu(x) = n(x)\, u^\mu(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2}\left(\varepsilon(x) + p(x)\right)\{u^\mu(x), u^\mu(x)\} - p(x)\, g^{\mu\nu}$$

$n(x)$, $\varepsilon(x)$, $p(x)$, $u^\mu(x)$ matrices! $\quad u^\mu(x)\, u_\mu(x) = 1$

To close the system of equations: $\nabla p = 0$ or $\varepsilon = 3p \Leftarrow T^\mu_\mu = 0$

Linear response approximation

Small perturbation of the space-time homogeneous & colorless state

$$n(x) = \tilde{n} + \delta n(x), \quad \varepsilon(x) = \tilde{\varepsilon} + \delta\varepsilon(x),$$

$$p(x) = \tilde{p} + \delta p(x), \quad u^{\mu}(x) = \tilde{u}^{\mu} + \delta u^{\mu}(x)$$

$\tilde{n}, \tilde{\varepsilon}, \tilde{p}, \tilde{u}^{\mu}$ unit matrices in color space

$$\tilde{n} >> \delta n, \quad \tilde{\varepsilon} >> \delta\varepsilon, \quad \tilde{p} >> \delta p, \quad \tilde{u}^{\mu} >> \delta u^{\mu}$$

$$F^{\mu\nu} \sim A^{\mu} \sim \delta n$$

Solutions of the linearized equations

- $D^{\mu} \to \partial^{\mu}$ full linearization $A^{\mu} \sim \delta n$
- Fourier transformations
- $\partial^{\nu}\delta p \approx 0$

continuity	$k_{\mu}\tilde{u}^{\mu}\,\delta n(k) + \tilde{n}\,k_{\mu}\delta u^{\mu}(k) = 0$
Euler	$i(\tilde{\varepsilon} + \tilde{p})\tilde{u}^{\mu}k_{\mu}\delta u^{\nu}(k) - g\tilde{n}\tilde{u}_{\mu}F^{\mu\nu}(k) = 0$

Solutions

$$\delta n(k) = ig\,\frac{\tilde{n}^2}{\tilde{\varepsilon} + \tilde{p}}\,\frac{\tilde{u}_{\nu}\,k_{\mu}}{(\tilde{u}\cdot k)^2}\,F^{\mu\nu}(k)$$

$$\delta u^{\nu}(k) = ig\,\frac{\tilde{n}}{\tilde{\varepsilon} + \tilde{p}}\,\frac{\tilde{u}_{\mu}}{\tilde{u}\cdot k}\,F^{\mu\nu}(k)$$

Color current & polarization tensor

$$j^{\mu}(x) = -\frac{g}{2}\left(n(x)\, u^{\mu}(x) - \frac{1}{N_c}\,\mathrm{Tr}[\, n(x)\, u^{\mu}(x)] \right)$$

$$j^{\mu}(x) = \tilde{j}^{\mu} + \delta j^{\mu}(x), \qquad \tilde{j}^{\mu} = 0$$

$$\delta j^{\mu}(x) = -\frac{g}{2}\left(\tilde{n}\, \delta u^{\mu}(x) + \tilde{u}^{\mu}\, \delta n(x) \right)$$

$\mathrm{Tr}[F^{\mu\nu}] = 0$

polarization tensor $\Pi^{\mu\nu}(x, y) = -\dfrac{\delta j^{\mu}(x)}{\delta A_{\nu}(y)}$

Polarization tensor

$$\Pi^{\mu\nu}(k) = -\frac{g^2}{2}\frac{\tilde{n}^2}{\tilde{\varepsilon}+\tilde{p}}\frac{(\tilde{u}\cdot k)(\tilde{u}^{\mu}k^{\nu}+\tilde{u}^{\nu}k^{\mu}) - k^2\tilde{u}^{\mu}\tilde{u}^{\nu} - (\tilde{u}\cdot k)^2 g^{\mu\nu}}{(\tilde{u}\cdot k)^2}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k)\,, \qquad k_{\mu}\Pi^{\mu\nu}(k) = 0$$

Dispersion equation

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$

chromodielectric tensor

$$k^\mu \equiv (\omega, \mathbf{k})$$

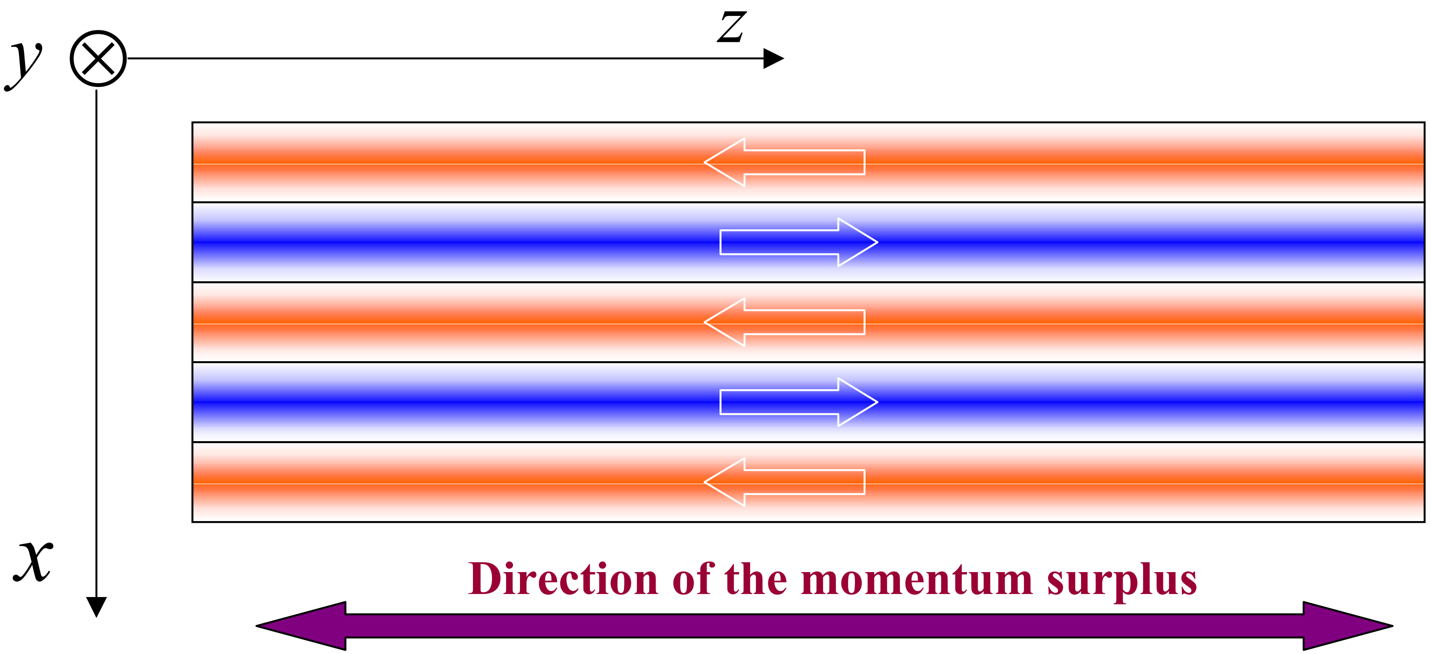
Dispersion equation

$$\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int \frac{d^3p}{(2\pi)^3} \frac{v^i}{\omega - \mathbf{kv} + i0^+} \frac{\partial f(\mathbf{p})}{\partial p^l}\left[\left(1 - \frac{\mathbf{kv}}{\omega}\right)\delta^{lj} + \frac{k^l v^j}{\omega}\right]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

Dispersion equation – configuration of interest



$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$

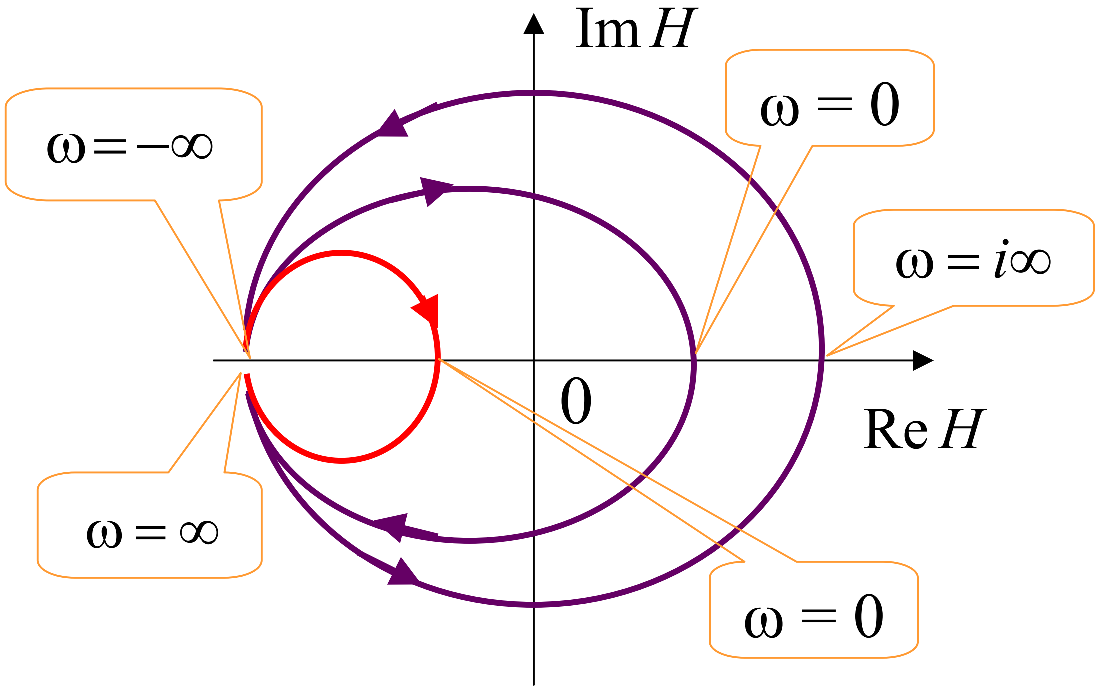
Dispersion equation

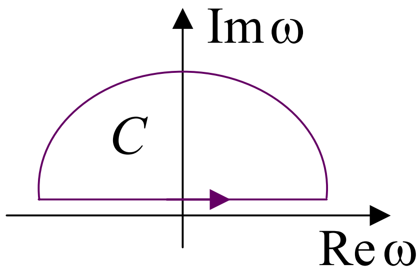
$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

Existence of unstable modes – Penrose criterion

$$H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i} \frac{d \ln H(\omega)}{d\omega} = \ln H(\omega) \Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$





There are unstable modes if

$$H(\omega = 0) < 0$$

Anisotropy!

Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho\sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2 + \sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

$\rho = 6\ \mathrm{fm}^{-3}$

$\alpha_s = g^2 / 4\pi = 0.3$

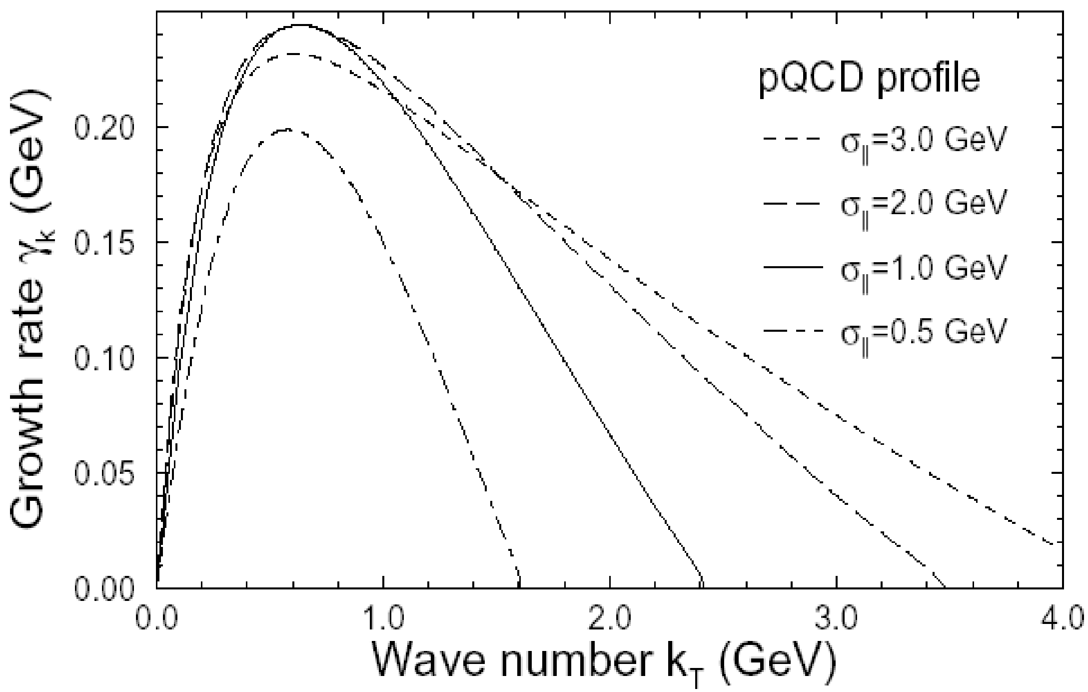
$\sigma_\perp = 0.3\ \mathrm{GeV}$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i \gamma_k$$

$$0 < \gamma_k \in \Re$$



J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

Hard-Loop dynamics

Soft fields in the passive background of hard particles

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\text{eff}} = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\Big[f(\mathbf{p}) F^a_{\mu\nu}(x) \left(\frac{p^\nu p^\rho}{(p\cdot D)^2}\right)_{ab} F^{b\mu}_\rho(x)$$

$$+ i\,\frac{C_F}{3}\,\tilde{f}(\mathbf{p})\,\bar{\psi}(x)\,\frac{p\cdot\gamma}{p\cdot D}\,\psi(x)\Big]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\mu \Lambda^\mu(p,q,k) = \Sigma(p) + \Sigma(q)$$

St. M., A. Rebhan & M. Strickland, Phys. Rev. D **74**, 025004 (2004)

Growth of instabilities – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

1+1 dimensions

$$A_a^\mu = A_a^\mu(t,z)$$

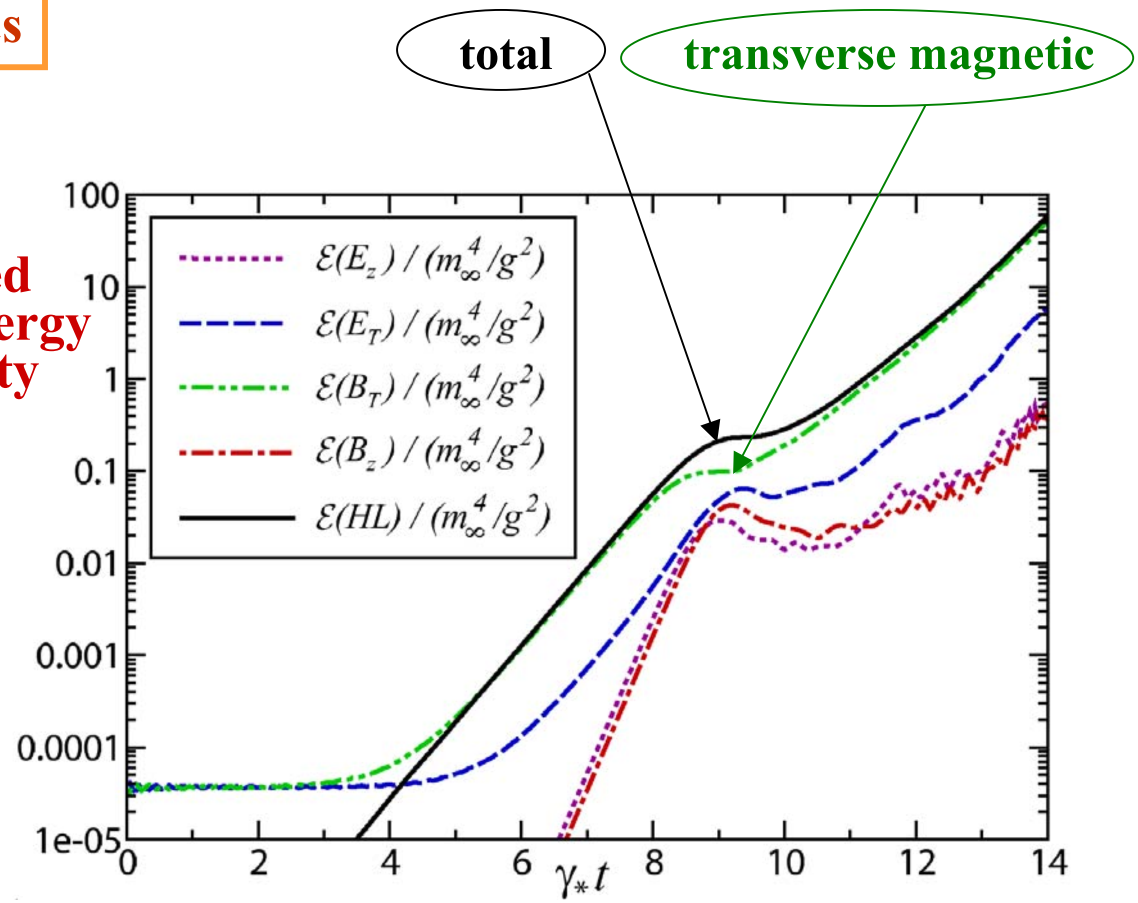
Scaled field energy density

Anisotropic particle's momentum distribution

$$f(\mathbf{p}) = f_{\rm iso}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi}\int_0^\infty dp\, p^2 \frac{df_{\rm iso}(p)}{dp}$$

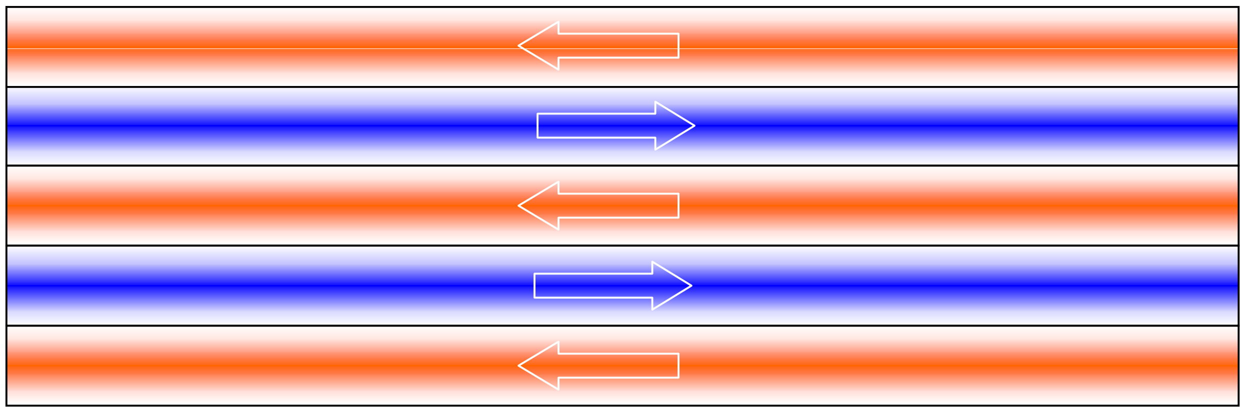
$$(m_D, \zeta)$$

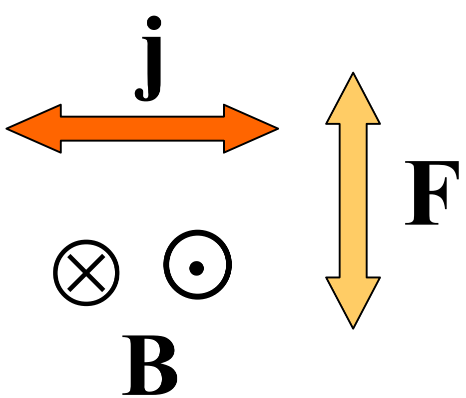


Strong anisotropy $\zeta = 10$ γ_* - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

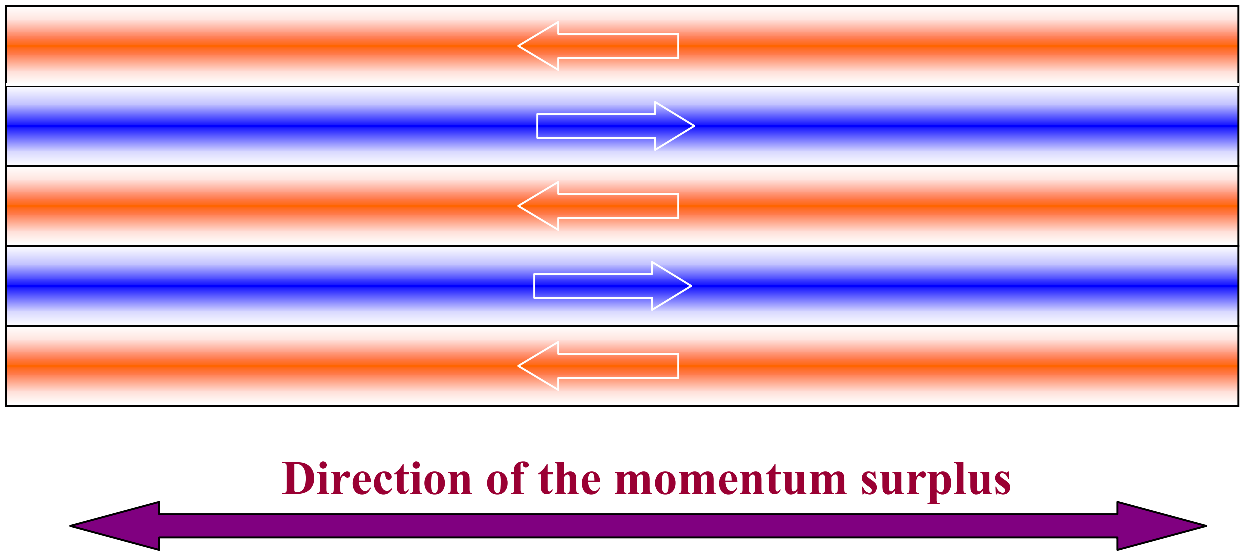
Isotropization - particles

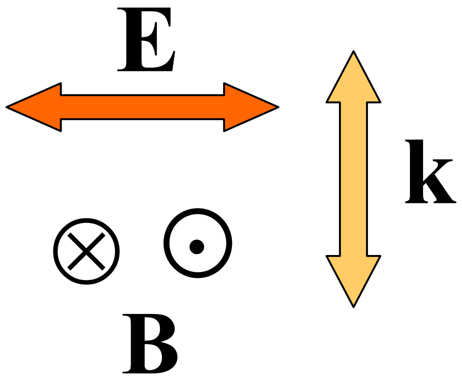




$$\Delta \mathbf{p} = \int dt\, \mathbf{F}$$

Isotropization - fields





$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^a \times \mathbf{E}^a \sim \mathbf{k}$$

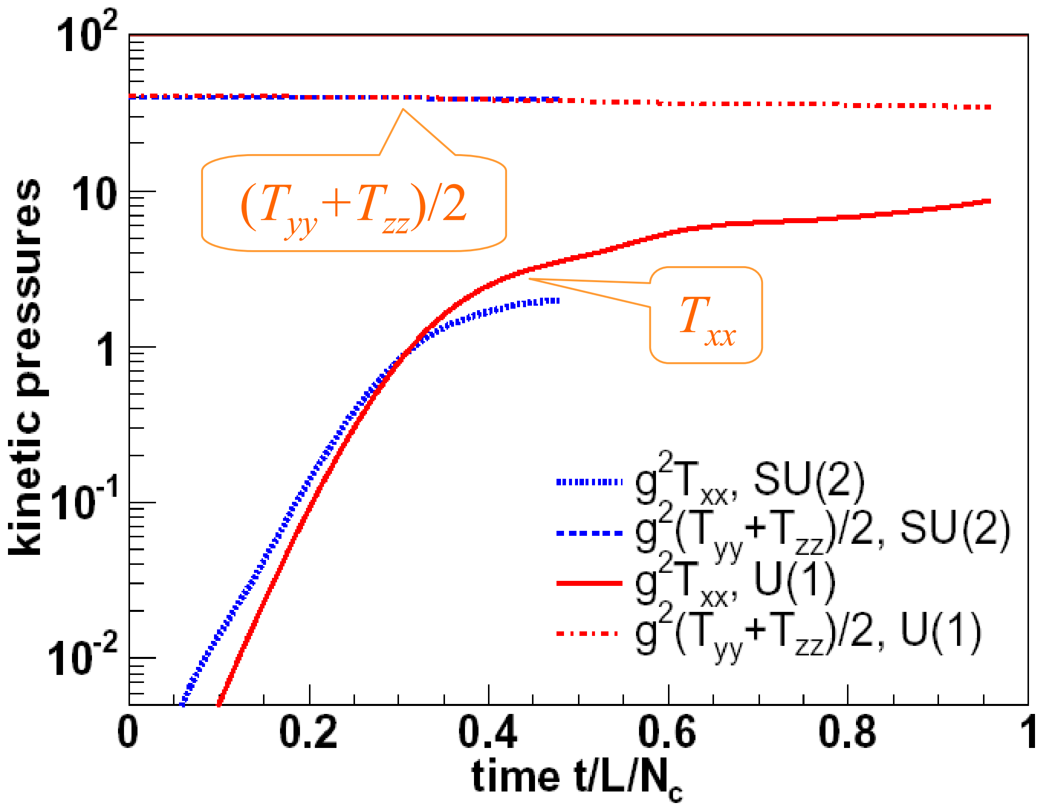
Isotropization – numerical simulation

Classical system of colored particles & fields

$$T_{ij} = \int \frac{d^3p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

$$T_{xx} = (T_{yy} + T_{zz})/2$$



A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

Instabilities @ LHC

RHIC ⇒ LHC

Energy density grows $\Rightarrow$

- $\alpha_s \searrow$ — weaker coupling
- $m_D \nearrow$
- $t_{\text{collec}} \searrow$ — faster instabilities

Conclusions

- Instabilities generate chromodynamic fields – QGP is not a gas of partons
- Instabilities are fast, faster than collisions
- Instabilities strongly influence plasma dynamics