Elements of QCD for hadron colliders

theoretical concepts and phenomenology

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What is QCD?

Quantum Chromodynamics

The theory describing the dynamics and interactions of quarks and gluons.

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the ingredients of QCD

- quarks and anti-quarks
  $\{u, d, s, c, b, t\}$ come in 3 colors
- gluons, bit like the photons in QED
  eight of them, carry colour charge
- a coupling, α_s
  not so small, running fast

high-energy collider relevance

- strong-interaction processes
- phenomena such as hadronic jets
- short-distance structure of hadrons

$\rightsquigarrow$ detailed understanding crucial

I'll try to give you a feel for:

How QCD works

How theorists handle QCD at hadron colliders

How hadron collisions get simulated

Contents of this course

- Basics of QCD
  - The QCD Lagrangian
  - Perturbation Theory & The running coupling
- Soft & collinear singularities
- The concepts of parton showers and jets
- QCD for processes with incoming protons
- Monte-Carlo event generators

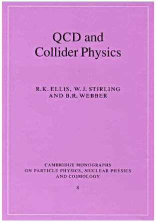
Disclaimer

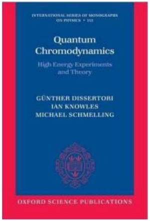
These lectures are *not* exhaustive!

- lacks most of the details, derivations, ...

Recommended reading:

- Ellis, Stirling & Webber
  "QCD and Collider Physics"
  Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **8**, 1 (1996)
- Dissertori, Knowles & Schmelling
  "Quantum Chromodynamics"
  International Series of Monographs on Physics · 115
- ... papers cited on the way





Basics of QCD

Basics of QCD: the naïve parton model

1st tenet: hadronic matter made of quarks

- the fermionic quarks carry fractional charges $Q_{u,c,t} = +\frac{2}{3},\ Q_{d,s,b} = -\frac{1}{3}$
- three-quark states form baryons: $|B\rangle = |\, q_1\ q_2\ q_3\rangle$

  $\rightsquigarrow$ baryons are fermions, follow Fermi-Dirac statistics

  $\rightsquigarrow$ wave functions must be totally anti-symmetric
- mesons thought of as bound states of quark & anti-quark: $|M\rangle = |\, q_1\ \bar{q}_2\rangle$

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obstacle at the time: how to account for spin-3/2 baryons?

- consider the resonance $|\Delta^{++}\rangle = |\, u_\uparrow\ u_\uparrow\ u_\uparrow\rangle$

  $\rightsquigarrow$ symmetrical state in space, spin & flavour
- introduce new degree of freedom: colour index $a \in \{1, 2, 3\}$

  $\rightsquigarrow$ $|\Delta^{++}\rangle = \epsilon_{abc}\,|\, u_{a\uparrow}\ u_{b\uparrow}\ u_{c\uparrow}\rangle$

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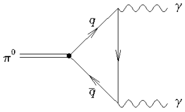
2nd tenet: hadronic matter must be colour-singlet states

Basics of QCD: colour degree of freedom

- consider the decay $\pi^0 \to \gamma\gamma$ $\left[|\pi^0\rangle = \frac{1}{\sqrt{2}}\left(|u\bar{u}\rangle - |d\bar{d}\rangle\right)\right]$



$$\Gamma^{\rm theo}(\pi^0 \to \gamma\gamma) = \xi^2 \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{64\pi} \frac{m_\pi^3}{f_\pi^2} = 7.6\,\xi^2\,\mathrm{eV}$$

electric charge and colour factor ξ given by

$$\xi = N_c \left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right] = \frac{N_c}{3}$$

experimental value is $\Gamma^{\rm exp} = 7.74 \pm 0.55\,\mathrm{eV}$ [PDG] $\rightsquigarrow \xi = 1 \rightsquigarrow N_c = 3$

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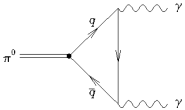

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- consider the ratio $R = \sum_q \sigma_{\rm tot}(e^+e^- \to q\bar{q})/\sigma_{\rm tot}(e^+e^- \to \mu^+\mu^-)$

u, d, s only $R = N_c \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = N_c\,\frac{2}{3}$

u, d, s, c, b only $R = N_c \left[2 \times \left(\frac{2}{3}\right)^2 + 3 \times \left(-\frac{1}{3}\right)^2\right] = N_c\,\frac{11}{9}$

$\rightsquigarrow$ data consistent with $N_c = 3$

Basics of QCD: the $SU(3)$ colour group

the group of unitary 3×3 matrices U with det(U) = +1

$\rightsquigarrow$ $SU(3)$ generators, hermitian & traceless Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

by convention we define $t^A_{ab} \equiv \frac{1}{2}\lambda^A_{ab}$ $\quad\rightsquigarrow\quad$ $U = \exp\{i\alpha_A t^A\}$

$$[t^A, t^B] = if_{ABC}t^C$$

with f_{ABC} the $SU(3)$ structure constants (anti-symmetric in all indices)

$\rightsquigarrow$ **SU(3) is a non-abelian group**

Note: The analogs of $SU(2)$ you know well, the Pauli matrices & ϵ_{ijk}

Basics of QCD: The Lagrangian

the free quark part

denote quark fields by ψ_q^a, where a denotes a colour index $a \in \{1, 2, 3\}$

$$\mathcal{L}_{\text{free Dirac}} = \sum_q \bar{\psi}_q^a \, i\delta_{ab}\gamma^\mu \partial_\mu \, \psi_q^b - m_q \bar{\psi}_q^a \psi_q^a \quad \text{with} \quad q \in \{u, d, s, c, b, t\}$$

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the pure gluon part

denote the gluon fields by A_μ^A, with $A \in \{1, .., 8\}$

$$\mathcal{L}_{\text{pure Gluon}} = -\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} \quad \text{with} \quad F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - g_s f_{ABC} A_\mu^B A_\nu^C$$

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coupling the quarks to gluons

minimal coupling of quarks with gluons, consistent with local gauge invariance

$$\mathcal{L}_{\text{interaction}} = \sum_q g_s \bar{\psi}_q^a \, \gamma^\mu t_{ab}^a A_\mu^a \, \psi_q^b \quad \text{with} \quad g_s^2 = 4\pi\alpha_s$$

the classical QCD Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{QCD}} &= \mathcal{L}_{\text{free Dirac}} + \mathcal{L}_{\text{interaction}} + \mathcal{L}_{\text{pure Gluon}} \\ &= \sum_q \bar{\psi}^a_q \left(i\delta_{ab}\gamma^\mu\partial_\mu + g_s\gamma^\mu t^a_{ab}A^A_\mu\right)\psi^b_q - m_q\bar{\psi}^a_q\psi^a_q - \frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu}\end{aligned}$$

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with $(D_\mu)_{ab} = \delta_{ab}\partial_\mu - ig_s t^A_{ab}A^A_\mu$ the $SU(3)_c$ covariant derivative

Basics of QCD: The Lagrangian

the classical QCD Lagrangian

$$\begin{aligned}
\mathcal{L}_{\rm QCD} &= \mathcal{L}_{\text{free Dirac}} + \mathcal{L}_{\text{interaction}} + \mathcal{L}_{\text{pure Gluon}} \\
&= \sum_q \bar{\psi}^a_q \left(i\delta_{ab}\gamma^\mu\partial_\mu + g_s\gamma^\mu t^a_{ab}A^A_\mu\right)\psi^b_q - m_q\bar{\psi}^a_q\psi^a_q - \frac{1}{4}F^A_{\mu\nu}F^{A\mu\nu} \\
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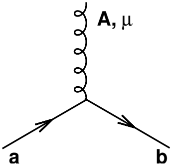
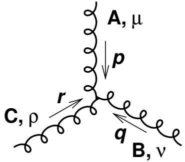
- ⇝ interacting field theory
- ⇝ non-abelian gauge field theory [to be quantized]
- ⇝ quark & gluon propagators
- ⇝ quark–gluon & gluon self interactions

Basics of QCD: The Feynman rules

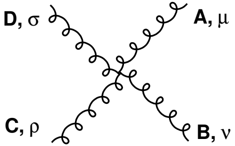
$$\mathcal{L}_{\rm QCD} \supset \bar{\psi}^a_q \left(-ig_s\gamma^\mu t^A_{ab}A^A_\mu\right)\psi^b_q - g_s f^{ABC}(\partial_\mu A^A_\nu)A^{B\mu}A^{C\nu} - \frac{1}{4}g_s^2 f^{XAB}f^{XCD}A^{A\mu}A^{B\nu}A^C_\mu A^D_\nu$$



$$-ig_s t^A_{ba}\gamma^\mu$$

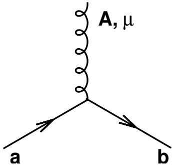


$$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$$



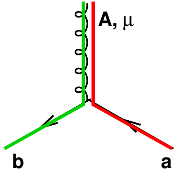
$$-ig_s^2 f^{XAC}f^{XBD}[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$$

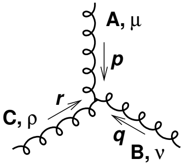
Basics of QCD: The Feynman rules



$\propto g_s t^A_{ba}$

$\Longleftrightarrow$





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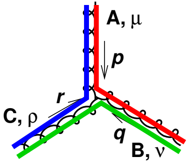
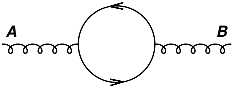
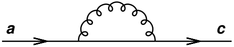
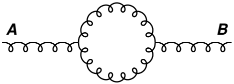


- gluon emission repaints the quark colour
- gluon carries colour and anti-colour
- gluon emission repaints the gluon colours

Basics of QCD: Quick guide to colour algebra

some useful $\mathrm{SU(N_c)}$ colour algebra relations

$\hookrightarrow$ appear when summing squared amplitudes colours

trace relation	corresponding diagram
$\mathrm{Tr}\{t^A t^B\} = T_R\,\delta^{AB}\,, \quad T_R = \frac{1}{2}$	
$\sum_A t^A_{ab} t^A_{bc} = C_F\,\delta_{ac}\,, \quad C_F = \frac{N_c^2 - 1}{2N_c}$	
$\sum_{C,D} f^{ACD} f^{BCD} = C_A\,\delta^{AB}\,, \quad C_A = N_c$	

QCD: $N_c \equiv$ number of colours $= 3 \quad \rightsquigarrow \quad C_A = 3 \quad \& \quad C_F = \frac{4}{3}$

Basics of QCD: Perturbation Theory I

given the Lagrangian we could start calculating, e.g. cross sections

- numerical solution in discretized space time (lattice QCD)
  - $\rightsquigarrow$ suitable for static properties of hadrons, e.g. hadron masses
  - $\rightsquigarrow$ not practicable for dynamical LHC collision events
- have to rely on perturbative techniques
  - $\rightsquigarrow$ relies on order-by-order expansion in small coupling, $\alpha_s \ll 1$

$$\mathcal{O} \approx C_0 + C_1\alpha_s + C_2 \underbrace{\alpha_s^2}_{\text{small}} + C_3 \underbrace{\alpha_s^3}_{\text{smaller}} + \underbrace{\ldots}_{\text{negligible?}}$$

  - $\rightsquigarrow$ calculational complexity grows extremely fast with powers of α_s, thus α_s better is small!?

Basics of QCD: The running coupling

as other couplings/parameters α_s is scale dependent [momentum scale μ^2]

$$\frac{d\alpha_s(\mu^2)}{d\ln\mu^2} = \beta(\alpha_s(\mu^2))\,, \qquad \beta(\alpha_s) = -\alpha_s^2(b_0 + b_1\alpha_s + b_2\alpha_s^2 + \ldots)$$

where

$$b_0 = \frac{11C_A - 2n_f}{12\pi}\,, \qquad b_1 = \frac{17C_A^2 - 5C_A n_f - 3C_F n_f}{24\pi^2} = \frac{153 - 19n_f}{24\pi^2}$$

Note sign of β: **Asymptotic freedom**, due to gluon self interaction

[Nobel prize in 2004 to Gross, Politzer & Wilczek]

- $\rightsquigarrow$ at high scales μ^2 coupling becomes small, quarks & gluons are almost free interactions weak, perturbation theory works
- $\rightsquigarrow$ at low scales coupling becomes strong, quarks & gluons interact strongly perturbation theory fails

Basics of QCD: The running coupling cont'd

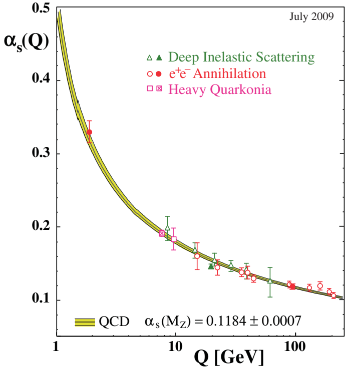
as other couplings/parameters α_s is scale dependent [momentum scale μ^2]

$\rightsquigarrow$ ignoring all terms other than b_0, we get for $\alpha_s(\mu^2)$

$$\frac{d\alpha_s(\mu^2)}{d\ln\mu^2} = -b_0\alpha_s^2 \quad \rightsquigarrow \quad \alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + b_0\alpha_s(\mu_0^2)\ln\frac{\mu^2}{\mu_0^2}} = \frac{1}{b_0\ln\frac{\mu^2}{\Lambda^2_{\rm QCD}}}$$

result expressed in terms of

- reference scale μ^0, e.g. M_Z^2
- non-perturbative constant $\Lambda_{\rm QCD} \simeq 0.2$ GeV
  - fundamental scale of QCD
  - sets scale for hadron masses



- perturbation theory valid for $\mu \gg \Lambda_{\rm QCD}$
- non-perturbative description $\mu \simeq \Lambda_{\rm QCD}$

Basics of QCD: Perturbation Theory II

QCD perturbation theory for the LHC?

- "New Physics" searched for at scales $\mu \sim p_T \sim 50\,\mathrm{GeV} - 5\mathrm{TeV}$

  The coupling is certainly small!

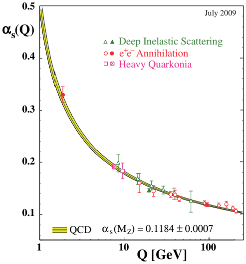
- but we're colliding protons $m_p \simeq 0.94$ GeV

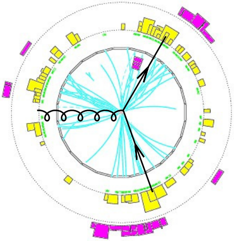
  The coupling is large!

- in the detectors we see hadrons

  perturbation theory doesn't apply

  lots of them – no one-to-one hadron-parton correspondence, when limiting ourselves to 1 or 2 orders in perturbation theory





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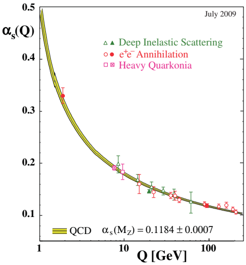

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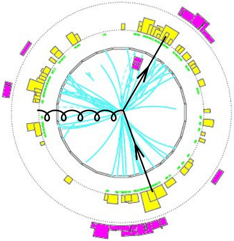

PT only offers no full solution for QCD at colliders

use perturbative QCD + non-perturbative modelling

key ingredient:

factorization

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Next items on the menue:

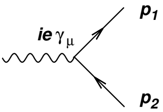
- what perturbation theory tells us about the structure of QCD events
  - soft- & collinear singularities & jets
  - parton distribution functions (PDFs)
- introduce methods available to carry out QCD predictions
  - fixed-order perturbative calculations
  - Monte-Carlo event generators

Soft & Collinear Singularities

Singularities: soft-gluon amplitude

consider the process $(e^+e^- \rightarrow)\gamma^* \rightarrow q\bar{q}$

$\mathcal{M}_{q\bar{q}} =$



$$= \bar{u}_a(p_1) i e_q \gamma_\mu \delta_{ab} v_b(p_2)$$

Singularities: soft-gluon amplitude

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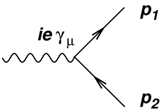
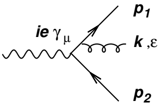


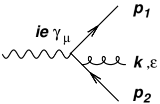
$$= \bar{u}_a(p_1)\, ie_q\gamma_\mu \delta_{ab} v_b(p_2)$$

emit a gluon (momentum k, polarization vector ϵ)

$\mathcal{M}_{q\bar{q}g} =$



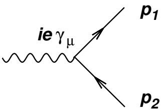
$+$



$$= -\bar{u}_a(p_1)\, ig_s \not{\epsilon}\, t^A_{ab} \frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2} ie_q\gamma_\mu v_b(p_2) + \bar{u}_a(p_1)\, ie_q\gamma_\mu \frac{i(\not{p}_2 + \not{k})}{(p_2+k)^2} ig_s \not{\epsilon}\, t^A_{ab} v_b(p_2)$$

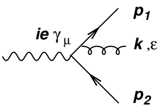
Singularities: soft-gluon amplitude

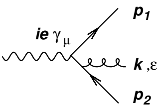
consider the process $(e^+e^- \to)\gamma^* \to q\bar{q}$

$\mathcal{M}_{q\bar{q}} =$ 

$$= \bar{u}_a(p_1) i e_q \gamma_\mu \delta_{ab} v_b(p_2)$$

emit a gluon (momentum k, polarization vector ϵ)

$\mathcal{M}_{q\bar{q}g} =$ 

$+$ 

$$= -\bar{u}_a(p_1) i g_s \not{\epsilon} t^A_{ab} \frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2} i e_q \gamma_\mu v_b(p_2) + \bar{u}_a(p_1) i e_q \gamma_\mu \frac{i(\not{p}_2 + \not{k})}{(p_2+k)^2} i g_s \not{\epsilon} t^A_{ab} v_b(p_2)$$

make gluon soft$\equiv k \ll p_1, p_2$, ignore terms suppressed by powers of k

$$\mathcal{M}_{q\bar{q}g} \simeq \bar{u}_a(p_1) i e_q \gamma_\mu t^A_{ab} v_b(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k}\right)$$

Singularities: squared amplitude

$$|\mathcal{M}_{q\bar{q}g}|^2 \simeq \sum_{A,a,b,\mathrm{pol}} \left| \bar{u}_a(p_1) i e_q \gamma_\mu t^A{}_{ab} v_b(p_2)\; g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

$$= -|M^2_{q\bar{q}}| C_F g_s^2 \left(\frac{p_1}{p_1.k} - \frac{p_2}{p_2.k} \right)^2 = |M^2_{q\bar{q}}| C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

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include phase-space factor $d\Phi_{q\bar{q}g} \simeq d\Phi_{q\bar{q}} \frac{d^3\vec{k}}{2E(2\pi)^3} = d\Phi_{q\bar{q}} \frac{E^2 dE \sin\theta d\theta d\phi}{2E(2\pi)^3}$
[factorizes as well]

$$|\mathcal{M}_{q\bar{q}g}|^2 d\Phi_{q\bar{q}g} \simeq |\mathcal{M}_{q\bar{q}}|^2 d\Phi_{q\bar{q}} d\mathcal{S}$$

$\rightsquigarrow$ factorization into hard $q\bar{q}$ piece & soft-gluon emission probability $d\mathcal{S}$

Singularities: squared amplitude

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$$d\mathcal{S} = \frac{2\alpha_s C_F}{\pi} \frac{dE}{E} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}, \quad \text{with } \theta = \theta_{p_1 k} \,\&\, \phi \text{ azimuth}$$

Singularities: squared amplitude

$$|\mathcal{M}_{q\bar{q}g}|^2 \simeq \sum_{A,a,b,\mathrm{pol}} \left| \bar{u}_a(p_1) i e_q \gamma_\mu t^A{}_{ab} v_b(p_2)\, g_s \left(\frac{p_1.\epsilon}{p_1.k} - \frac{p_2.\epsilon}{p_2.k} \right) \right|^2$$

$$= -|M^2_{q\bar{q}}| C_F g_s^2 \left(\frac{p_1}{p_1.k} - \frac{p_2}{p_2.k} \right)^2 = |M^2_{q\bar{q}}| C_F g_s^2 \frac{2p_1.p_2}{(p_1.k)(p_2.k)}$$

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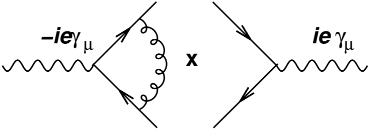
gluon emission singularity structure (universal/process independent)

- diverges for $E \to 0$ aka infrared/soft singularity
- diverges for $\theta \to 0$ and $\theta \to \pi$ aka collinear singularity

Singularities: Real–virtual cancellation

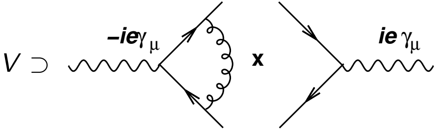
$\mathcal{O}(\alpha_s)$ correction to total cross section, sum of real & virtual contributions

$V \supset$ 

Total cross section must be finite. If real part it divergent, so must the virtual!

Singularities: Real–virtual cancellation

$\mathcal{O}(\alpha_s)$ correction to total cross section, sum of real & virtual contributions



Total cross section must be finite. If real part it divergent, so must the virtual!

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(1 + \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} R\left(\frac{E}{Q}, \theta\right) - \frac{2\alpha_s C_F}{\pi} \int \frac{dE}{E} \int \frac{d\theta}{\sin\theta} V\left(\frac{E}{Q}, \theta\right)\right)$$

- $R(E/Q, \theta)$ parametrizes full real-emission matrix element ($E \gg 0$)

$$\lim_{E \to 0} R(E/Q, \theta) = 1$$

- $V(E/Q, \theta)$ parametrizes virtual corrections for all momenta
- for every divergence $R(E/Q, \theta)$ and $V(E/Q, \theta)$ cancel

$$\lim_{E \to 0} (R - V) = 0 \quad \text{and} \quad \lim_{\theta \to 0, \pi} (R - V) = 0$$

The emerging picture

- corrections to $\sigma_{\rm tot}$ dominated by hard, large-angle gluons
- soft gluons play no role for $\sigma_{\rm tot}$
    - collision characterised by $t_{\rm hard} \sim 1/Q$
    - soft gluons emitted on long time scales $t_{\rm soft} \sim 1/(E\theta^2)$
      $\rightsquigarrow$ cannot influence cross section
    - transition to hadrons occurs on long time scales $t_{\rm had} \sim 1/\Lambda_{\rm QCD}$
      $\rightsquigarrow$ can thus be ignored
- with proper choice for scale of α_s, $\mu = Q$, perturbation theory works well

$$\sigma_{tot} = \sigma_{q\bar{q}} \left(\underbrace{1}_{\rm LO} \underbrace{+1.045\frac{\alpha_s(Q^2)}{\pi}}_{\rm NLO} \underbrace{+0.94\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2}_{\rm NNLO} \underbrace{-15\left(\frac{\alpha_s(Q^2)}{\pi}\right)^3}_{\rm NNNLO} + \cdots \right)$$

[coefficients given for $Q = M_Z$]

Total cross sections are inclusive quantities,
inclusive in the number of additional QCD partons!

Summary 1st lecture

The QCD Lagrangian

- gauge-field theory for the strong force
- dynamics and interactions of strongly interacting particles

  $\rightsquigarrow$ quarks & gluons carry colour charges
- renormalized coupling runs fast

The two faces of QCD

- confined phase: large coupling regime, physics of hadrons
- asymptotic free phase: coupling small, perturbation theory applicable

Soft & collinear divergences

- singularities associated with the emission of soft/collinear gluons
- generic feature of massless gauge-field theories
- divergences cancel between real & virtual corrections