Probabilistic Reasoning in Physics

– inference, forecasting, decision –

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“Probability is good sense reduced to a calculus” (Laplace)

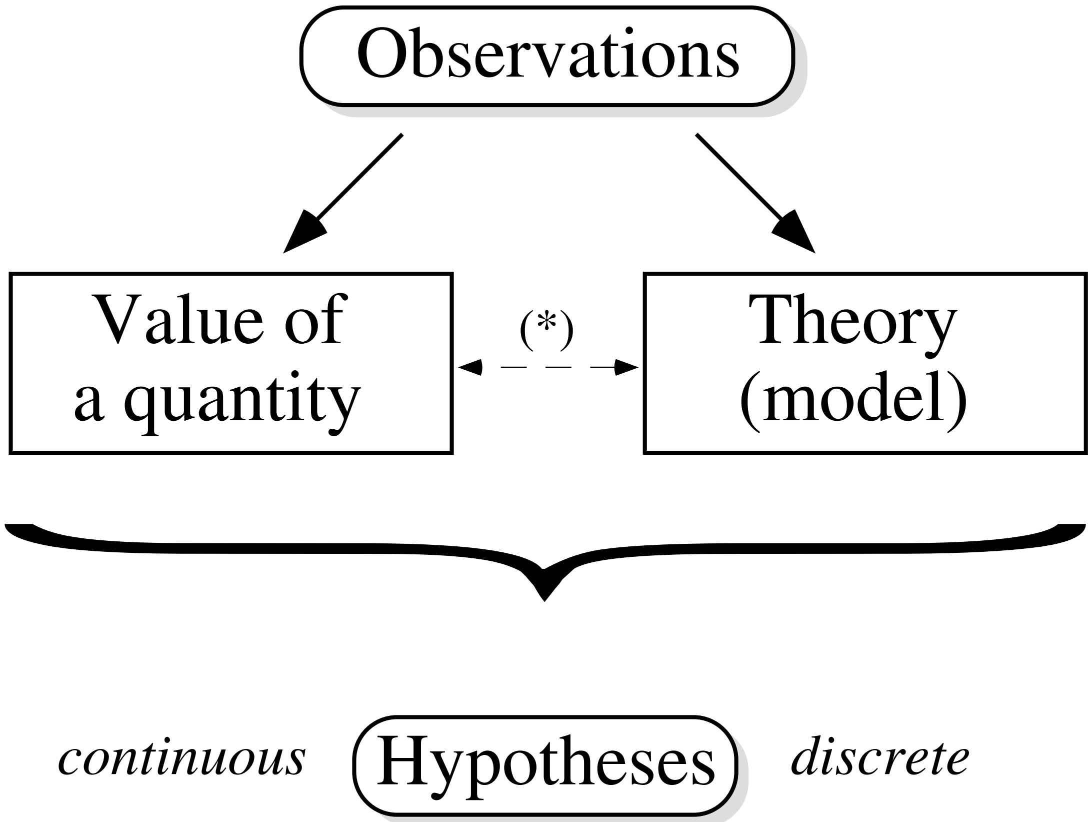
Outline

- "Science and hypothesis" (Poincaré)
- Uncertainty, probability, decision.
- Causes $\longleftrightarrow$ Effects

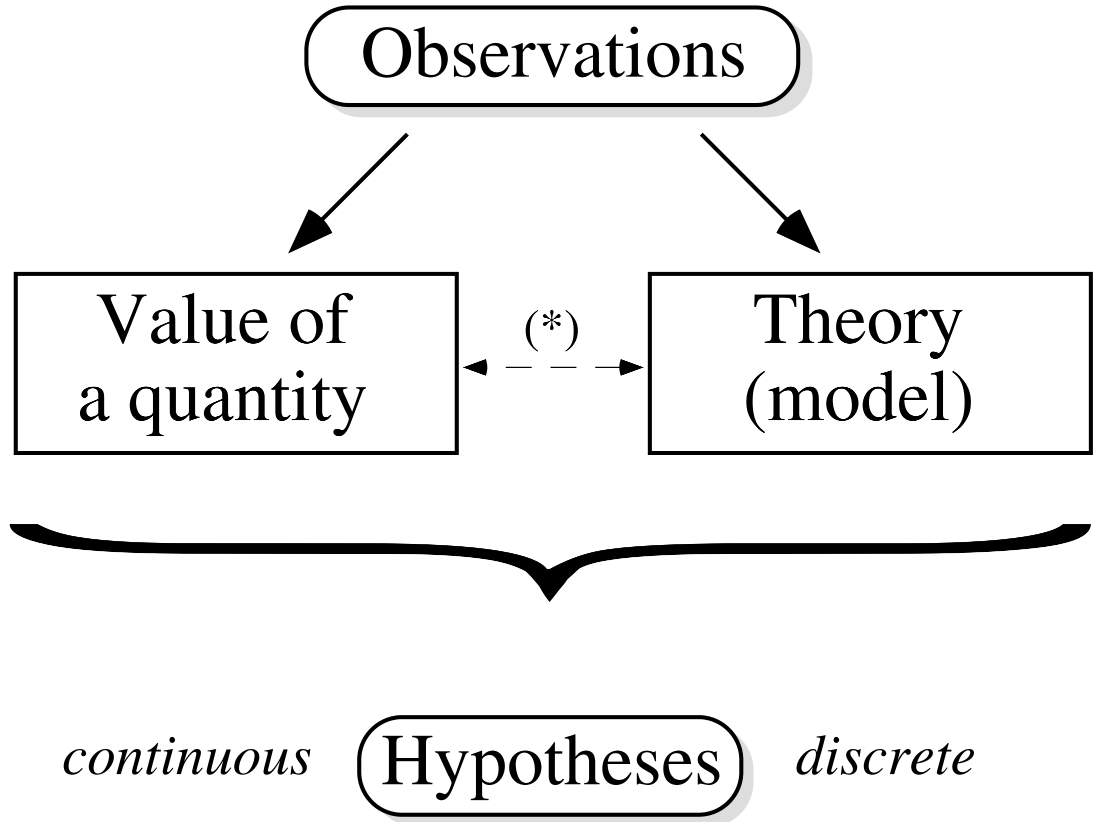
  "The essential problem of the experimental method" (Poincaré).
- A toy model and its physics analogy: the six box game

  "Probability is either referred to real cases or it is nothing" (de Finetti).
- Probabilistic approach [but . . . What is probability?]
- Basic rules of probability and Bayes rule.
- Bayesian inference and its graphical representation:
  $\Rightarrow$ Bayesian networks
- Some examples of applications in Physics
- Conclusions

Physics

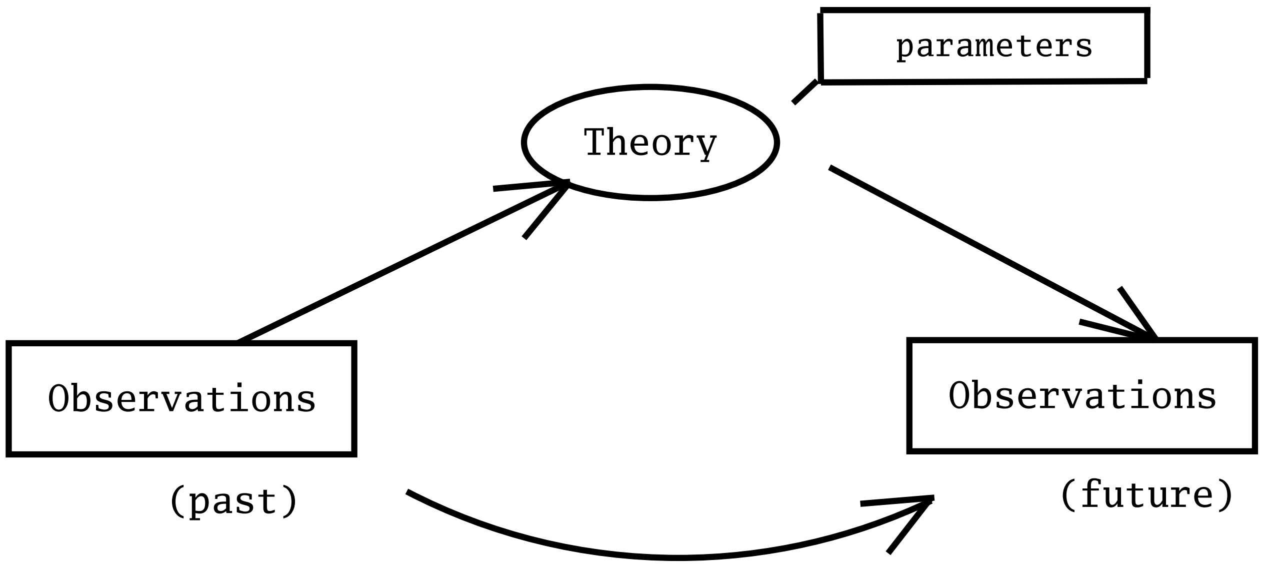

Observations
Value of a quantity
(*)
Theory (model)
continuous
Hypotheses
discrete

Physics



(*) A quantity might be meaningful only within a theory/model

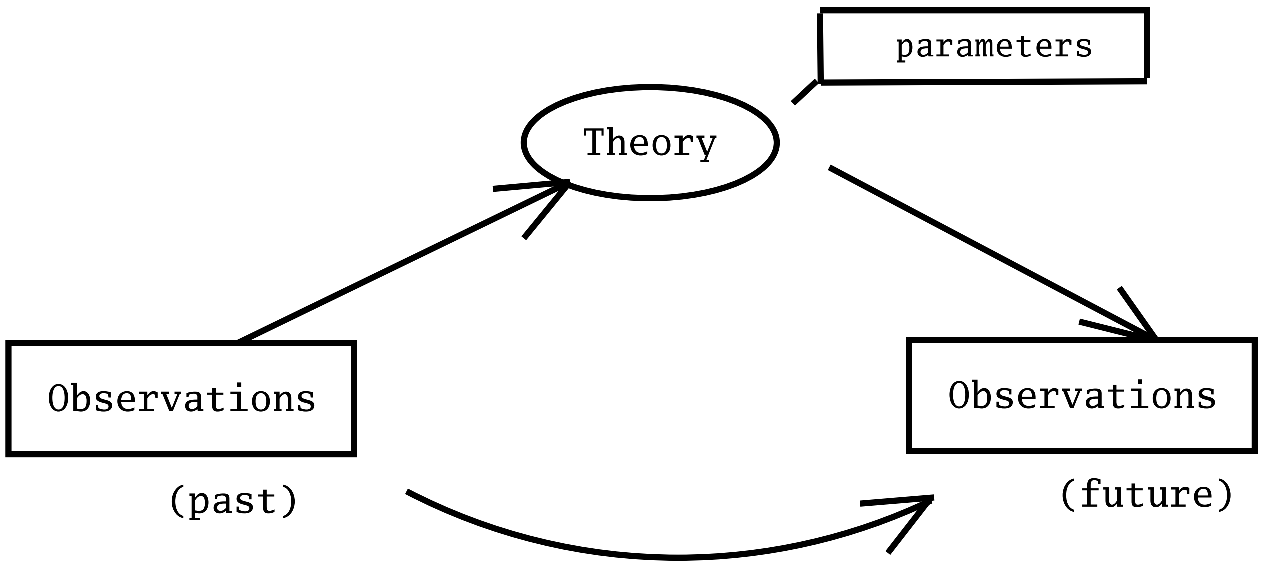
From past to future



Task of physicists:

- Describe/understand the physical world
  $\Rightarrow$ inference of laws and their parameters
- Predict observations
  $\Rightarrow$ forecasting

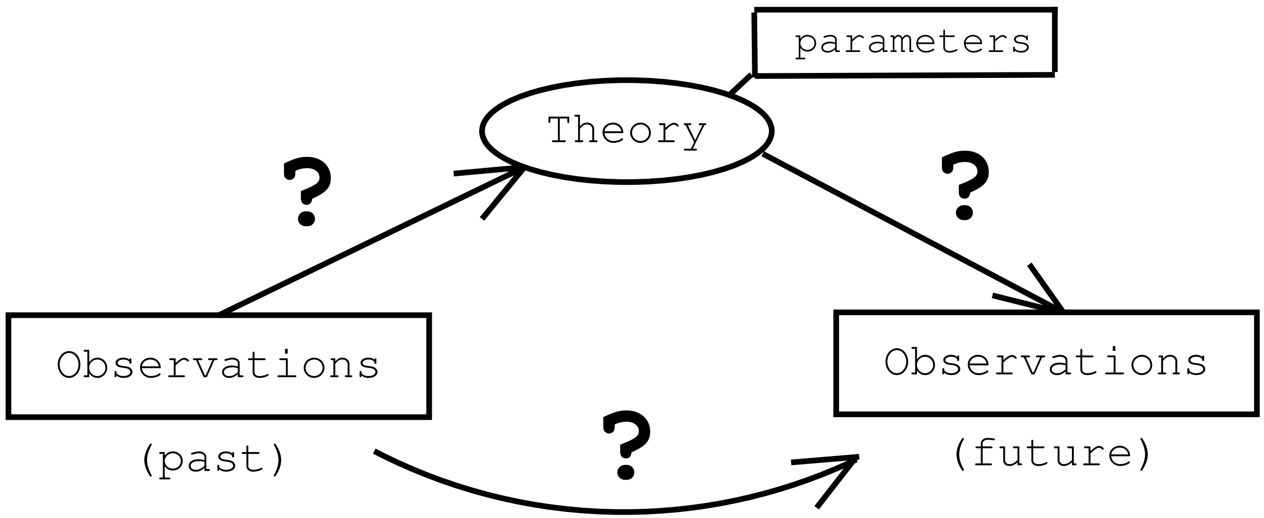
From past to future



Process

- neither automatic
- nor purely contemplative
  - → 'scientific method'
  - → planned experiments ('actions') ⇒ decision.

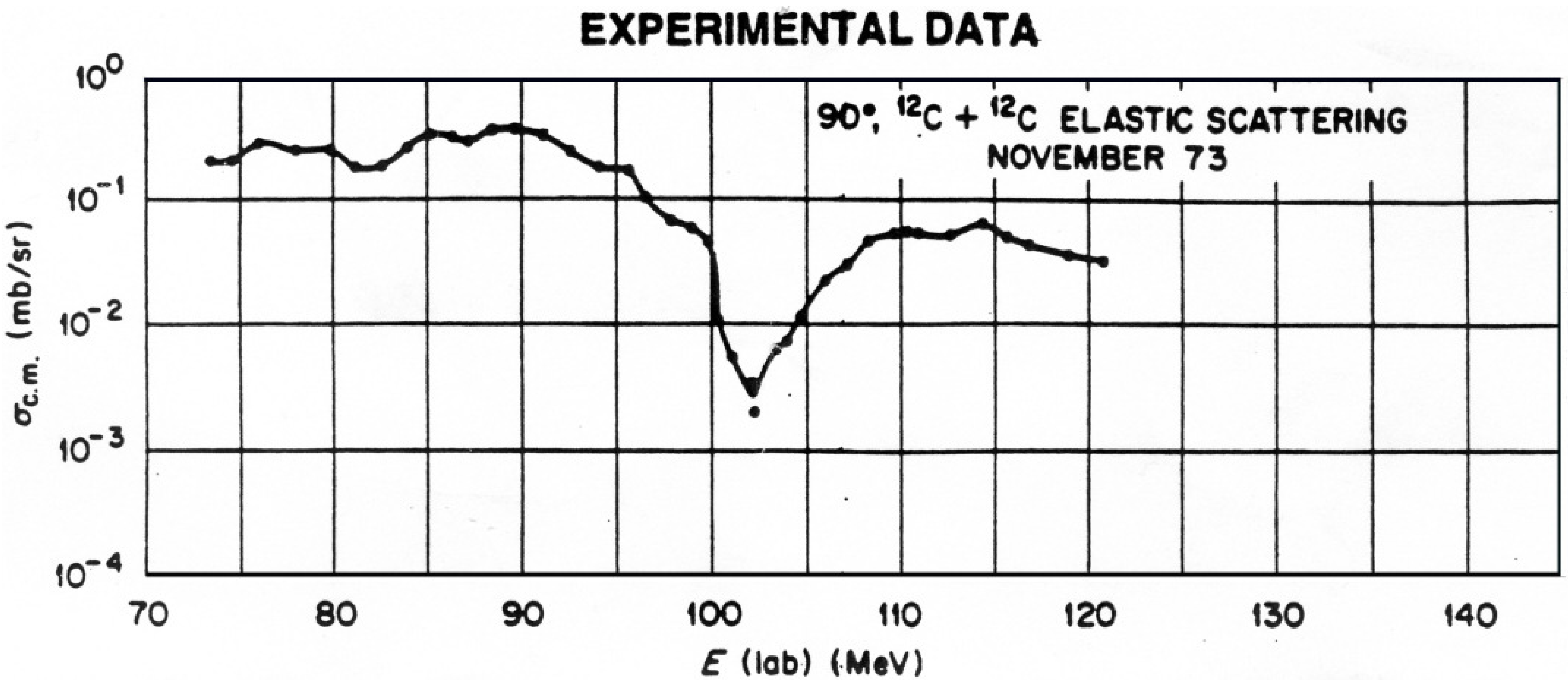
From past to future



⇒ Uncertainty:

1. Given the past observations, in general we are not sure about the theory parameters (and/or the theory itself)
2. Even if we were sure about theory and parameters, there could be internal (e.g. Q.M.) or external effects (initial/boundary conditions, ‘errors’, etc) that make the forecasting uncertain.

Inferential-predictive process



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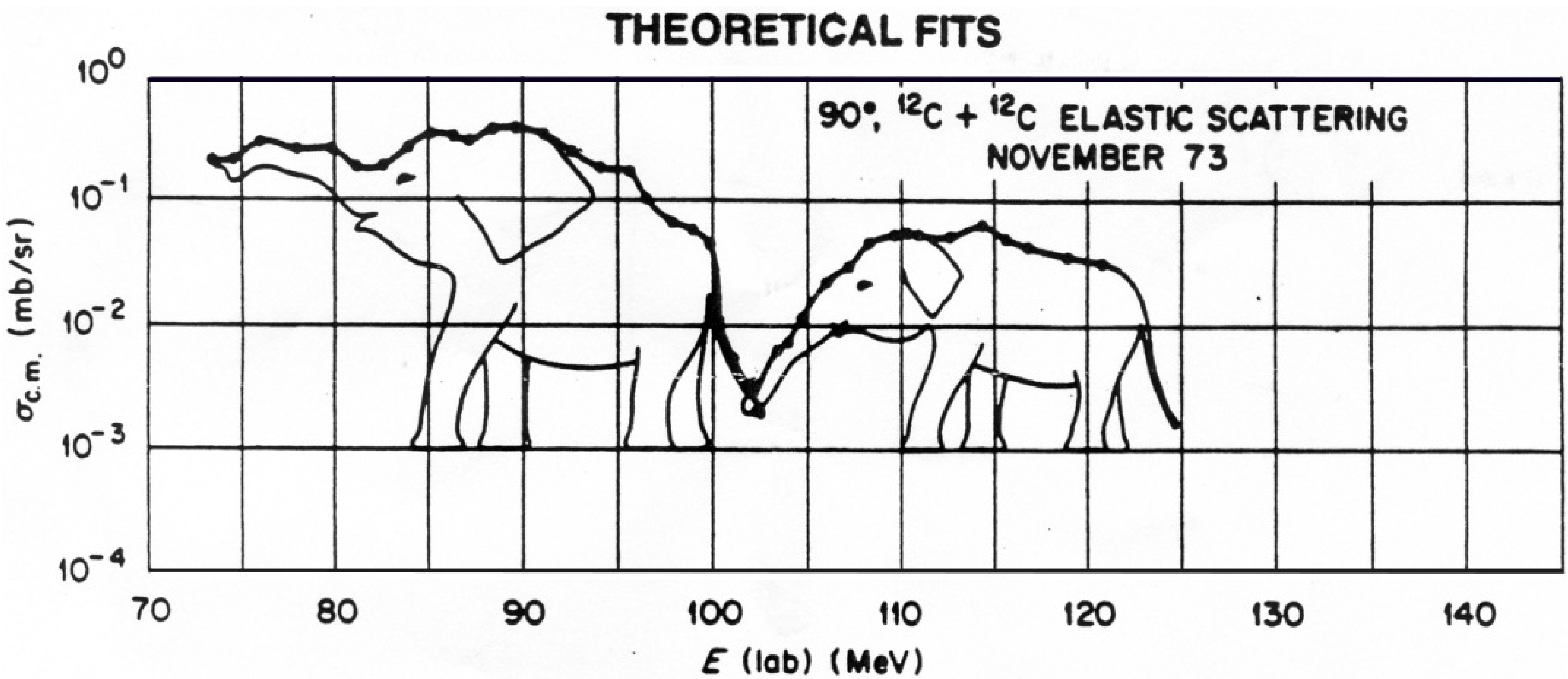

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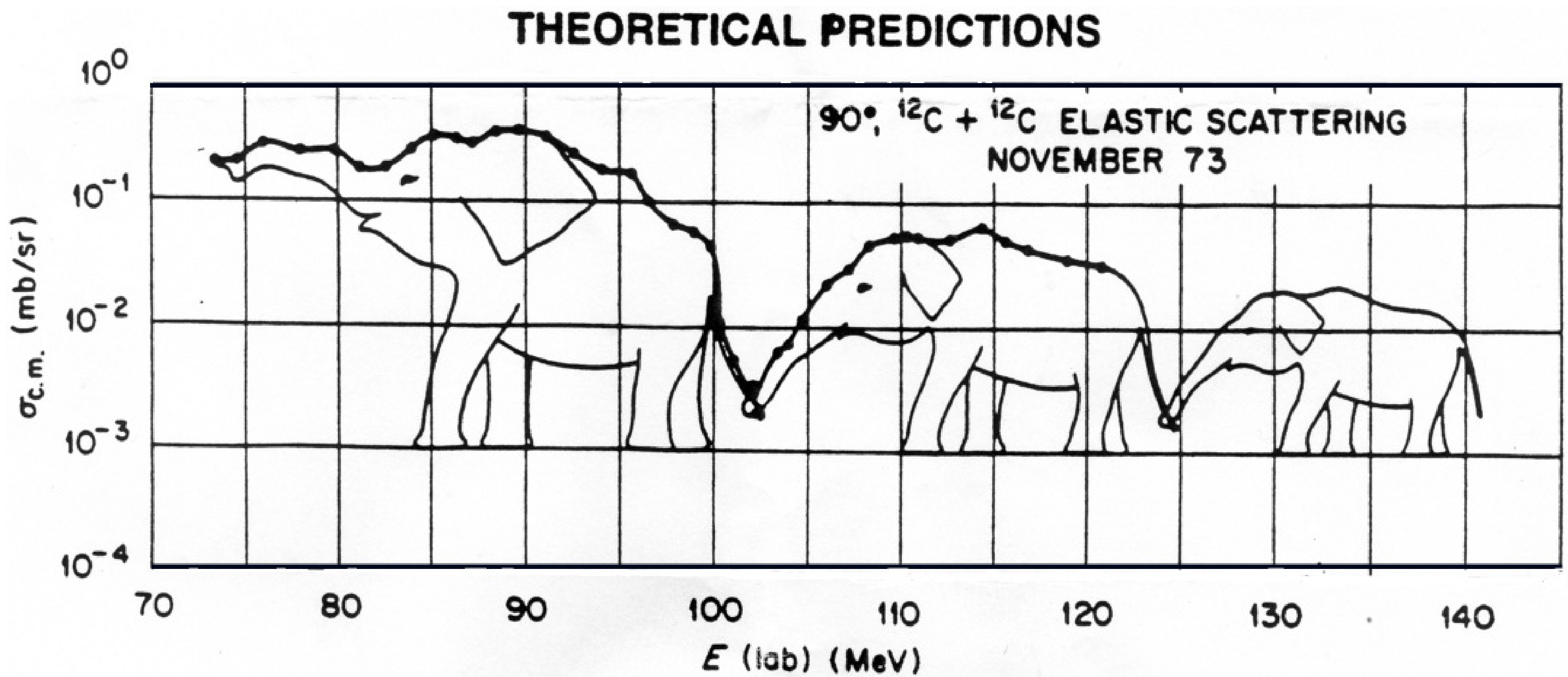

(S. Raman, *Science with a smile*)

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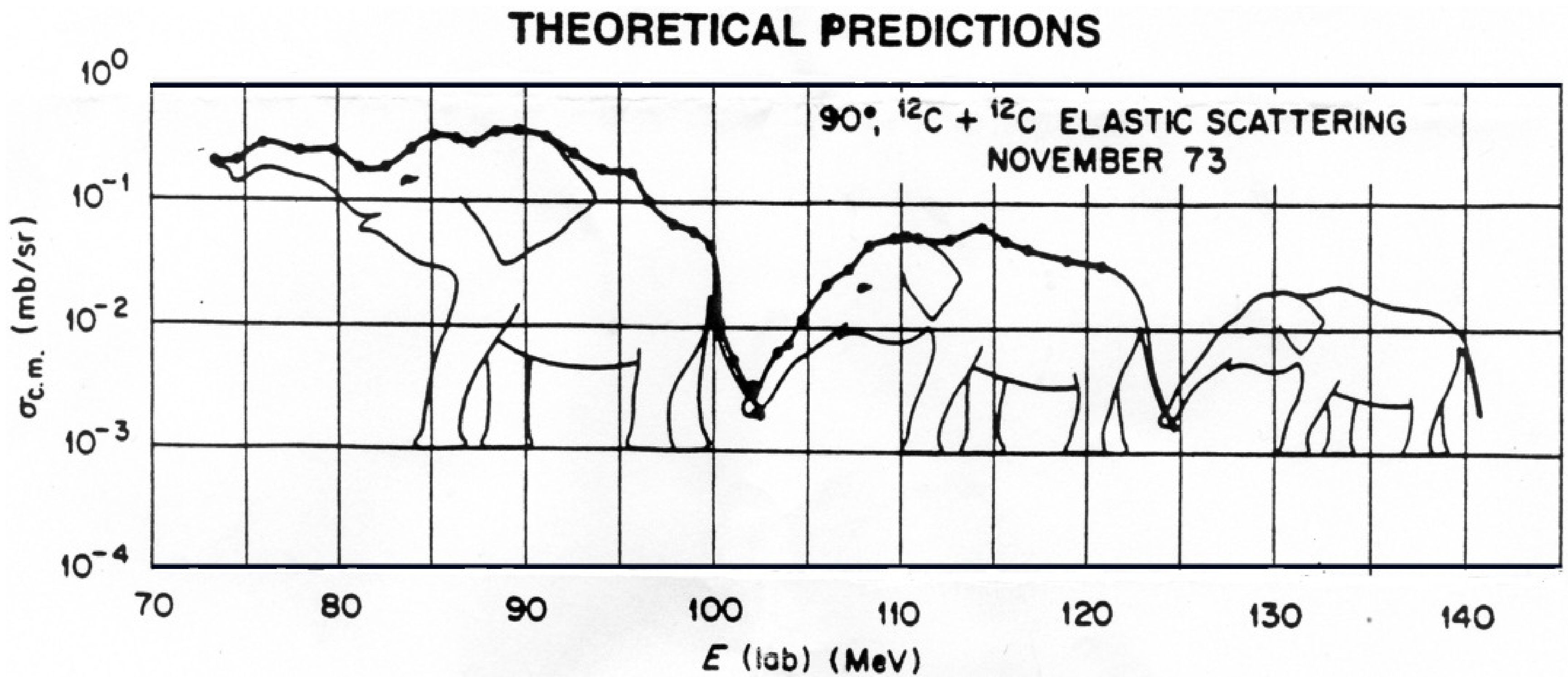

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Even if the (*ad hoc*) model fits perfectly the data,
we do not believe the predictions
because we don't trust the model!

[Many 'good' models are *ad hoc* models!]

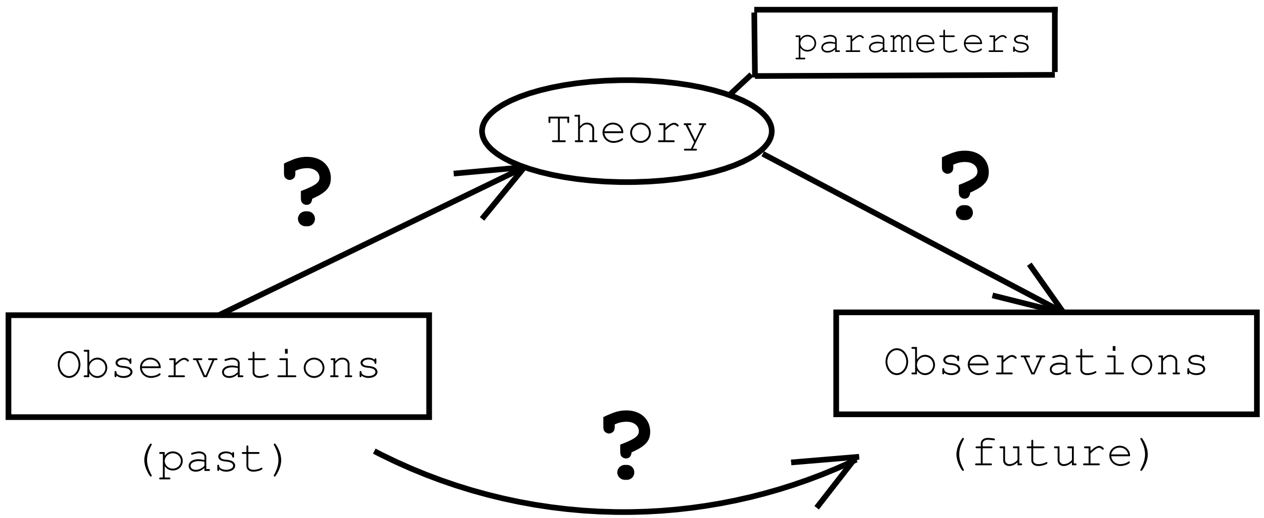
2011 IgNobel prize in Mathematics

- D. Martin of USA (who predicted the world would end in 1954)
- P. Robertson of USA (who predicted the world would end in 1982)
- E. Clare Prophet of the USA (who predicted the world would end in 1990)
- L.J. Rim of KOREA (who predicted the world would end in 1992)
- C. Mwerinde of UGANDA (who predicted the world would end in 1999)
- H. Camping of the USA (who predicted the world would end on September 6, 1994 and later predicted that the world will end on October 21, 2011)

2011 IgNobel prize in Mathematics

"For teaching the world to be careful when making mathematical assumptions and calculations"

Deep source of uncertainty



Uncertainty:

Theory	— ? ⟶	Future observations
Past observations	— ? ⟶	Theory
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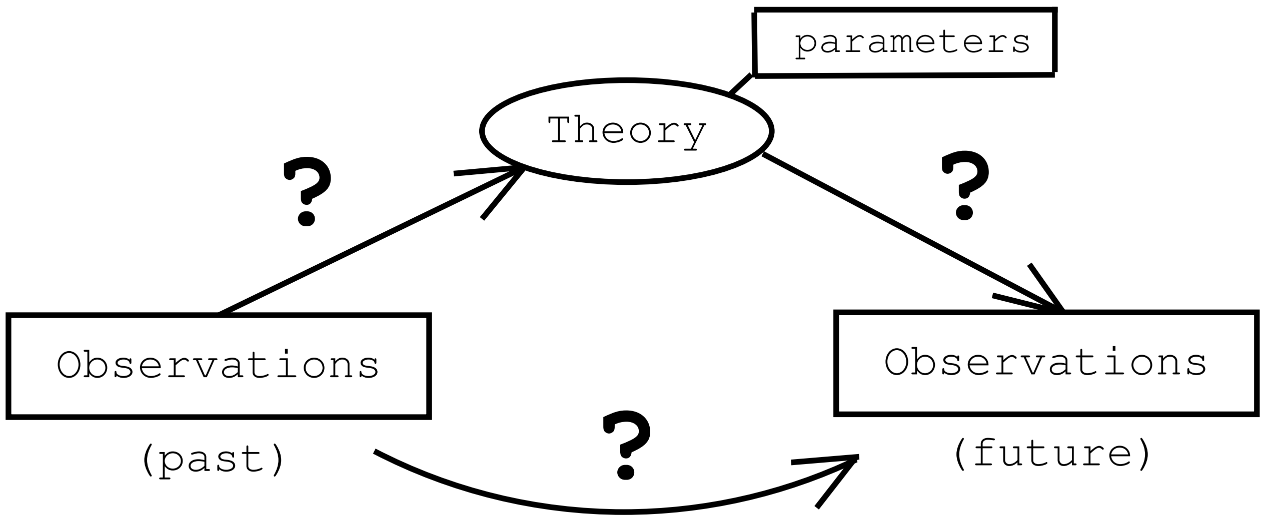

Uncertainty:

Theory	— ? $\longrightarrow$	Future observations
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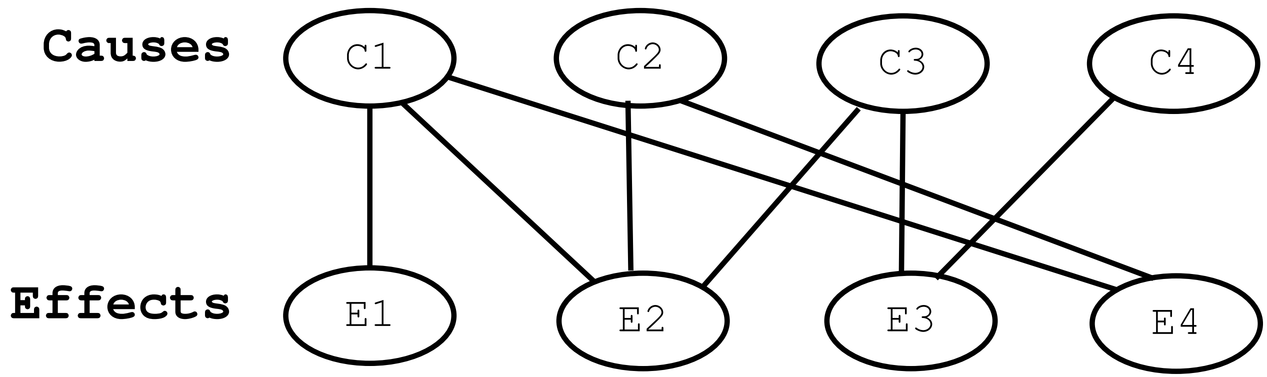
$\Longrightarrow$ **Uncertainty about causal connections**

CAUSE $\Longleftrightarrow$ **EFFECT**

Causes $\rightarrow$ effects

The same *apparent* cause might produce several,different effects



Given an observed effect, we are not sure about the exact cause that has produced it.

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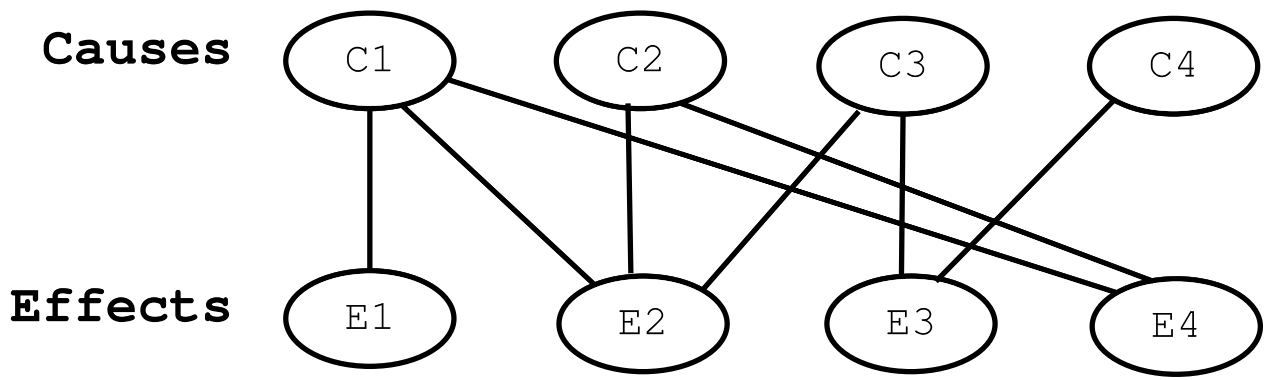

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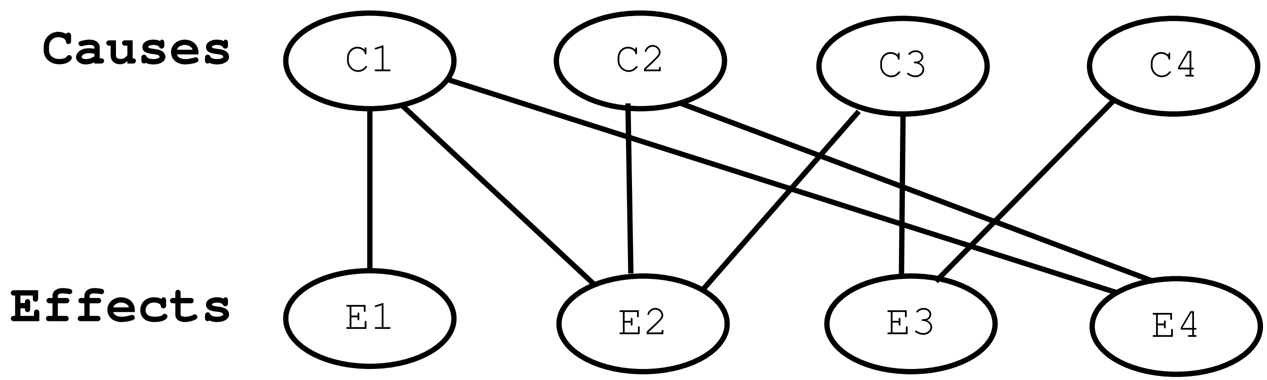

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$$\mathbf{E_2} \Rightarrow \{C_1,\, C_2,\, C_3\}?$$

The "essential problem" of the Sciences

"Now, these problems are classified as *probability of causes*, and are most interesting of all their scientific applications. I play at *écarté* with a gentleman whom I know to be perfectly honest. What is the chance that he turns up the king? It is 1/8. This is a problem of the probability of effects.

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I play with a gentleman whom I do not know. He has dealt ten times, and he has turned the king up six times. What is the chance that he is a sharper? This is a problem in the probability of causes. It may be said that it is the essential problem of the experimental method."

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(H. Poincaré – *Science and Hypothesis*)

Why physics students are not taught how to tackle this kind of problems?

Uncertainty and probability

We, as physicists, consider absolutely natural and meaningful statements of the following kind

- $P(-10 < \epsilon'/\epsilon \times 10^4 < 50) >> P(\epsilon'/\epsilon \times 10^4 > 100)$
- $P(172 \le m_{top}/\text{GeV} \le 174) \approx 70\%$
- $P(M_H < 125.5\,\text{GeV}) > P(M_H > 125.5\,\text{GeV})$

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[The fact that for several (most?) people in this audience this criticism is misterious is a clear indication of the confusion concerning this matter]

Doing Science in conditions of uncertainty

The constant status of uncertainty does not prevent us from doing Science (in the sense of Natural Science and not just Mathematics)

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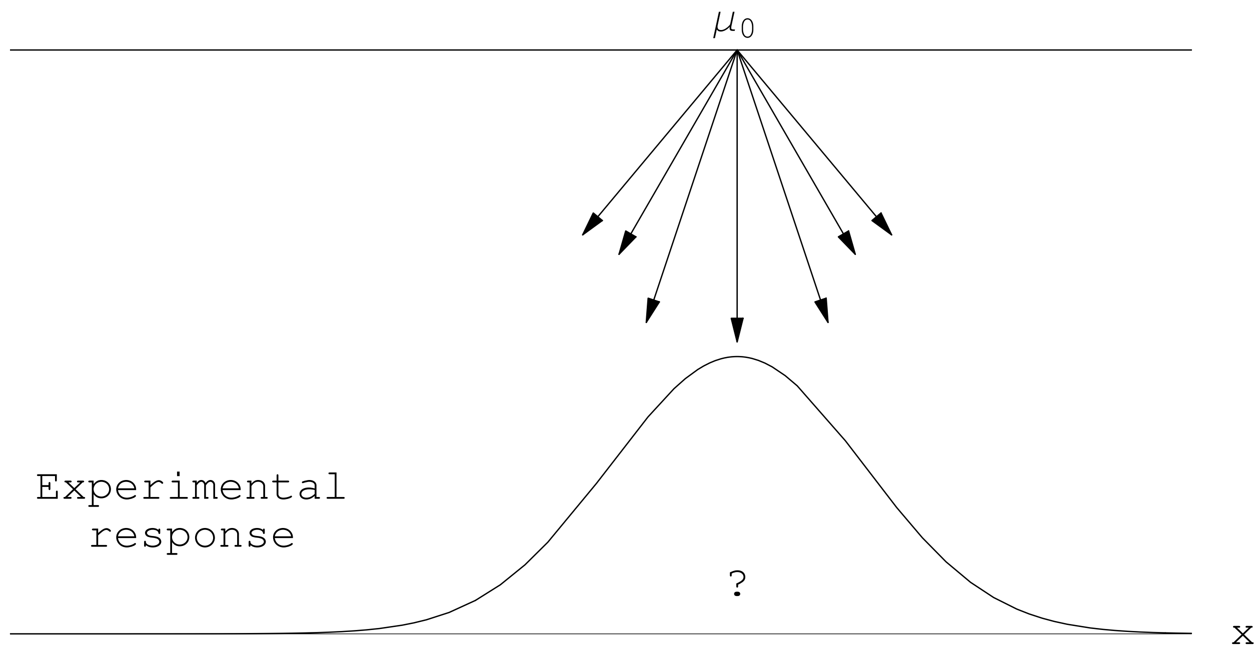
Indeed

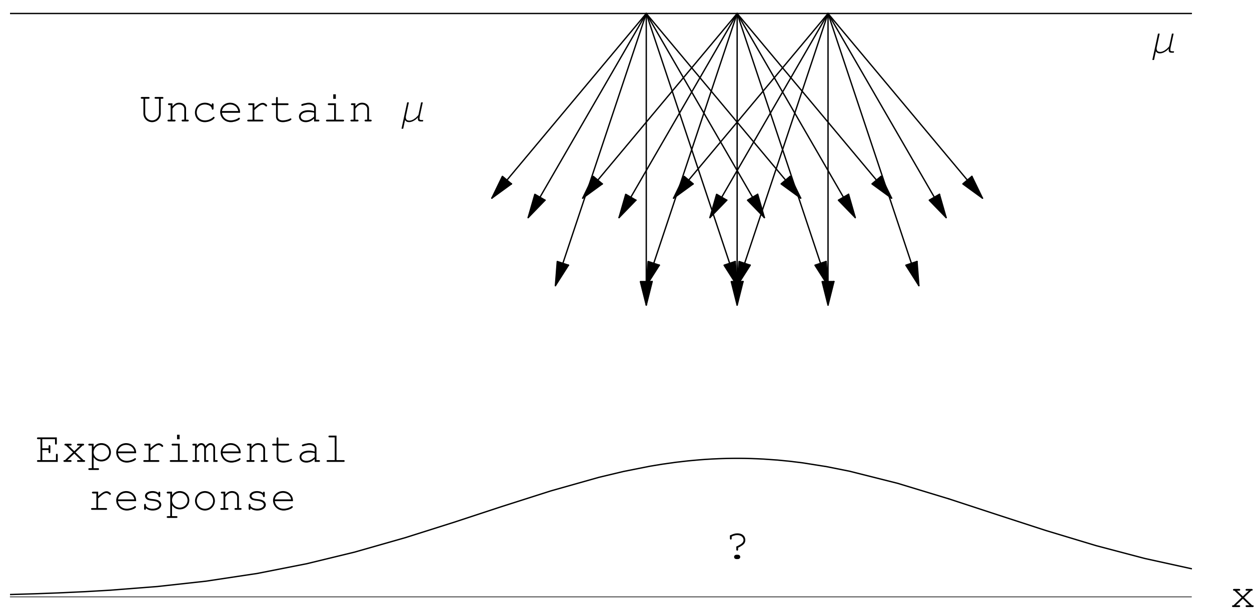
"It is scientific only to say what is more likely and what is less likely" (Feynman)

From ‘true value’ to observations



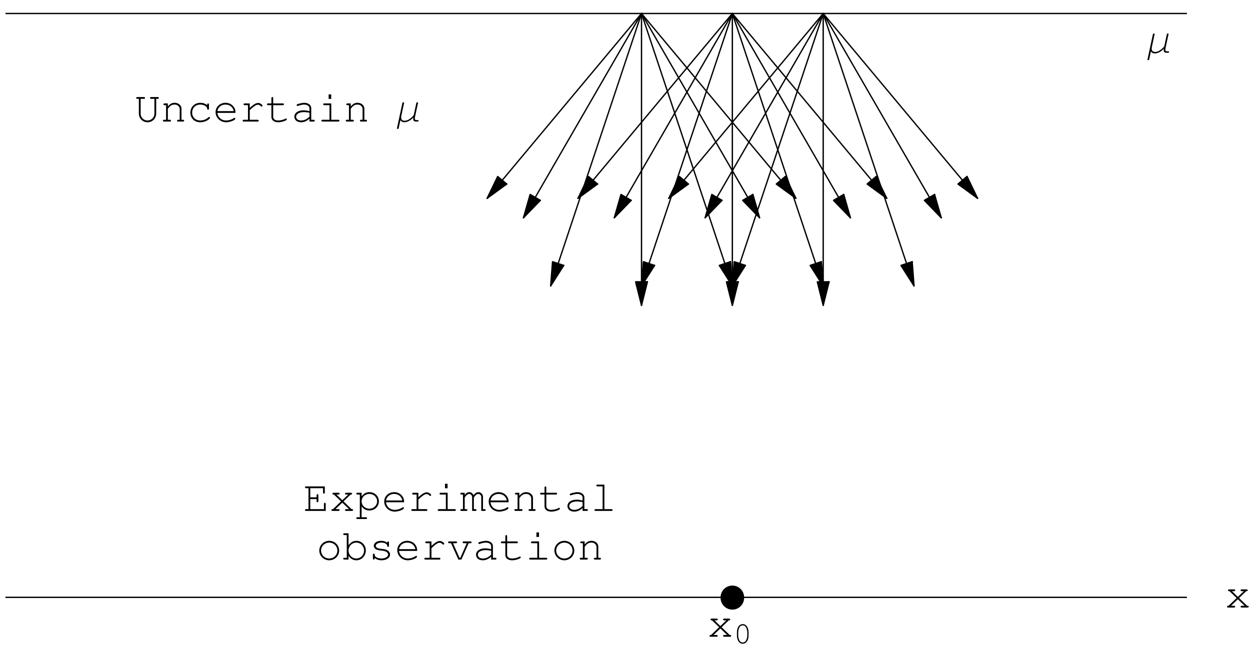
Given μ (exactly known) we are uncertain about x

From ‘true value’ to observations



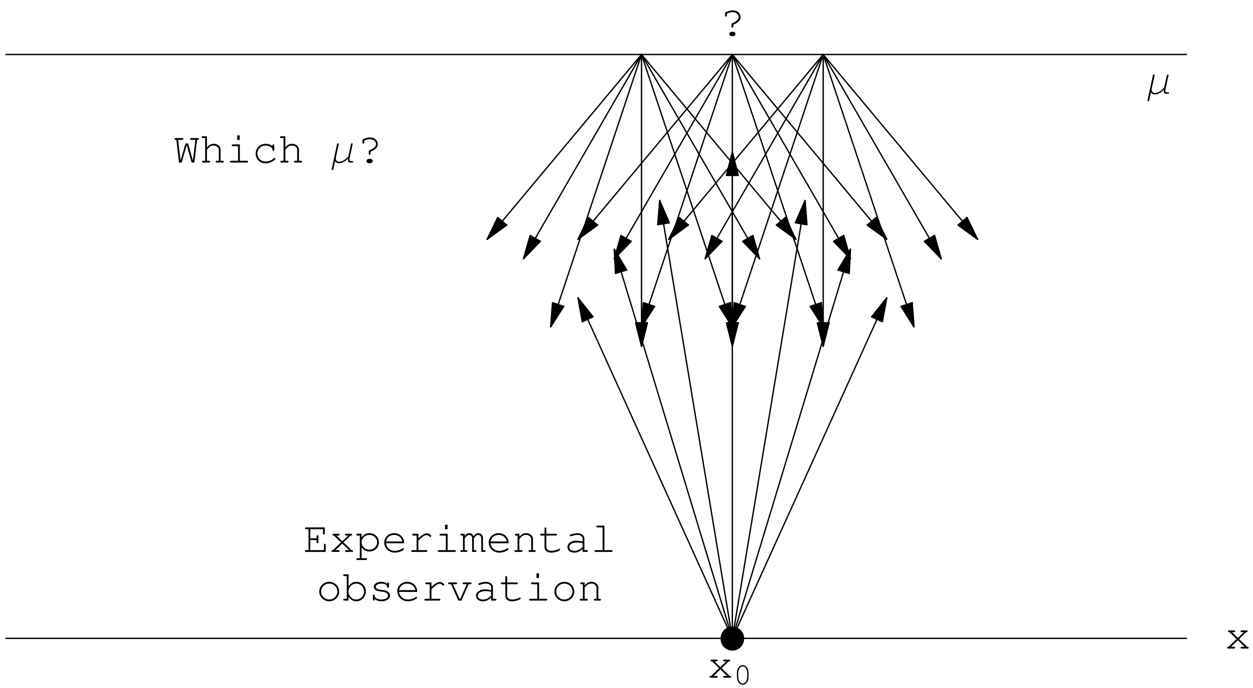
Uncertainty about μ makes us more uncertain about x

…and back: Inferring a true value



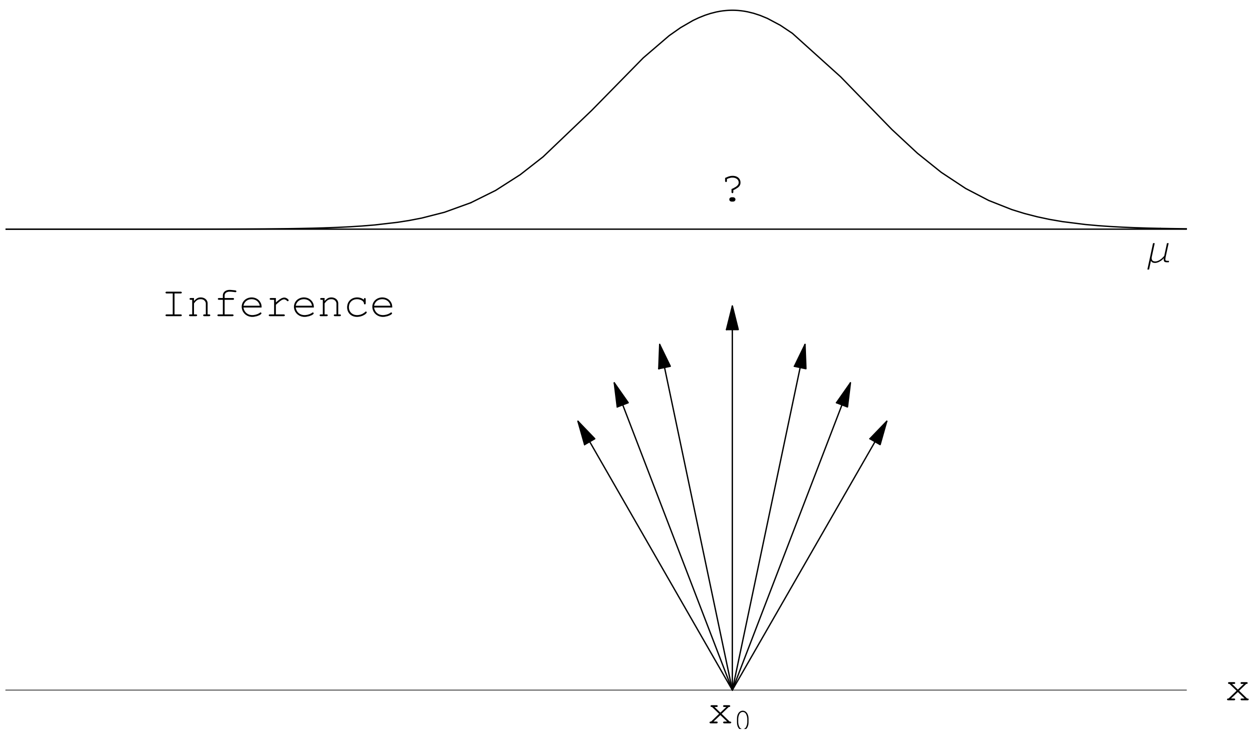
The observed data is certain: $\rightarrow$ 'true value' uncertain.

. . . and back: Inferring a true value



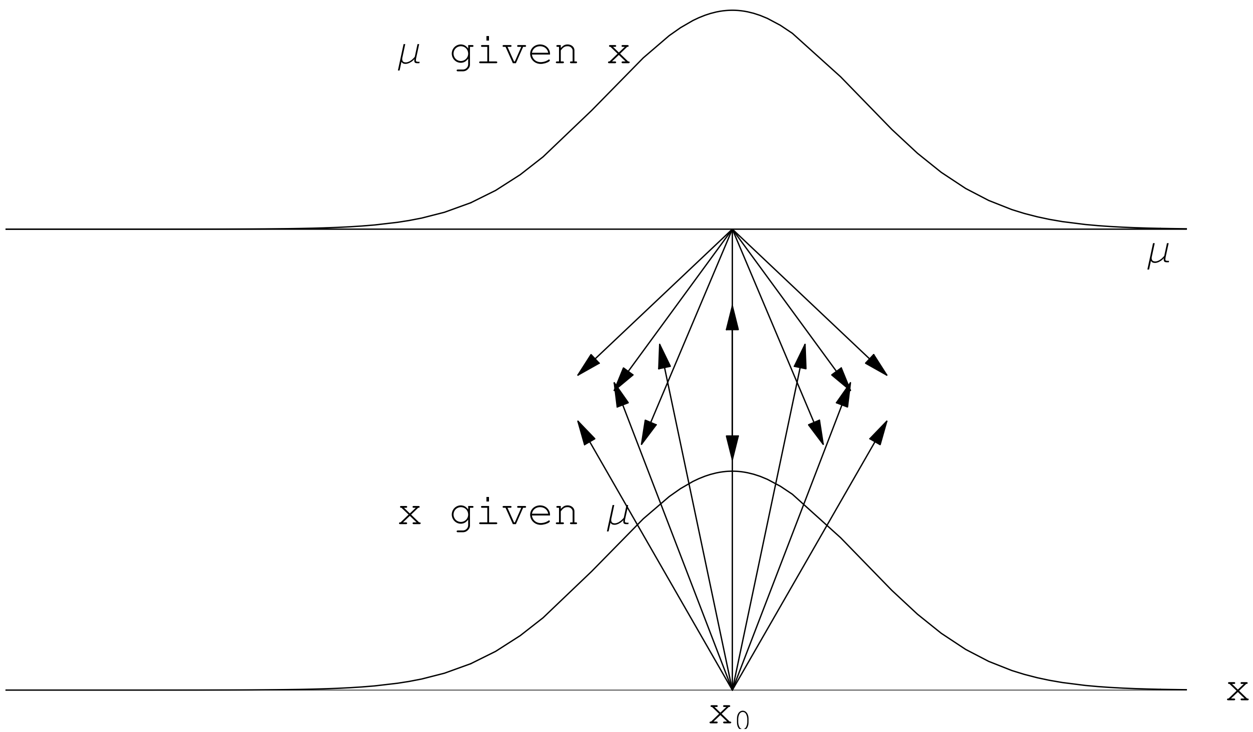
Where does the observed value of x comes from?

. . . and back: Inferring a true value



We are now uncertain about μ, given x.

…and back: Inferring a true value



Note the symmetry in reasoning.

A very simple experiment

Let’s make an experiment

A very simple experiment

Let's make an experiment

- Here
- Now

A very simple experiment

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For simplicity

- μ can assume only six possibilities:

$$\mathbf{0, 1, \ldots, 5}$$

- x is binary:

$$\mathbf{0, 1}$$

$$[\,(1,2);\ \text{Black/White};\ \text{Yes/Not};\ \ldots\,]$$

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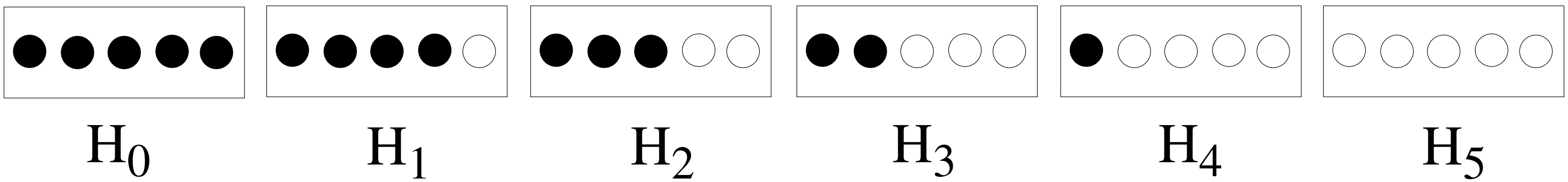
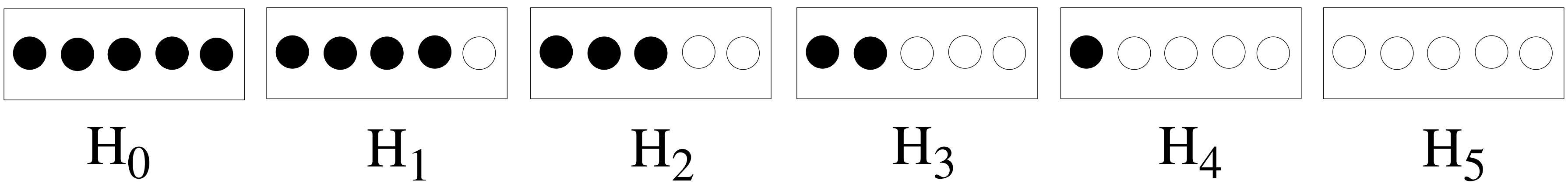
$\Rightarrow$ Later we shall make μ continous.

Which box? Which ball?



Let us take randomly one of the boxes.

Which box? Which ball?



Let us take randomly one of the boxes.

We are in a state of uncertainty concerning several *events*, the most important of which correspond to the following questions:

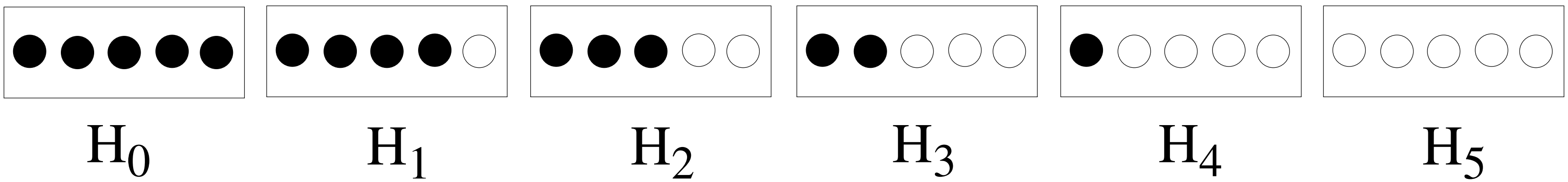
(a) Which box have we chosen, $H_0, H_1, \ldots, H_5$?

(b) If we extract randomly a ball from the chosen box, will we observe a white ($E_W \equiv E_1$) or black ($E_B \equiv E_2$) ball?

Our certainties:

$$\cup_{j=0}^{5} H_j = \Omega$$
$$\cup_{i=1}^{2} E_i = \Omega\,.$$

Which box? Which ball?



Let us take randomly one of the boxes.

- What happens after we have extracted one ball and looked its color?
  - Intuitively feel *how to* roughly *change* our opinion about
    - the possible cause
    - a future observation

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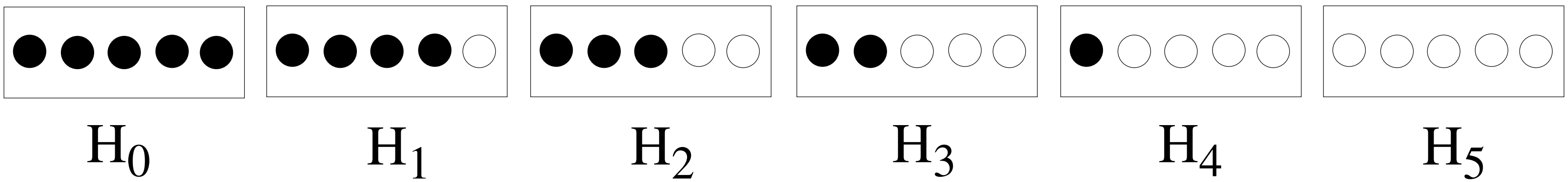

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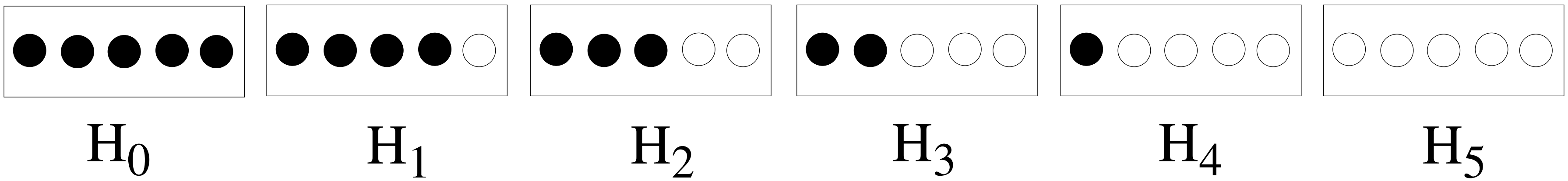

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  - Can we do it *quantitatively*, in an 'objective way'?
- And after a sequence of extractions?

The toy inferential experiment

The aim of the experiment will be to guess the content of the box without looking inside it, only extracting a ball, record its color and reintroducing in the box

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This toy experiment is conceptually very close to what we do in Physics

$\Rightarrow$ try to guess what we cannot see (the electron mass, a branching ratio, etc)

…from what we can see (somehow) with our senses.

The rule of the game is that we are not allowed to watch inside the box! (As we cannot open an electron and read its properties, unlike we read the MAC address of a PC interface.)

***Where is* probability?**

We all agree that the experimental results change

- the probabilities of the box compositions;
- the probabilities of a future outcomes,

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Certainly <u>not</u> *in* the box!

Subjective nature of probability

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Probability depends on the status of information of the *subject* who evaluates it.

Probability is <u>always</u> conditional probability

"Thus whenever we speak loosely of 'the probability of an event', it is always to be understood: probability with regard to a certain given state of knowledge"

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where I_s is the information available to *subject* s.

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$\longrightarrow$ Three boxes TV contests

What are we talking about?

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$\Rightarrow$ How much we believe something

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$\rightarrow$ 'Degree of belief' $\leftarrow$

Beliefs and 'coherent' bets

Remarks:

- Subjective does not mean arbitrary!

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  "The usual touchstone, whether that which someone asserts is merely his persuasion – or at least his subjective conviction, that is, his firm belief – is betting. It often happens that someone propounds his views with such positive and uncompromising assurance that he seems to have entirely set aside all thought of possible error. A bet disconcerts him. Sometimes it turns out that he has a conviction which can be estimated at a value of one ducat, but not of ten. For he is very willing to venture one ducat, but when it is a question of ten he becomes aware, as he had not previously been, that it may very well be that he is in error." (Kant)

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		1	X	2
17/07 19:00	SL. LIBEREC - S KARAGANDY	1,20	6,00	10,00
17/07 18:00	HJK HELSINK - KR REYKJAV	1,30	4,75	8,00
17/07 17:45	FLORA TALLINN - BASILEA	9,50	5,00	1,25
17/07 18:30	DUDELANGE - SALISBURGO	10,00	5,00	1,25
17/07 20:45	VALLETTA - PARTIZAN	10,00	5,00	1,25
18/07 18:00	BATE BORISOV - VARDAR	1,15	6,50	13,00

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$\rightarrow P(3477 \leq M_{Sun}/M_{Sat} \leq 3547 \,|\, I(\text{Laplace})) = 99.99\%$

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Hint: $P(\theta \leq \theta_{obs} \,|\, m_0) \neq P(m \geq m_o \,|\, \theta_{obs})$!!

$\Rightarrow$ more in second lecture.

Standard textbook definitions

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

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It is easy to check that 'scientific' definitions suffer of circularity

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Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equally possible cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

Note!: *"lorsque rien ne porte à croire que l'un de ces cas doit arriver plutot que les autres" (Laplace)*

Replacing 'equi-probable' by 'equi-possible' is just cheating students (as I did in my first lecture on the subject. . .).

Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity, plus other problems

$$p = \frac{\#\text{ favorable cases}}{\#\text{ possible equiprobable cases}}$$

$$p = \lim_{n\to\infty} \frac{\#\text{ times the event has occurred}}{\#\text{ independent trials under same condition}}$$

Future $\Leftrightarrow$ Past (belief!)

$n \to \infty$: $\to$ *"usque tandem?"*

$\to$ *"in the long run we are all dead"*

$\to$ It limits the range of applications

‘Definitions’ $\rightarrow$ evaluation rules

Very useful evaluation rules

$$A) \quad p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$B) \quad p = \frac{\#\,\text{times the event has occurred}}{\#\text{independent trials under same condition}}$$

If the implicit beliefs are well suited for each case of application.

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If the implicit beliefs are well suited for each case of application.

BUT they cannot define the concept of probability!

‘Definitions’ $\rightarrow$ evaluation rules

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In the probabilistic approach we are following

- Rule A is recovered immediately (under the assumption of equiprobability, when it applies).
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In the probabilistic approach we are following

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- Rule B results from a theorem (under well defined assumptions): $\Rightarrow$ Laplace’s rule of succession

Unifying role of subjective probability

- Wide range of applicability

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- Probability statements all have the same meaning no matter to what they refer and how the number has been evaluated.
  - $P(\text{rain next Saturday in Vienna}) = 68\%$
  - $P(\text{Usain Bolt will win the 100m in London}) = 68\%$
  - $P(M_H \leq 125.5\,\text{GeV}) = 68\%$
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  They all convey unambiguously the <u>same confidence</u> on something.

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- You might agree or disagree, but at least You know what this person has in his mind. (NOT TRUE with "C.L.'s"!)
- If a person has these beliefs and he/she has the chance to win a rich prize bound to one of these events, he/she is indifferent to the choice.

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We can talk very naturally about **probabilities of true values!**

Probability Vs “probability”…

Errors on ratios of small numbers of events
F. James$^{(*)}$ and M. Roos
Nucl. Phys. **B172** (1980) 475
(`http://ccdb4fs.kek.jp/cgi-bin/img_index?8101205`)

When the result of the measurement of a physical quantity is published as $R=R_0\pm\sigma_0$ without further explanation, it is implied that R is a Gaussian-distributed measurement with mean R_0 and variance σ_0^2. This allows one to calculate various confidence intervals of given "probability", i.e. the "probability" P that the true value of R is within a given interval. P is given by the area under the corresponding part of the Gaussian curve, and is the basis of well-known rules-of-thumb such as "the probability of exceeding two standard deviations is 5%".

$^{(*)}$ Influential CERN ’frequentistic guru’ of HEP community

Mathematics of beliefs

The good news:

The basic laws of degrees of belief are the same we get from the inventory of favorable and possible cases, or from events occurred in the past.

[Details skipped. . .]

Basic rules of probability

1. $0 \le P(A \,|\, I) \le 1$
2. $P(\Omega \,|\, I) = 1$
3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I) \quad [\text{if } P(A \cap B \,|\, I) = \emptyset]$

4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

Remember that probability is always conditional probability!

I is the background condition (related to information 'I'_s')

$\rightarrow$ usually implicit (we only care on 're-conditioning')

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Note: 4. does not define conditional probability.
(Probability is always conditional probability!)

Mathematics of beliefs

An even better news:

The fourth basic rule
can be fully exploided!

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(Liberated by a curious ideology that forbits its use)

A simple, powerful formula

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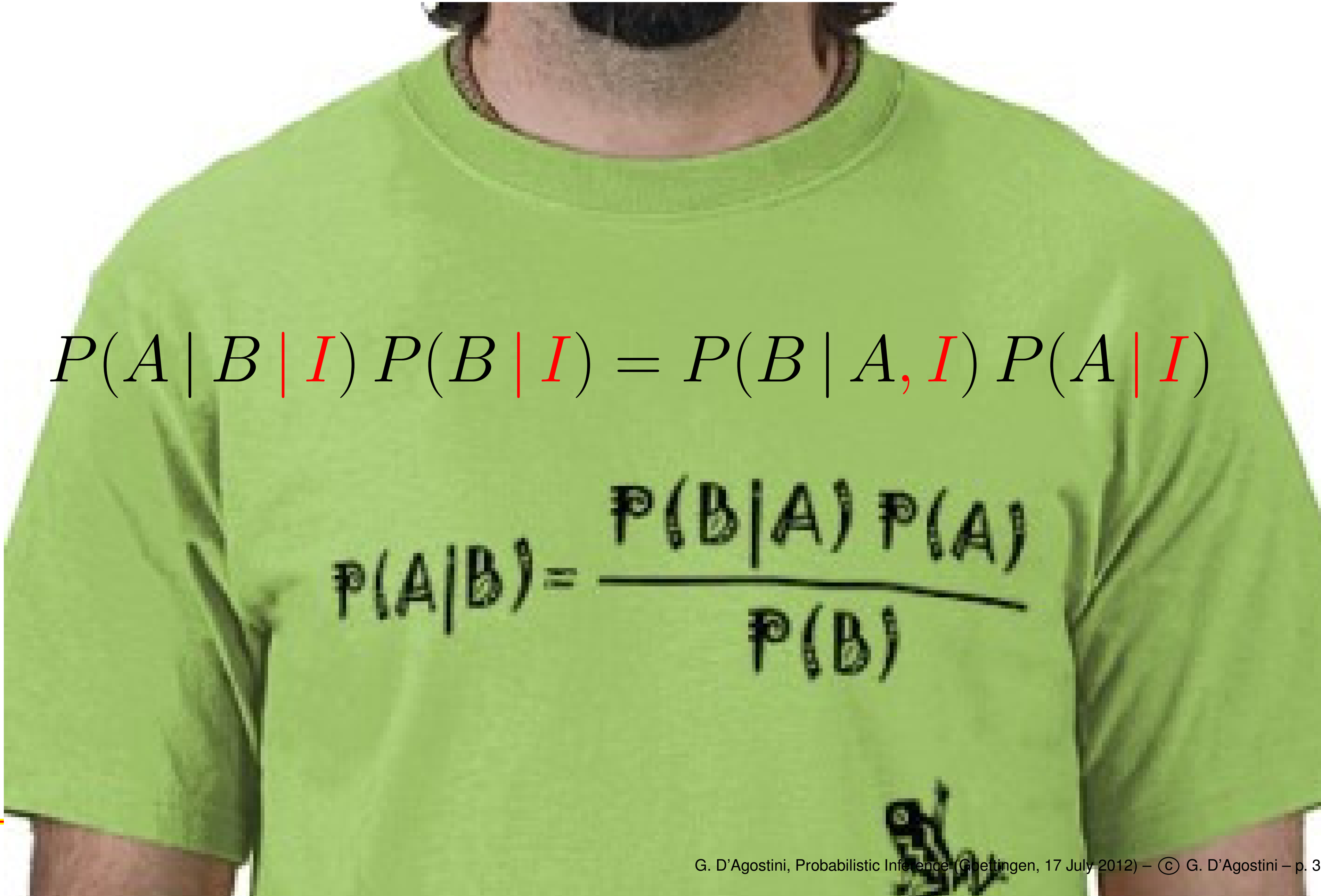
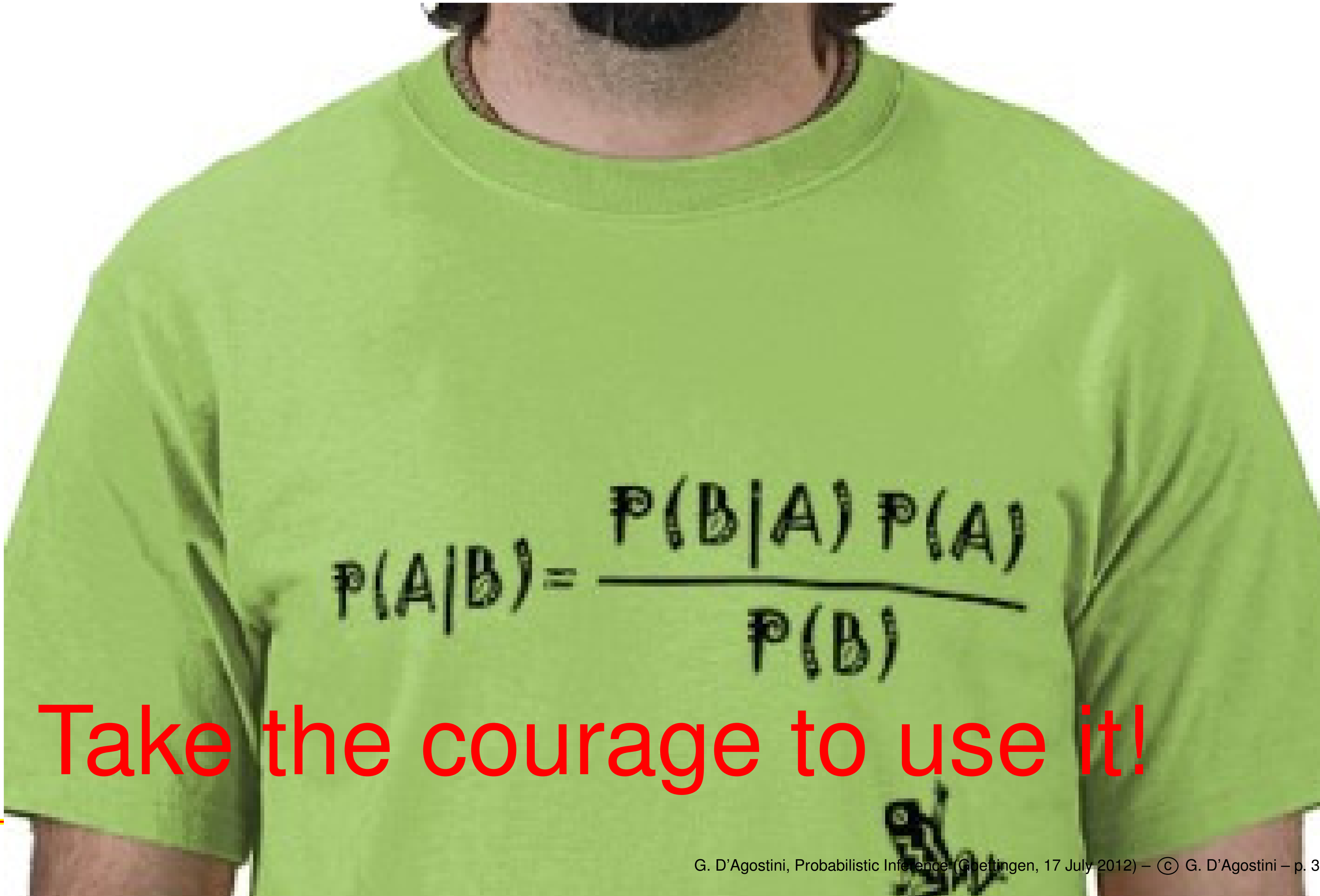
$$P(A\,|\,B\,|\,I)\,P(B\,|\,I) = P(B\,|\,A, I)\,P(A\,|\,I)$$

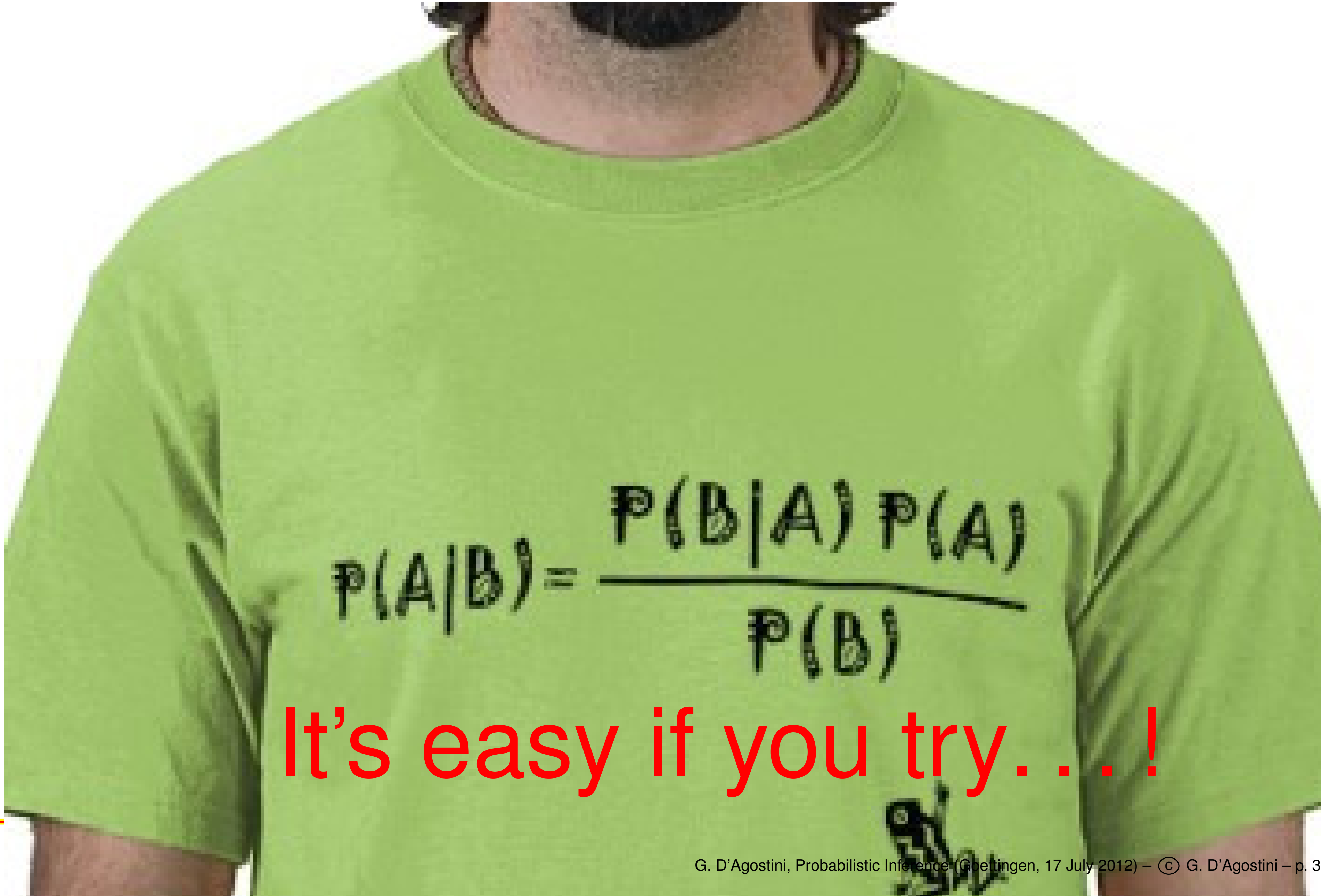


A simple, powerful formula



Take the courage to use it!

A simple, powerful formula



It's easy if you try. . . !

Laplace's "Bayes Theorem"

"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}.

$$P(C_i \,|\, E) \propto P(E \,|\, C_i)$$

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"The greater the probability of an observed event given any one of a number of causes to which that event may be attributed, the greater the likelihood of that cause {given that event}. The probability of the existence of any one of these causes {given the event} is thus a fraction whose numerator is the probability of the event given the cause, and whose denominator is the sum of similar probabilities, summed over all causes.

$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)}{\sum_j P(E \,|\, C_j)}$$

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$$P(C_i\,|\,E) = \frac{P(E\,|\,C_i)\,P(C_i)}{\sum_j P(E\,|\,C_j)\,P(C_j)}$$

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$$P(C_i \,|\, E) = \frac{P(E \,|\, C_i)\, P(C_i)}{\sum_j P(E \,|\, C_j)\, P(C_j)}$$

"This is the fundamental principle (*) of that branch of the analysis of chance that consists of reasoning *a posteriori* from events to causes"

(*) In his "Philosophical essay" Laplace calls 'principles' the 'fondamental rules'.

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(*) In his "Philosophical essay" Laplace calls 'principles' the 'fondamental rules'.

Note: denominator is just a normalization factor.

$$\Rightarrow \qquad P(C_i \,|\, E) \propto P(E \,|\, C_i)\, P(C_i)$$

Most convenient way to remember Bayes theorem

Telling it with Gauss' words

A reference to the Princeps Mathematicorum (Prince of Mathematicians) is a must in this town and in this place.

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$$P(C_i \,|\, \text{data}) = \frac{P(\text{data} \,|\, C_i)}{P(\text{data})} \, P_0(C_i)$$

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"post illa observationes" *"ante illa observationes"*

(Gauss)

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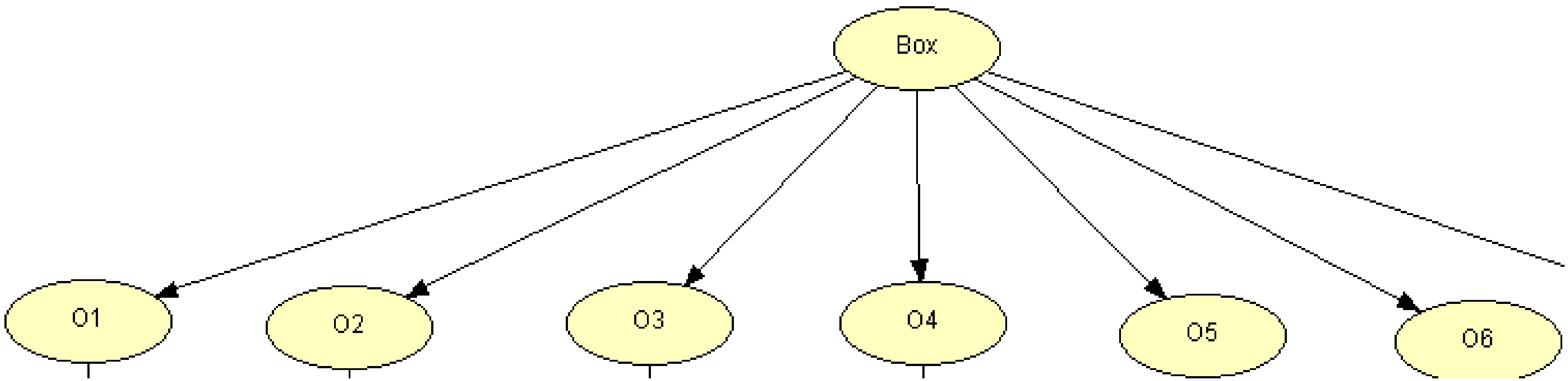
Arguments used to derive Gaussian distribution

- $f(\mu \,|\, \{x\}) \propto f(\{x\} \,|\, \mu) \cdot f_0(\mu)$
- $f_0(\mu)$ 'flat' (all values a priory equally possible)
- posterior maximized at $\mu = \overline{x}$

Cause-effect representation

box content → observed color



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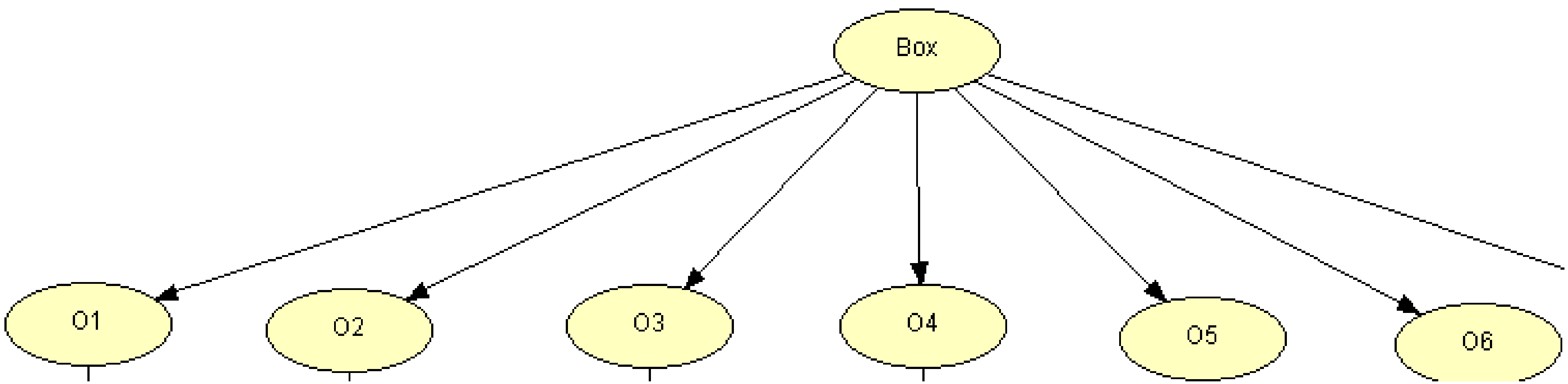
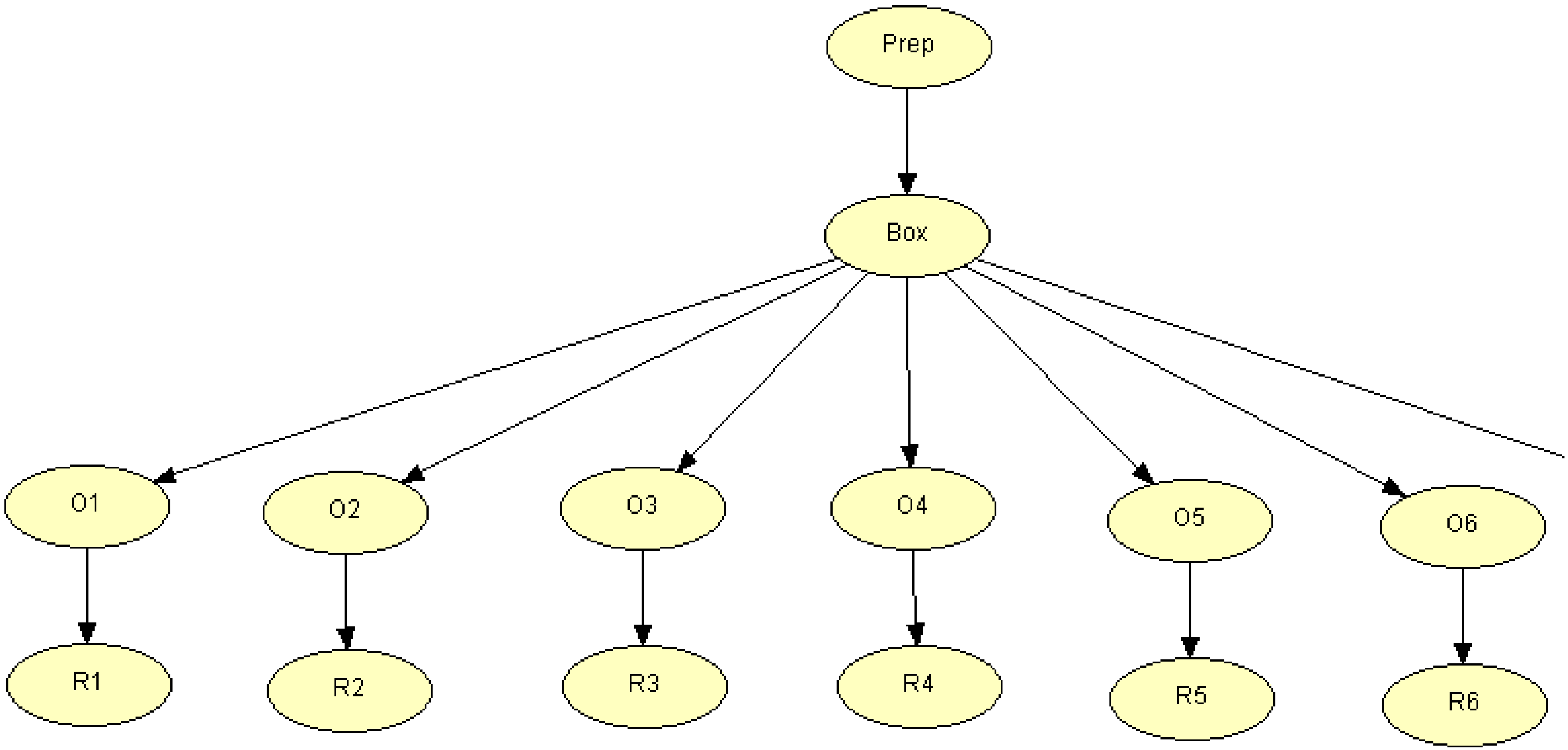


An effect might be the cause of another effect $\Longrightarrow$

A network of causes and effects



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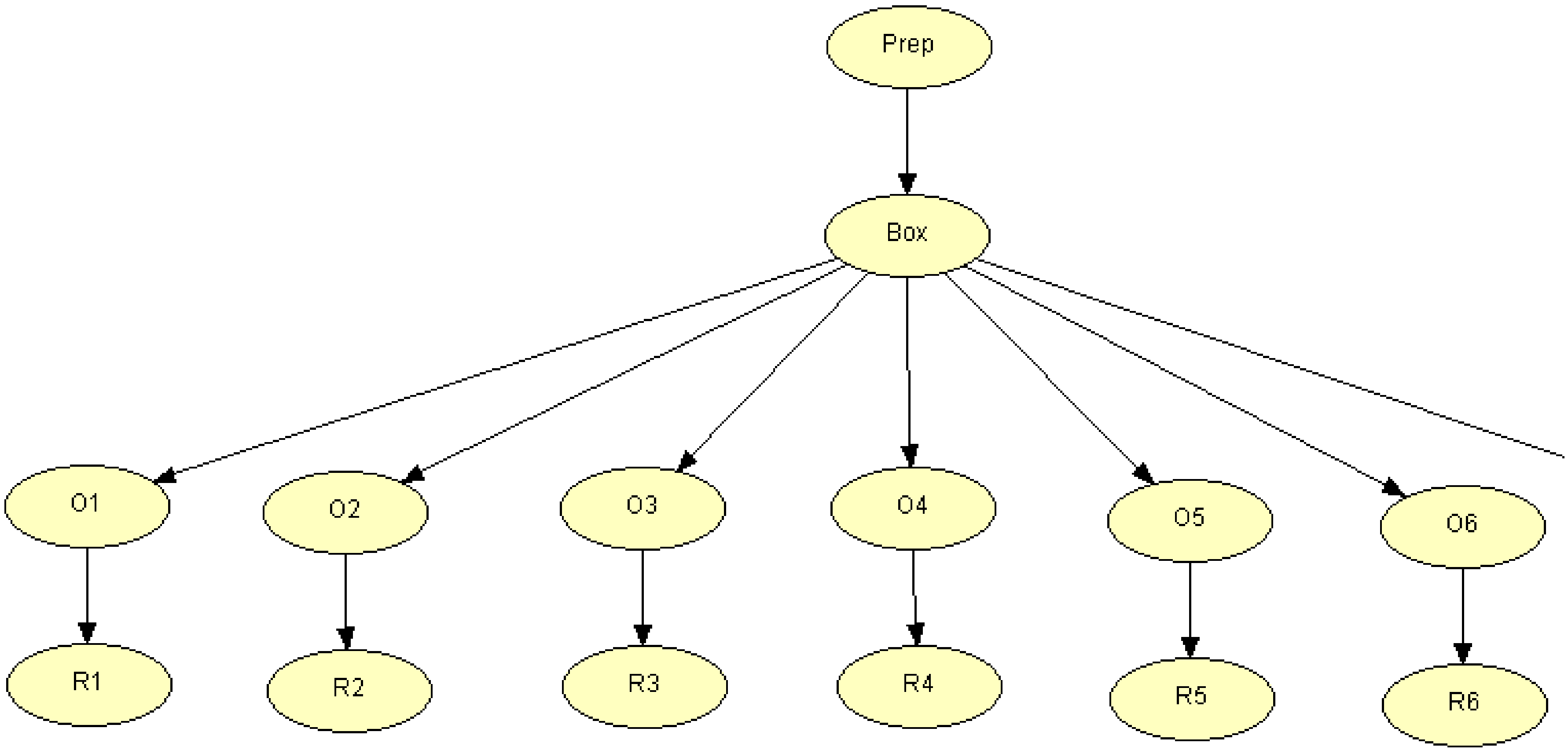

and so on. . .

$\Rightarrow$ Physics applications

Inferring ‘proportions’

Let’s turn the toy experiment to a ‘serious’ physics case:

- Inferring H_j is the same as inferring the proportion of white balls:

$$H_j \quad \longleftrightarrow \quad j \quad \longleftrightarrow \quad p = \frac{j}{5}$$

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$$n: \quad 6 \rightarrow \infty$$

$\Rightarrow p$ continous in $[0, 1]$

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$$n: \quad 6 \to \infty$$

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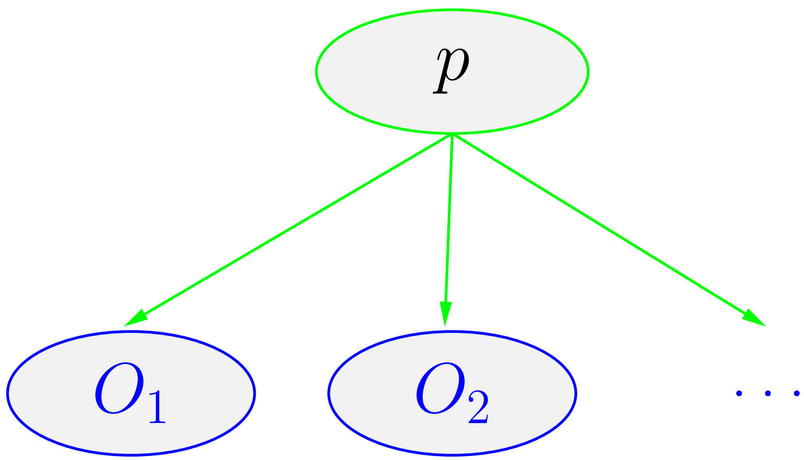
- Generalize White/Black $\longrightarrow$ Success/Failure

$\Rightarrow$ efficiencies, branching ratios, ...

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p



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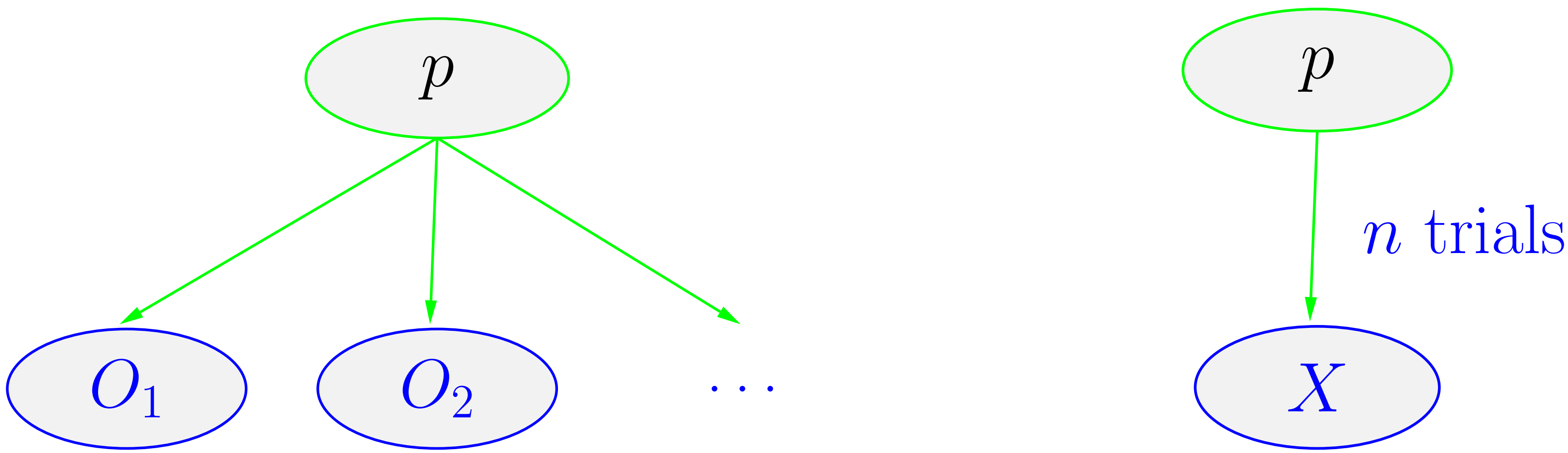

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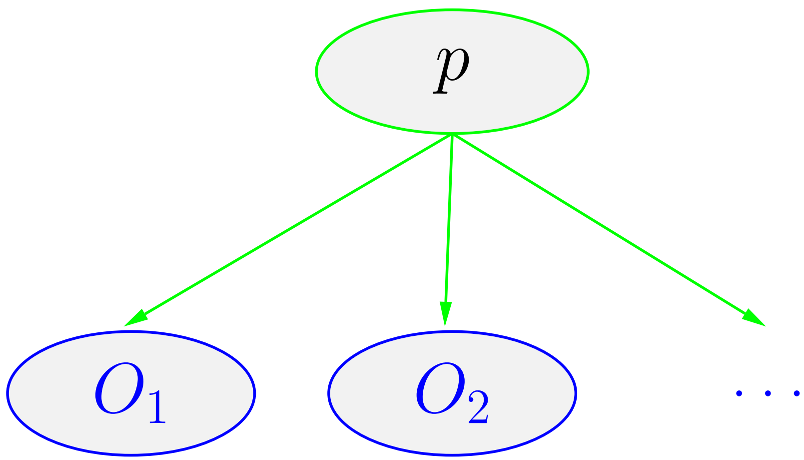
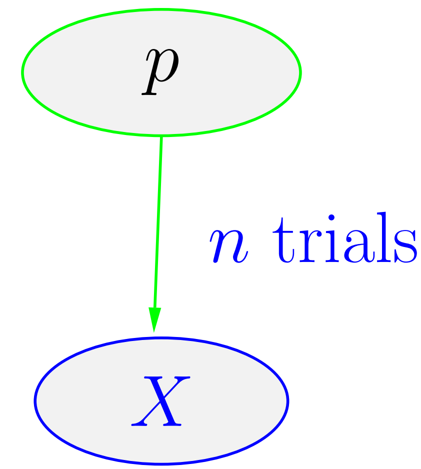


"independent Bernoulli trials"



"binomial distribution"

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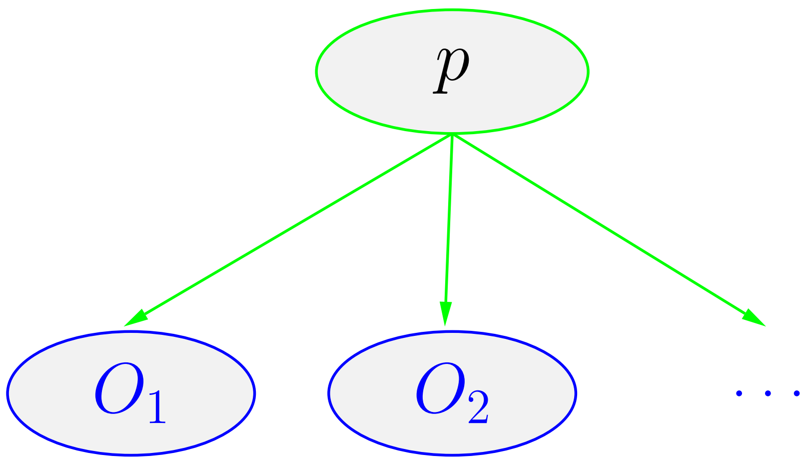

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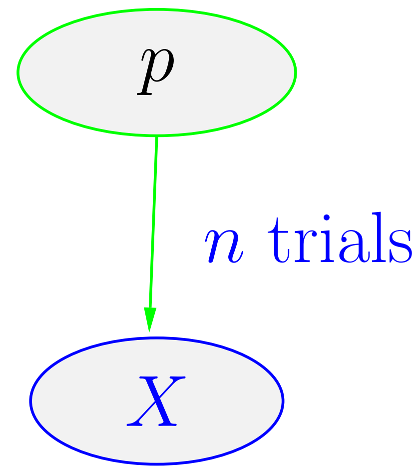
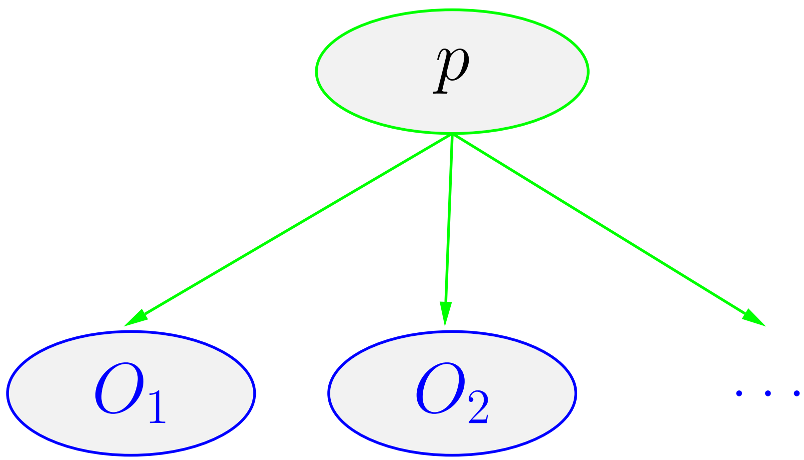


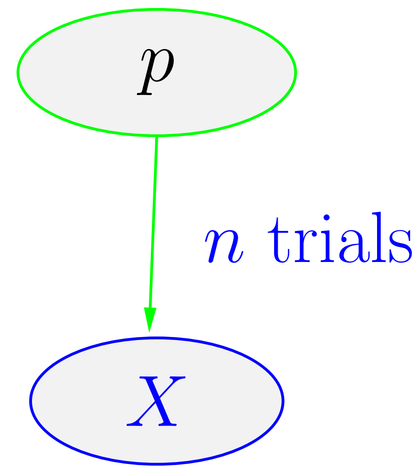
"binomial distribution"

$\Rightarrow$ In the light of the experimental information
there will be values of p we shall believe more,
and others we shall believe less.

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p





"independent Bernoulli trials"

"binomial distribution"

$$P(p_i \,|\, O_1, O_2, \ldots)$$
$$f(p \,|\, O_1, O_2, \ldots)$$

$$P(p_i \,|\, X, n)$$
$$f(p \,|\, X, n)$$

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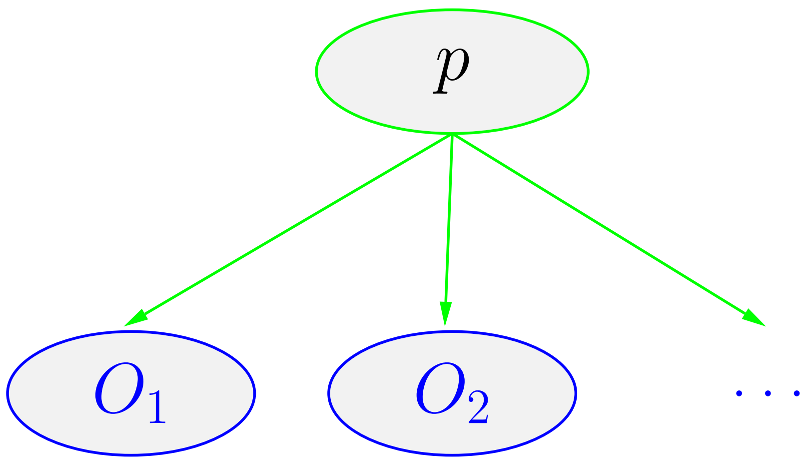
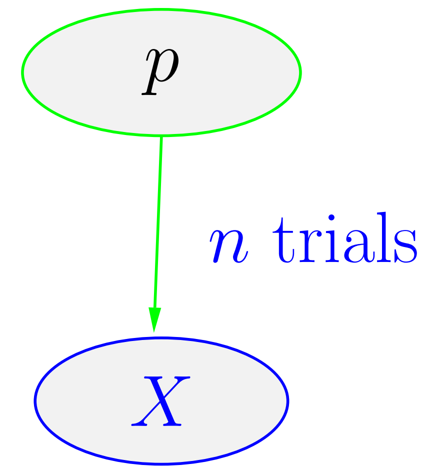




"independent Bernoulli trials"

$$P(p_i \,|\, O_1, O_2, \ldots)$$
$$f(p \,|\, O_1, O_2, \ldots)$$

$$\propto f(O_1, O_2, \ldots \,|\, p) \cdot f_0(p)$$

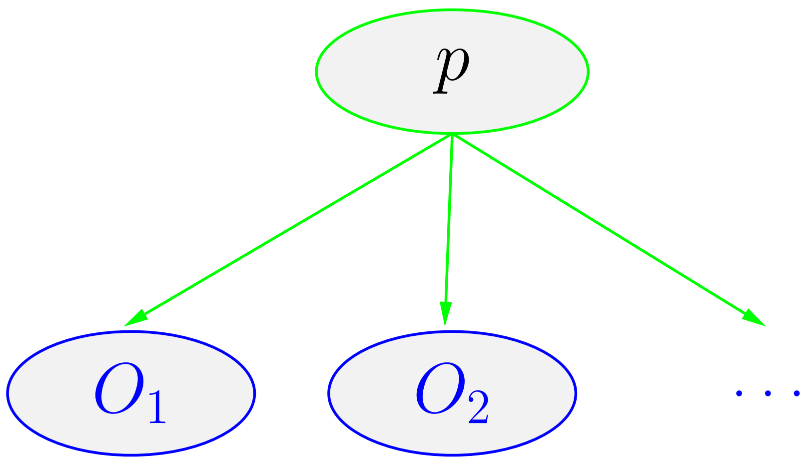
"binomial distribution"

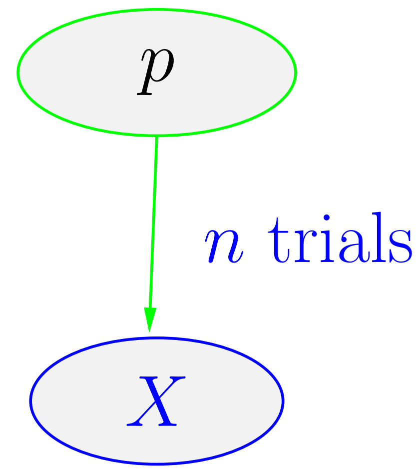
$$P(p_i \,|\, X, n)$$
$$f(p \,|\, X, n)$$

$$\propto f(X \,|\, n, p) \cdot f_0(p)$$

Inferring Bernoulli's trial parameter p

Making several independent trials *assuming* the same p





"independent Bernoulli trials" "binomial distribution"

$$P(p_i\,|\,O_1, O_2, \ldots) \qquad P(p_i\,|\,X, n)$$
$$f(p\,|\,O_1, O_2, \ldots) \qquad f(p\,|\,X, n)$$

Are the two inferences the same?
(not obvious in principle)

Graphical models

Before analysing in some detail this case let's make an overview of other important cases in physics

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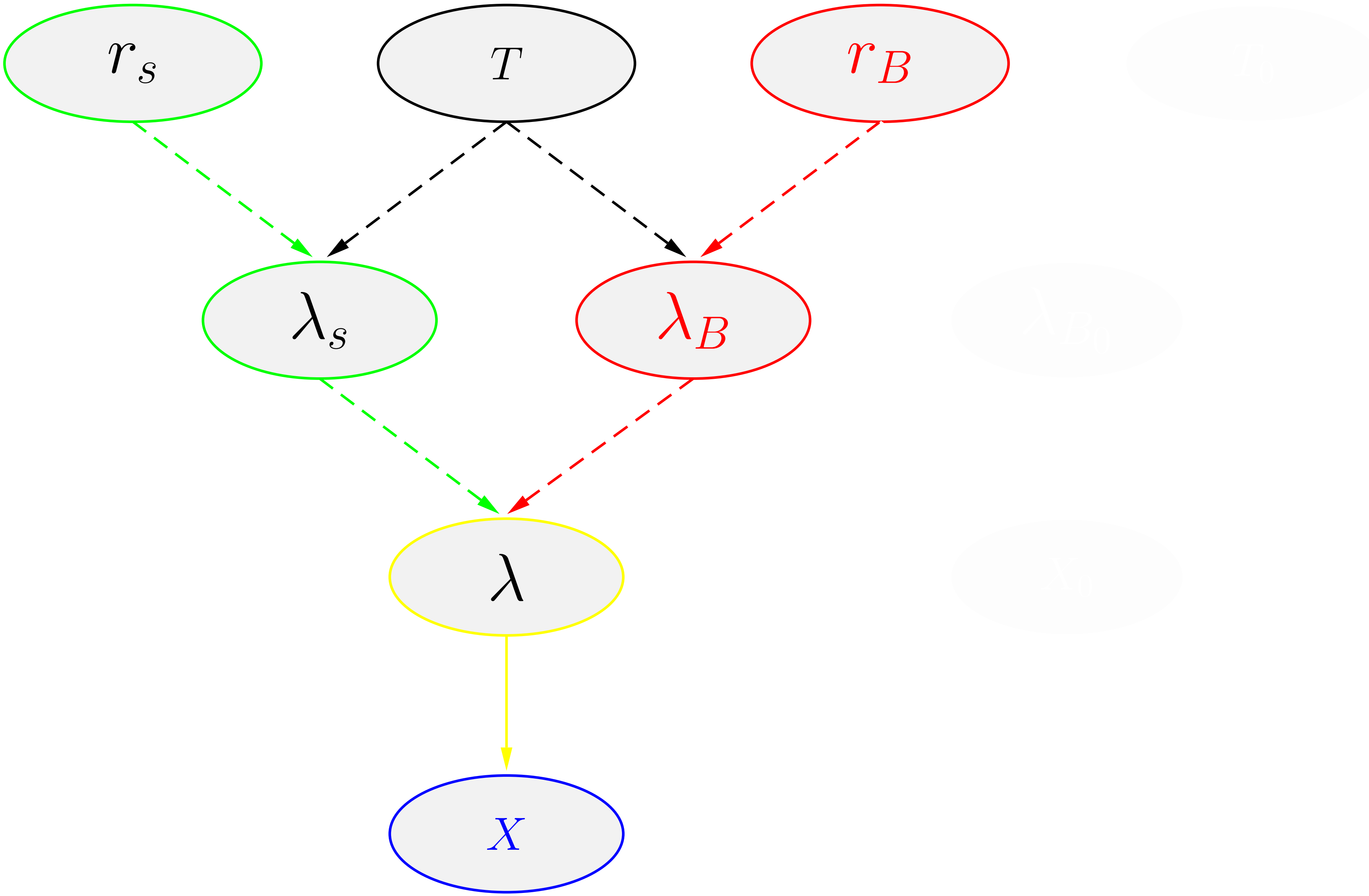
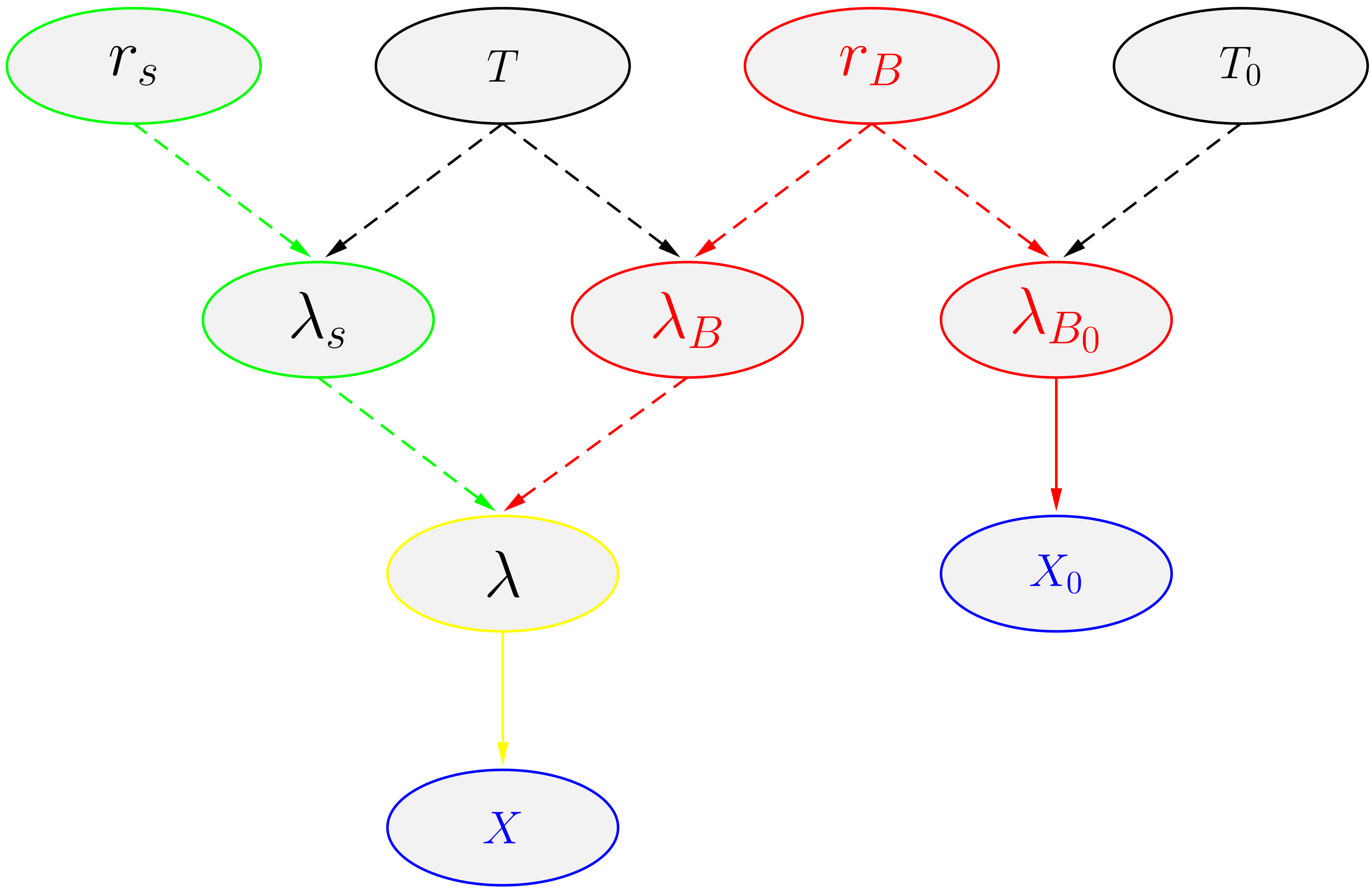
⇒ Nowadays, thanks to progresses in mathematics and computing, drawing the problem as a 'belief network' is more than 1/2 step towards its solution!

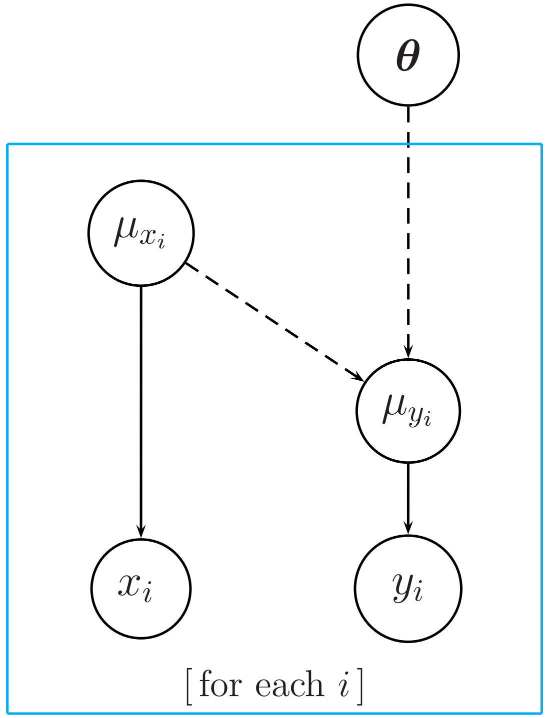
Signal and background



Signal and background



A different way to view fit issues



Determistic link μ_x's to μ_y's

Probabilistic links $\mu_x \rightarrow x$, $\mu_y \rightarrow y$

(errors on both axes!)

$\Rightarrow$ aim of fit: $\{\boldsymbol{x}, \boldsymbol{y}\} \rightarrow \boldsymbol{\theta}$

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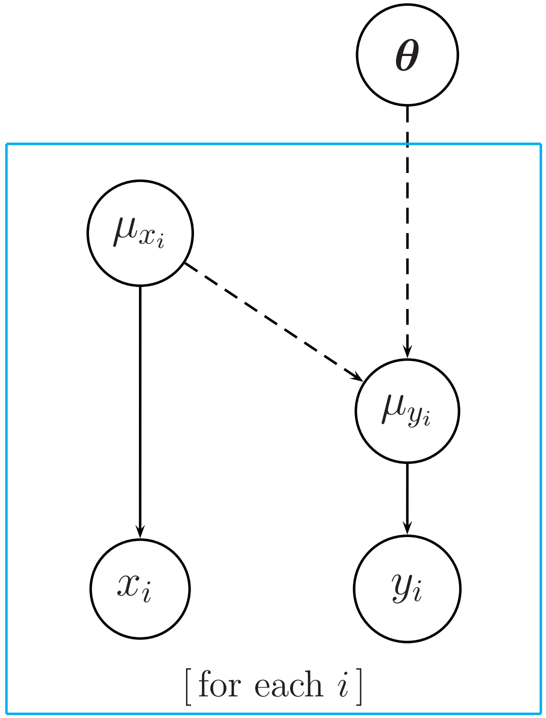

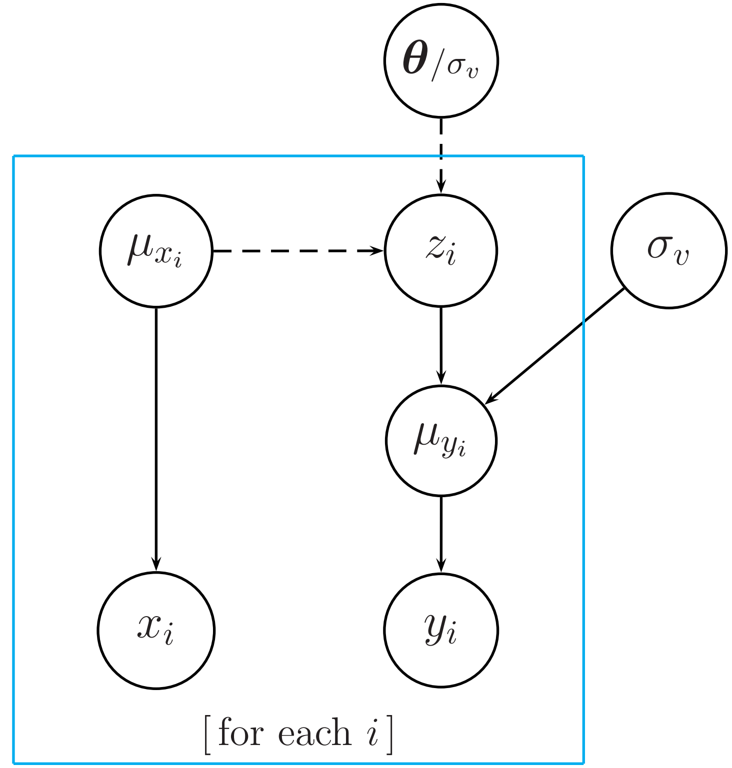

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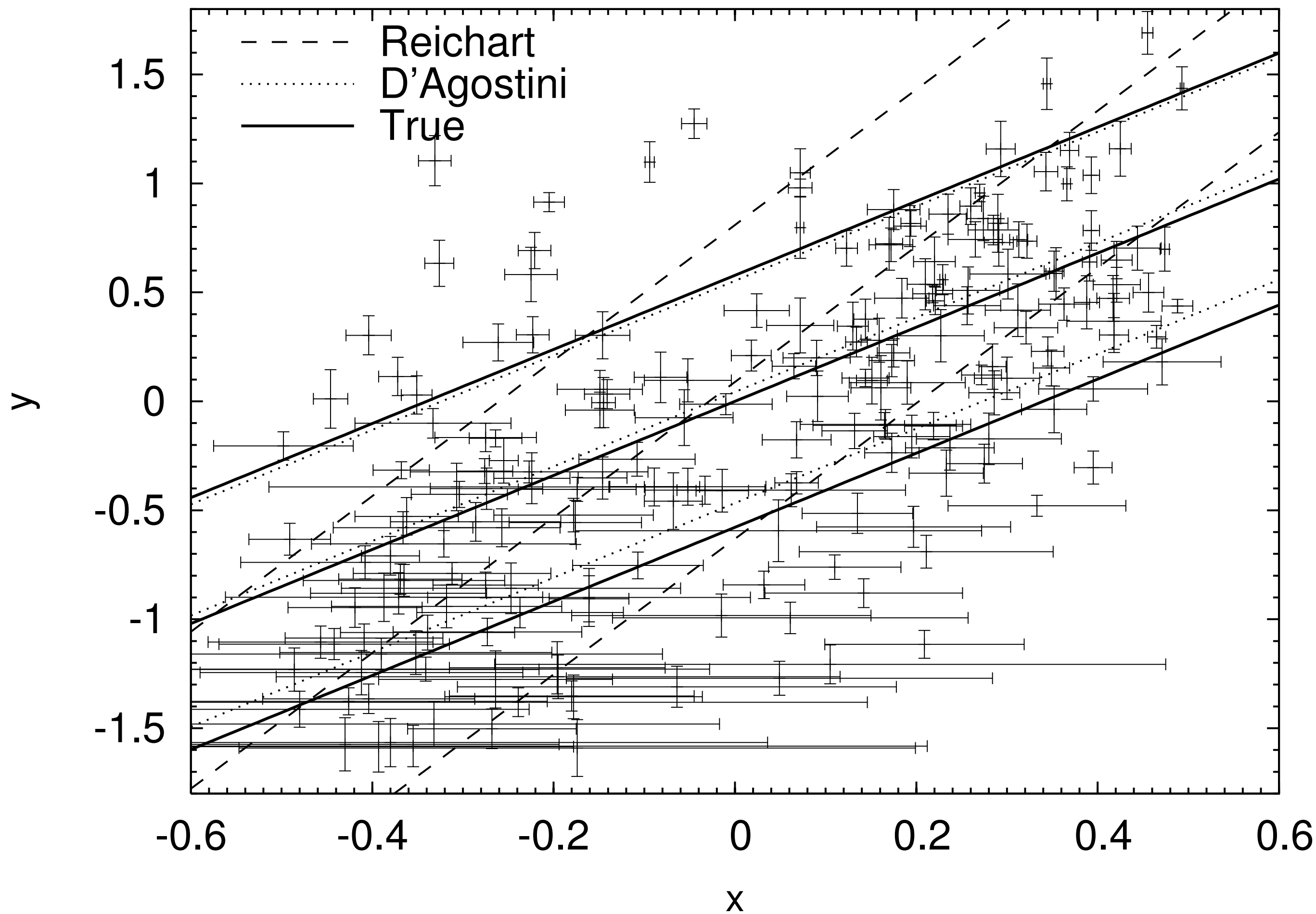
Probabilistic links $\mu_x \to x$, $\mu_y \to y$
(errors on both axes!)
$\Rightarrow$ aim of fit: $\{\boldsymbol{x}, \boldsymbol{y}\} \to \boldsymbol{\theta}$

Extra spread
of the data points

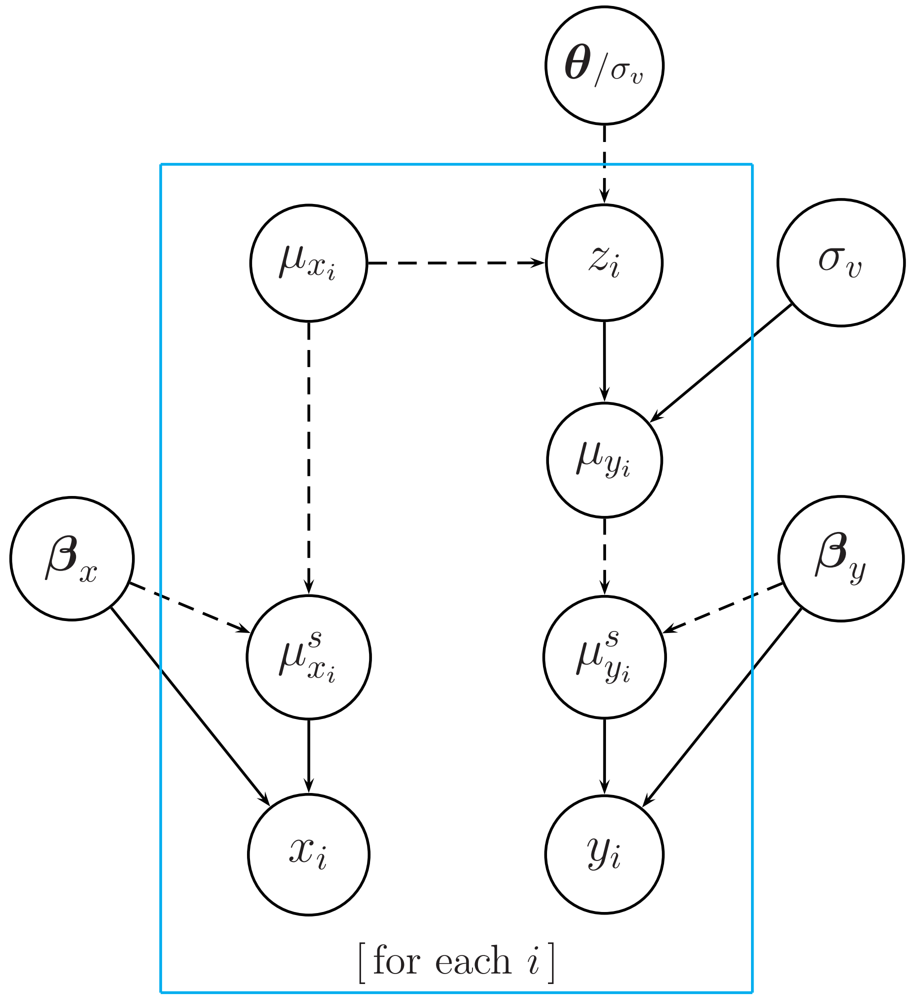
A different way to view fit issues

A physics case (from Gamma ray burts):



(Guidorzi et al., 2006)

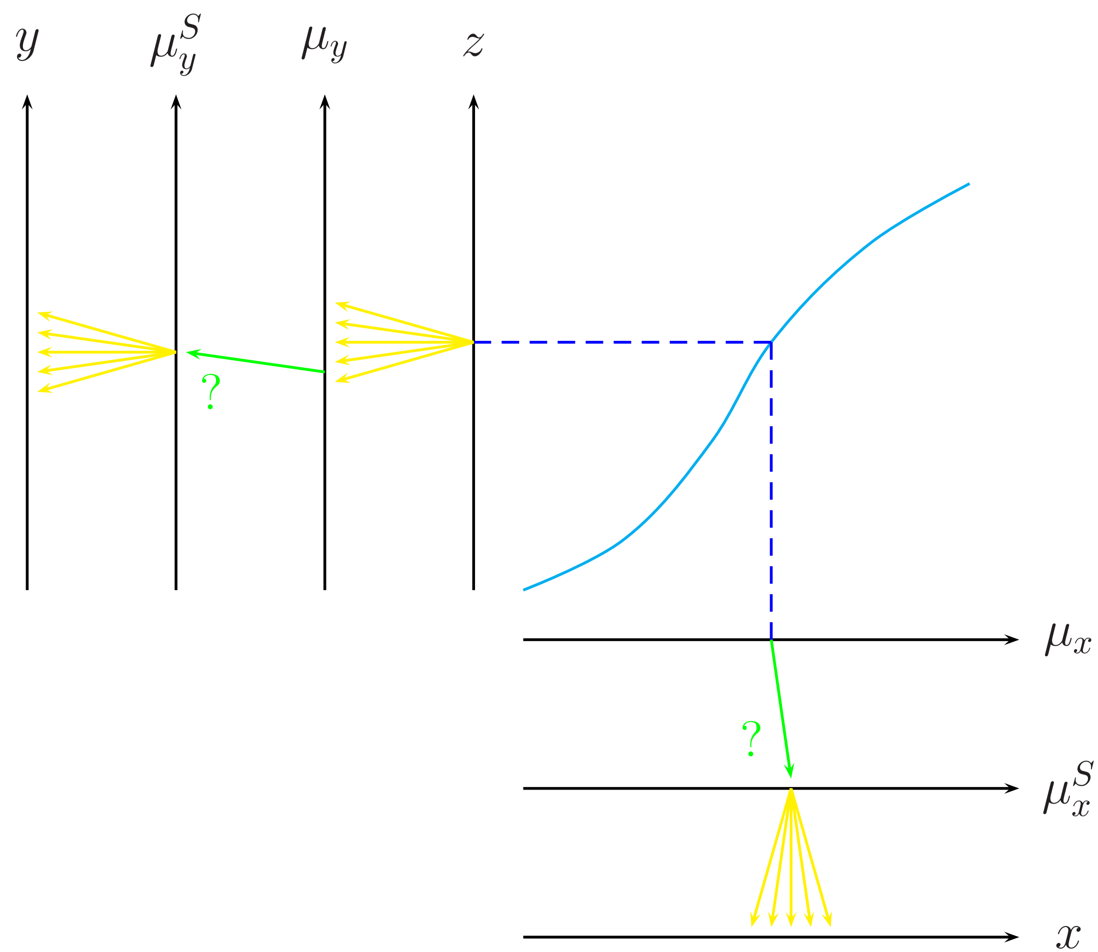
A different way to view fit issues



Adding systematics

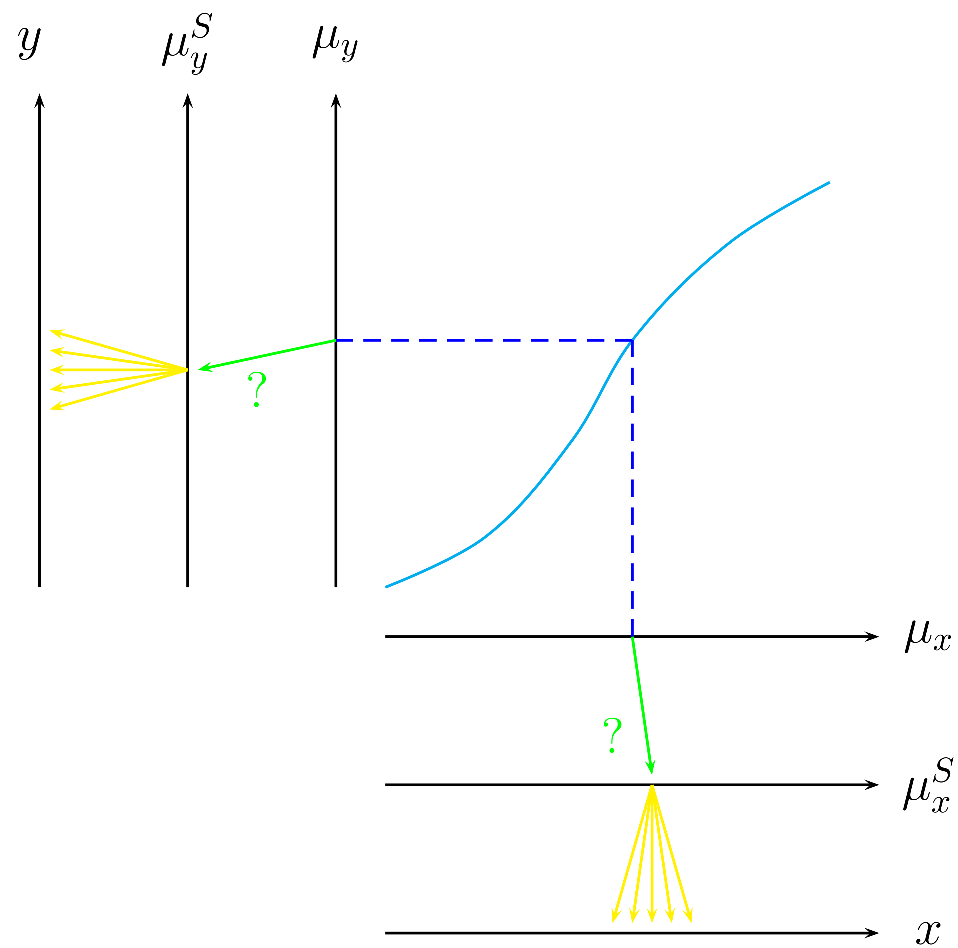
A different way to view fit issues

Stated differently:



A different way to view fit issues

Only systematics (on both axes)



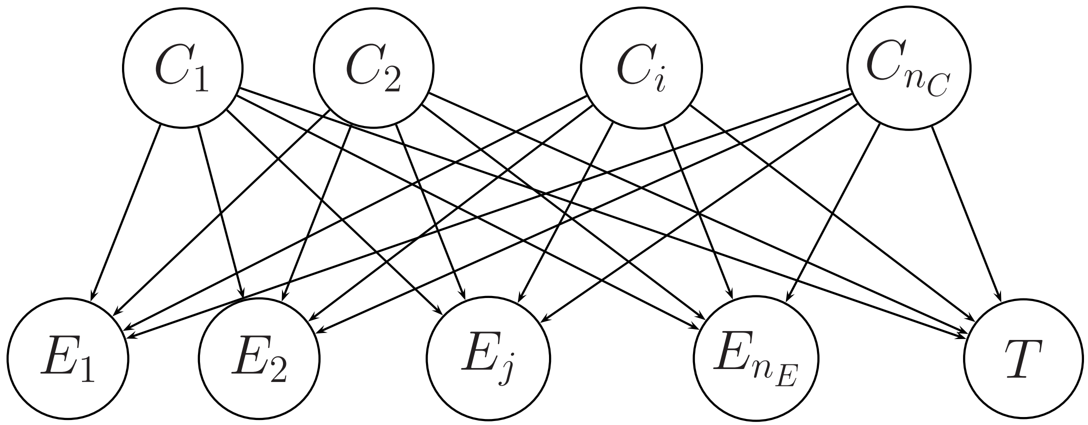
A different way to view fit issues

In this approach systematic effects reflect our uncertainty

$\Rightarrow$ they can be handled rigorously using probability theory!

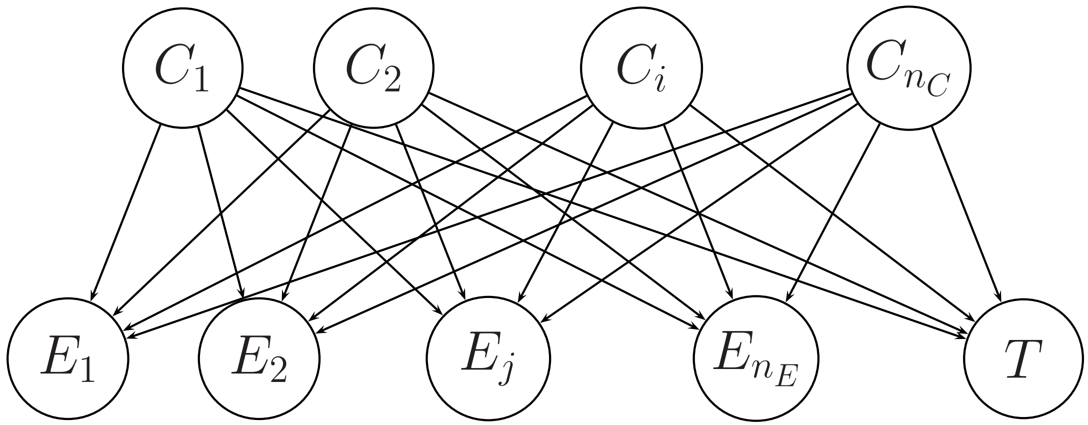
Unfolding a discretized spectrum

Probabilistic links: Cause-bins $\leftrightarrow$ effect-bins

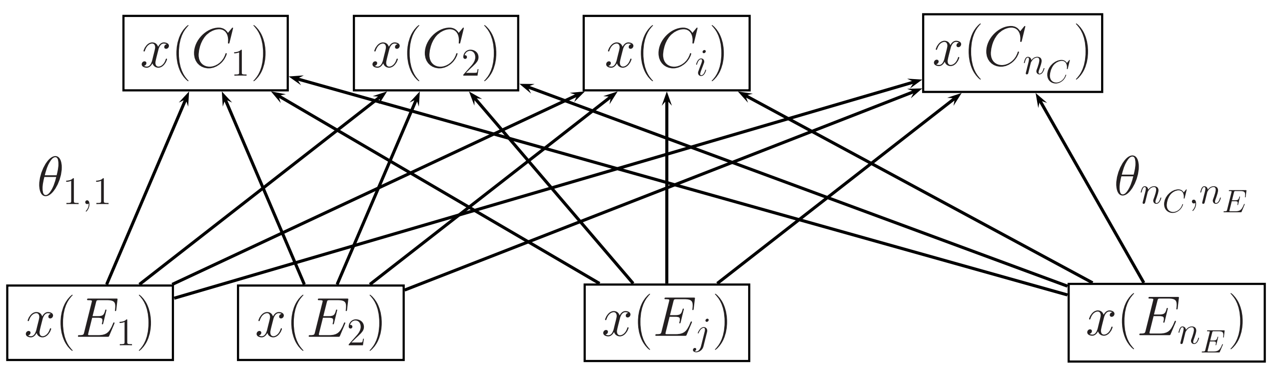


Unfolding a discretized spectrum

Probabilistic links: Cause-bins $\leftrightarrow$ effect-bins

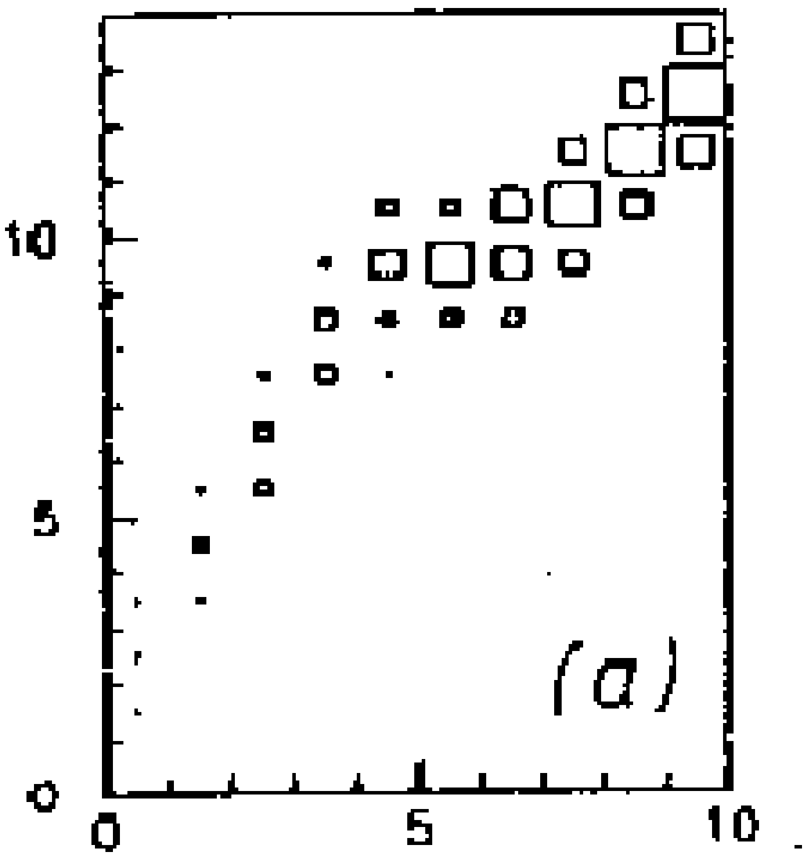


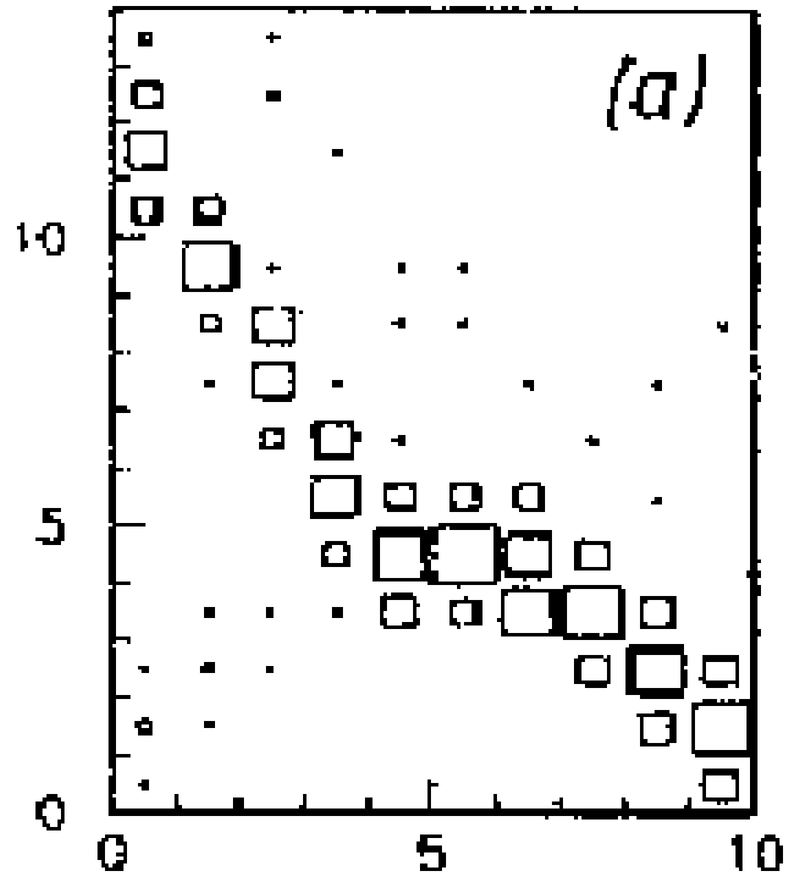
Sharing the observed events among the cause-bins



Unfolding a discretized spectrum

Academic (quite nasty!) smearing matrices:





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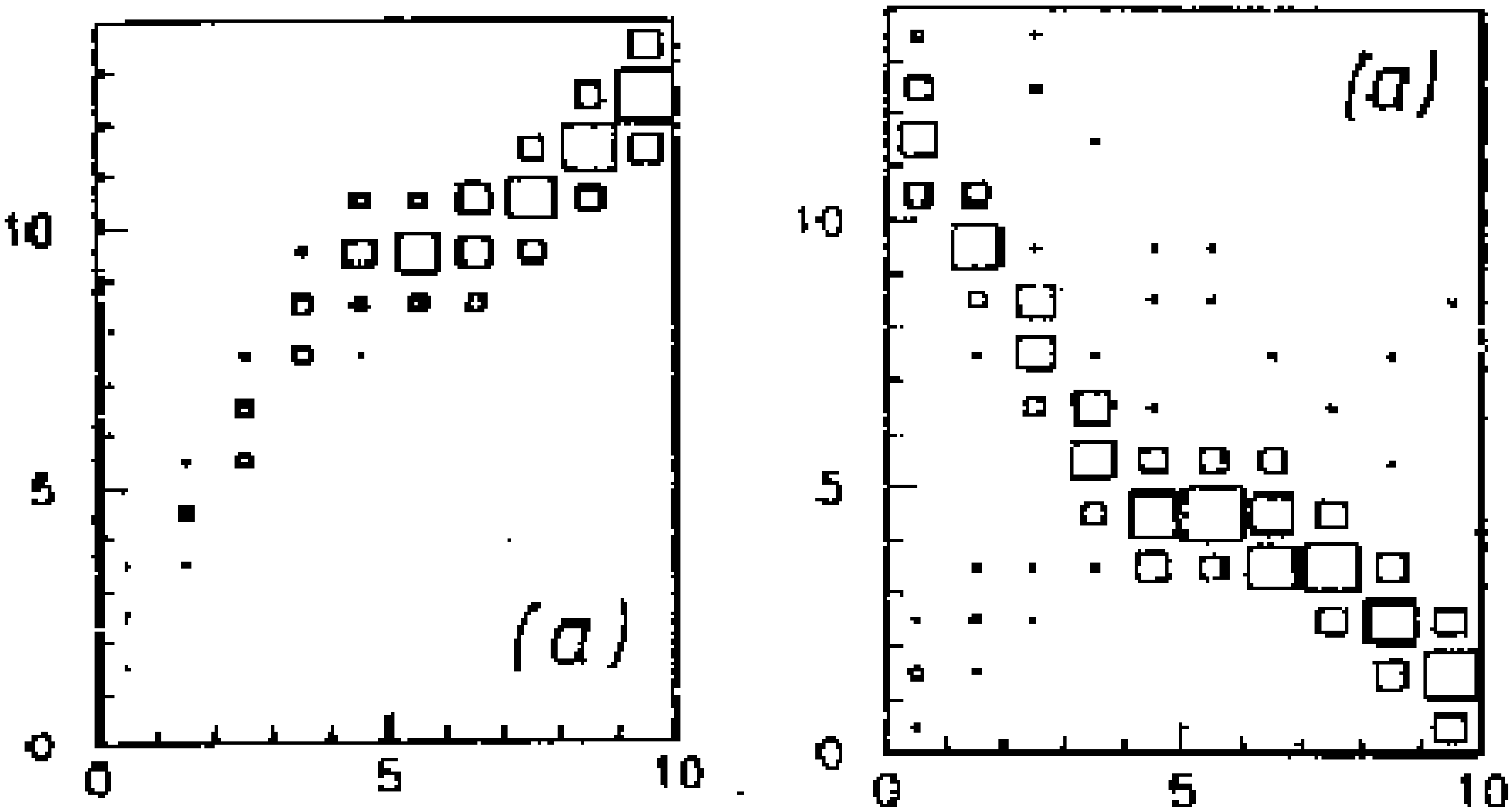
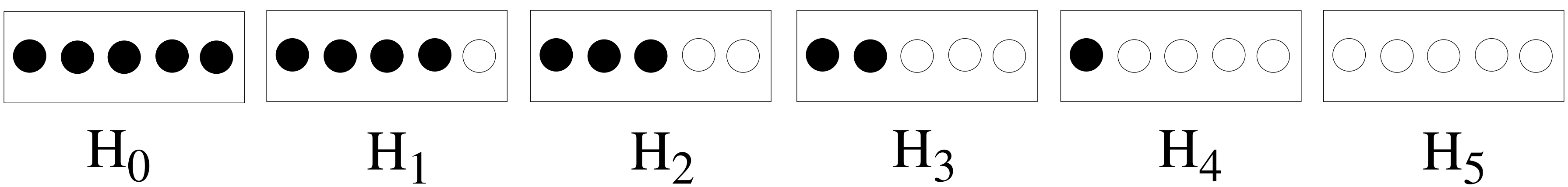


Some demos ⇒

Application to the six box problem



Remind:

- $E_1 =$ White
- $E_2 =$ Black

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)}\, P(H_j \,|\, I)$$

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Our prior belief about H_j

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the ‘response of the apparatus in measurements.
$\rightarrow$ likelihood (traditional, rather confusing name!)

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Probability of E_i taking account all possible H_j
$\rightarrow$ How much we are confident that E_i will occur.

Collecting the pieces of information we need

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$$P(H_j \mid E_i, I) = \frac{P(E_i \mid H_j, I)}{P(E_i \mid I)} P(H_j \mid I)$$

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We can rewrite it as

$P(E_i \mid I) = \sum_j P(E_i \mid H_j, I) \cdot P(H_j \mid I)$

We are ready

Now that we have set up our formalism, let's play a little

- analyse real data
- some simulations

Then

- $H_j \longleftrightarrow j \longleftrightarrow p_j$
- extending p to a continuum:

  $\Rightarrow$ Bayes' billiard

  (prototype for all questions related to efficiencies, branching ratios)
- On the meaning of p

Which box? Which ball?

Inferential/forecasting history:

1. $k = 0$
   $P_0(H_j) = P(H_j \,|\, I_0)$ (priors)
2. `begin loop`:
   $k = k + 1$
   $\Rightarrow E^{(k)}$ (k-th extraction)
3. $P_k(H_j \,|\, I_k) \propto P(E^{(k)} \,|\, H_j) \times P_{k-1}(H_j \,|\, I_k)$

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Let's play!

Bayes' billiard

This is the original problem in the theory of chances solved by Thomas Bayes in late '700:

- imagine you roll a ball at random on a billiard;
- you mark the relative position of the ball along the billiard's length (l/L) and remove the ball
- then you roll at random other balls
  - write down if it stopped left or right of the first ball;
  - remove it and go on with n balls.
- Somebody has to guess the position of the first ball knowing only how mane balls stopped left and how many stoppe right

Bayes' billiard and Bernoulli trials

It is easy to recongnize the analogy:

- Left/Right $\rightarrow$ Success/Failure
- if Left $\leftrightarrow$ Success:
  - $l/L \leftrightarrow p$ of binomial (Bernoulli trials)

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Imagine a sequence $\{S, S, F, S, \ldots\}$ [f_0 is uniform]:

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f(p\,|\,x,n) &\propto p^{x}(1-p)^{(n-x)} \qquad\qquad [x=\#S]
\end{aligned}
$$

Parametric inference

$\rightarrow$ Choose a model and infer its parameter(s).

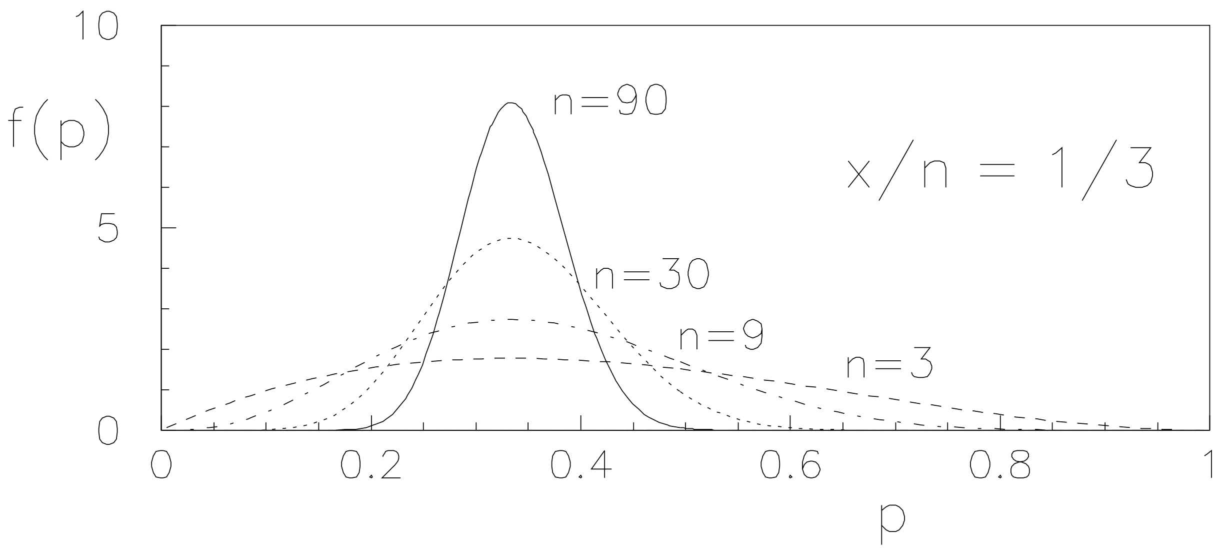
Bayes theorem for continuous variables has following structure

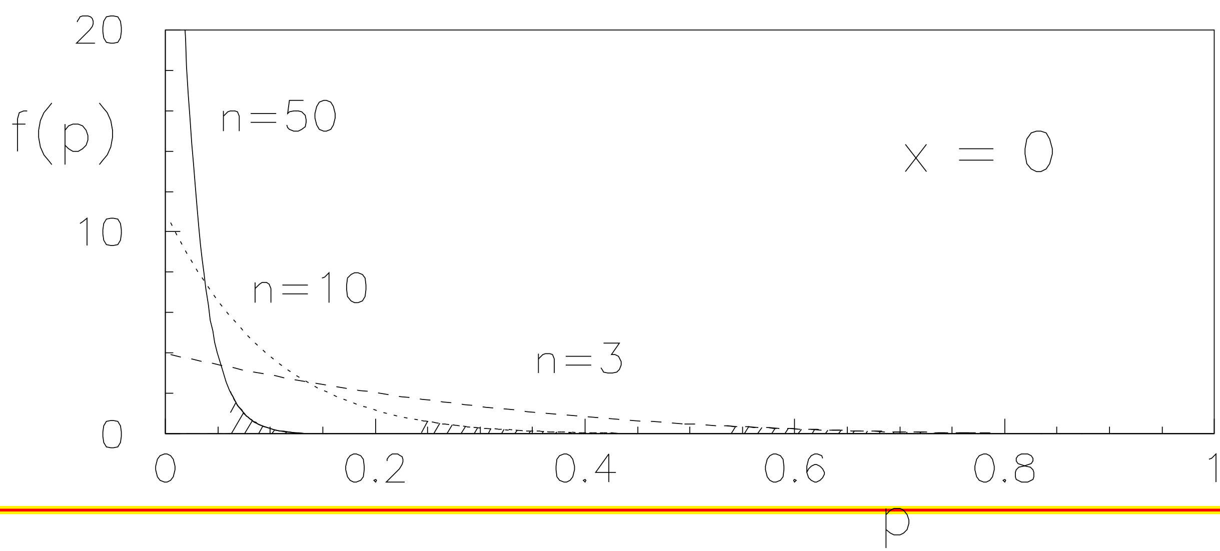
$$f(\theta\,|\,\text{data}) \propto f(\text{data}\,|\,\theta)\, f_0(\theta)$$

$$\begin{aligned}
f(p\,|\,x,n,\mathcal{B}) &= \frac{f(x\,|\,\mathcal{B}_{n,p})\, f_\circ(p)}{\int_0^1 f(x\,|\,\mathcal{B}_{n,p})\, f_\circ(p)\, dp} \\
&= \frac{\frac{n!}{(n-x)!\,x!}\, p^x\,(1-p)^{n-x}\, f_\circ(p)}{\int_0^1 \frac{n!}{(n-x)!\,x!}\, p^x\,(1-p)^{n-x}\, f_\circ(p)\, dp} \\
&= \frac{p^x\,(1-p)^{n-x}}{\int_0^1 p^x\,(1-p)^{n-x}\, dp}\,,
\end{aligned}$$

Inferring the Binomial p

$$f(p\,|\,x,n,\mathcal{B}) = \frac{(n+1)!}{x!\,(n-x)!}\,p^x\,(1-p)^{n-x}\,,$$





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$$f(p \,|\, x, n, \mathcal{B}) = \frac{(n+1)!}{x!\,(n-x)!}\, p^x\, (1-p)^{n-x}\,,$$

$$\begin{aligned} \mathsf{E}(p) &= \frac{x+1}{n+2} \\ \mathsf{Var}(p) &= \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2} \\ &= \mathsf{E}(p)\;(1-\mathsf{E}(p))\;\frac{1}{n+3}. \end{aligned}$$

Laplace's rule of successions

Interpretation of E(p)

Think at any future event $E_{i>n}$
$\Rightarrow$ if we were sure of p, then our confidence on $E_{i>n}$ will be exactly p, i.e.

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$$\begin{aligned} P(E_{i>n}\,|\,x,n,\mathcal{B}) &= \int_0^1 P(E_i\,|\,p)\,f(p\,|\,x,n,\mathcal{B})\,\mathrm{d}p \\ &= \int_0^1 p\,f(p\,|\,x,n,\mathcal{B})\,\mathrm{d}p \\ &= \mathrm{E}(p) \\ &= \frac{x+1}{n+2} \quad \text{(for uniform prior)}\,. \end{aligned}$$

From frequencies to probabilities

$$\mathsf{E}(p) = \frac{x+1}{n+2}$$

Laplace's rule of successions

$$\mathsf{Var}(p) = \mathsf{E}(p)\,(1-\mathsf{E}(p))\,\frac{1}{n+3}.$$

For 'large' n, x and $n-x$: asymptotic behaviors of $f(p)$:

$$\mathsf{E}(p) \approx p_m = \frac{x}{n} \quad [\text{with } p_m \text{ mode of } f(p)]$$

$$\sigma_p \approx \sqrt{\frac{p_m\,(1-p_m)}{n}} \xrightarrow[n\to\infty]{} 0$$

$$p \sim \mathcal{N}(p_m, \sigma_p)\,.$$

Under these conditions the frequentistic "definition" (evaluation rule!) of probability (x/n) is recovered.

Special case with $x = 0$

$$
\begin{aligned}
f(p \,|\, 0, n, \mathcal{B}) &= (n+1)\,(1-p)^n \\
F(p \,|\, 0, n, \mathcal{B}) &= 1 - (1-p)^{n+1} \\
p_m &= 0 \\
\mathrm{E}(p) &= \frac{1}{n+2} \longrightarrow \frac{1}{n} \\
\sigma(p) &= \sqrt{\frac{(n+1)}{(n+3)(n+2)^2}} \longrightarrow \frac{1}{n}
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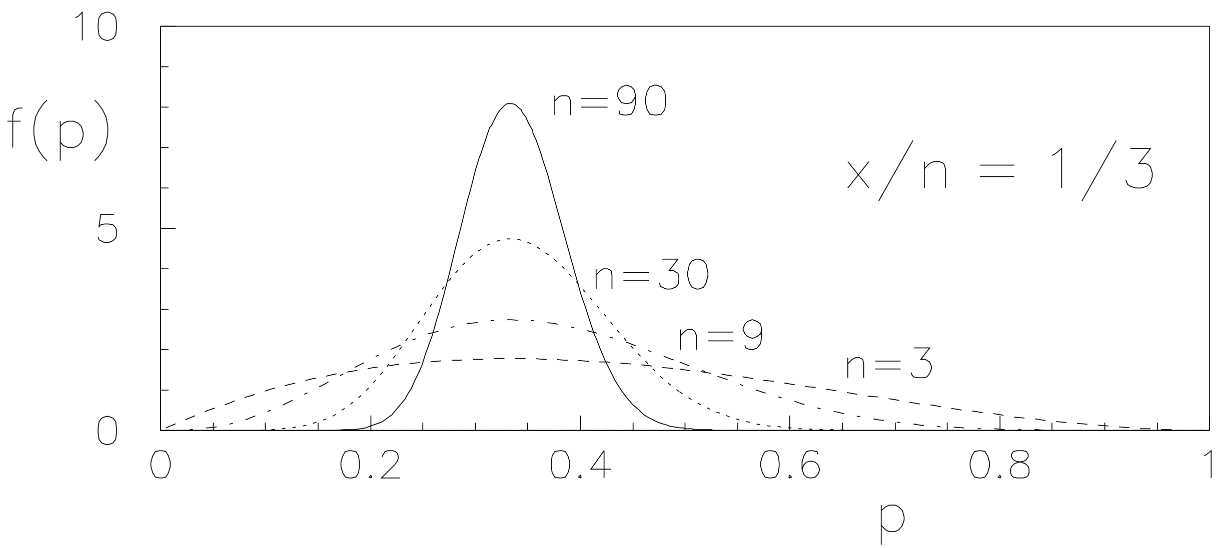
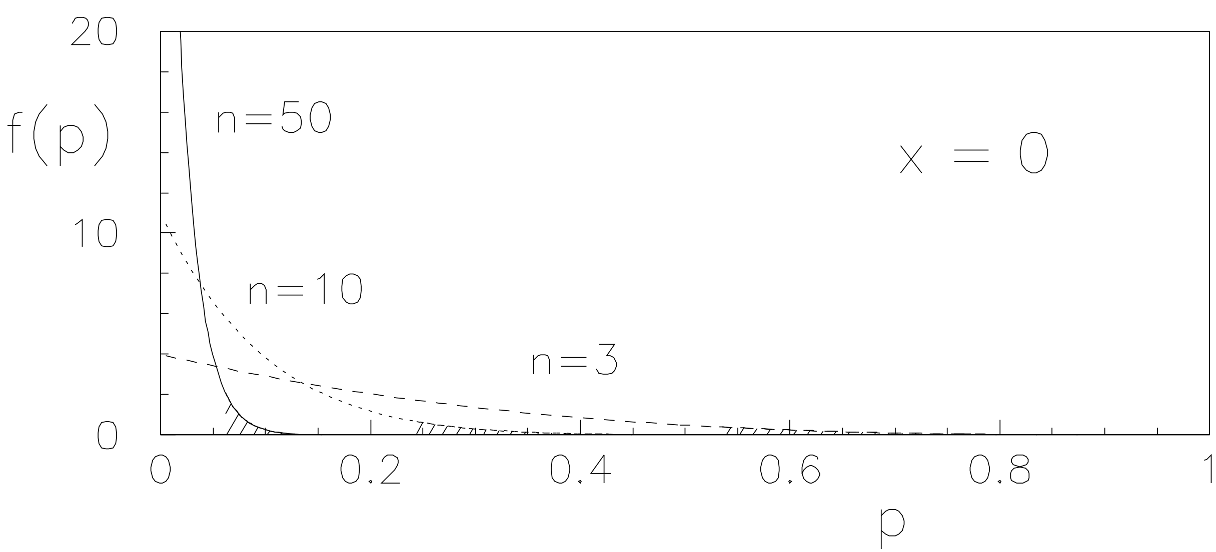
$$P(p \le p_u\,|\,0,n,\mathcal{B}) = 95\%$$
$$\Rightarrow \quad p_u = 1 - \sqrt[n+1]{0.05} \;:$$

Probabilistic upper bound

Special case with $x = 0$





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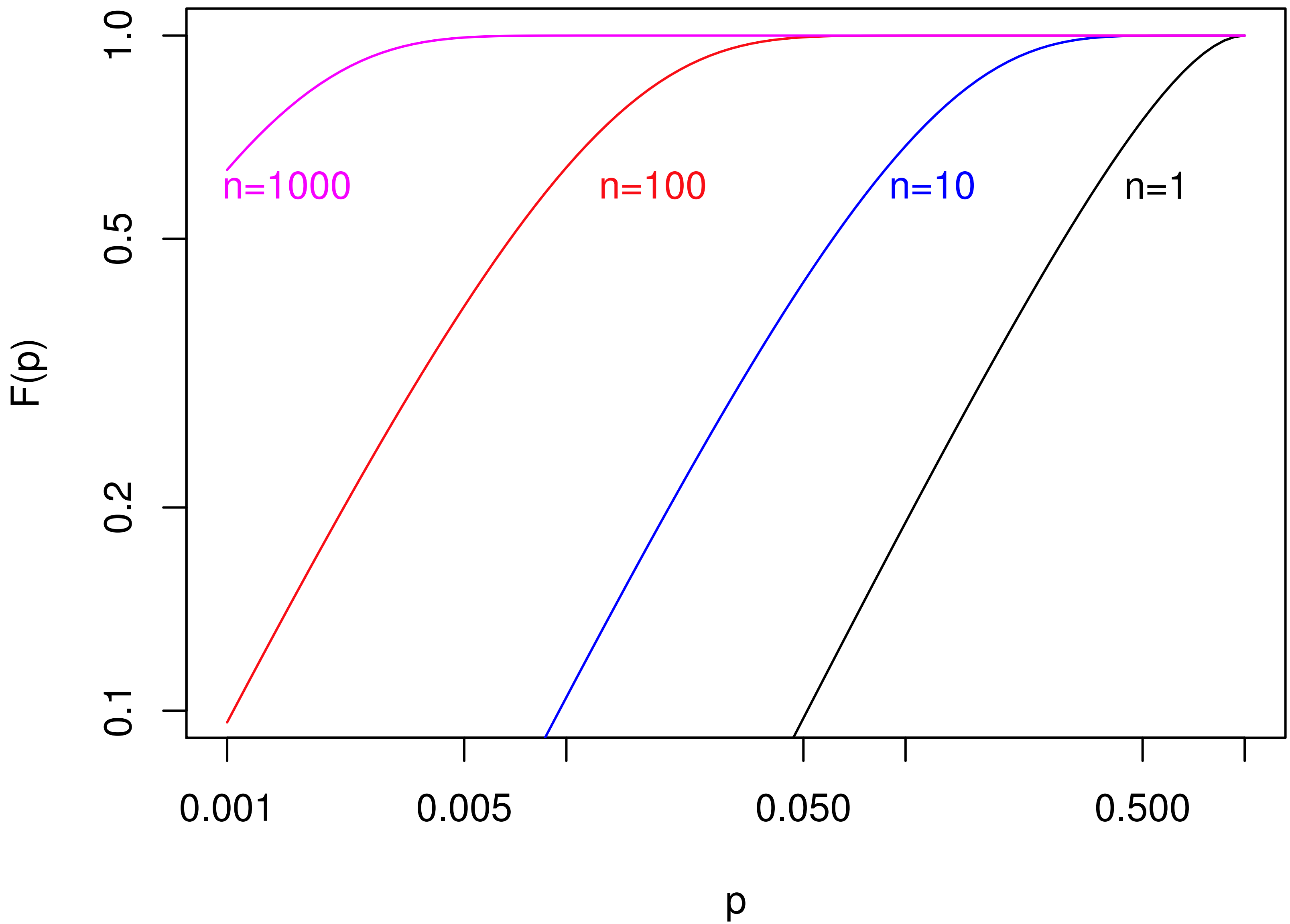

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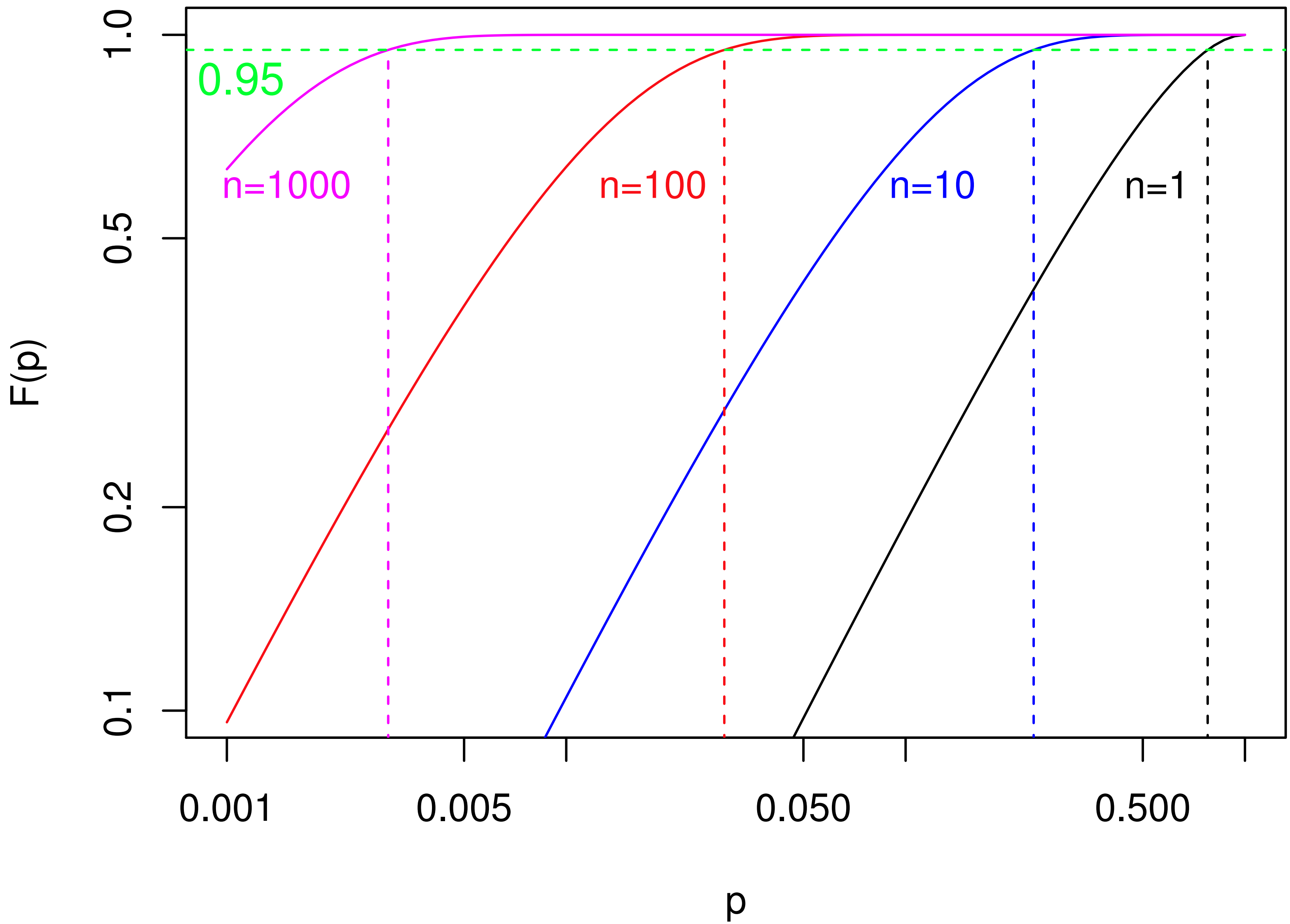

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$\Rightarrow F(p \,|\, x = n, \mathcal{B}) = \int_0^p f(p')dp' = p^{n+1}$ [its cumulative]

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$\Rightarrow F(p \,|\, x = n, \mathcal{B}) = \int_0^p f(p')dp' = p^{n+1}$ [its cumulative]

$\Rightarrow p_m = 1$ [mode of posterior]

$\Rightarrow \mathrm{E}[p] = \int_0^1 p f(p)\, dp = \frac{n+1}{n+2}$ [expected value]

Special case with $x = n$

$$\begin{aligned}\sigma^2(p) &= \mathsf{E}[(p-\mathsf{E}[p])^2] = \mathsf{E}[p^2] - \mathsf{E}^2[p] \\ &= \int_0^1 p^2 f(p)\,dp - \left(\frac{n+1}{n+2}\right)^2 \\ &= \frac{n+1}{n+3} - \frac{(n+1)^2}{(n+2)^2} = \frac{n+1}{(n+3)(n+2)^2} \\ &\rightarrow \frac{1}{n^2}\,.\end{aligned}$$

$\Rightarrow$ Asymptotically ($n \rightarrow \infty$) the variance is the same for the two cases $x = 0$ and $x = n$ (just a question of symmetry)

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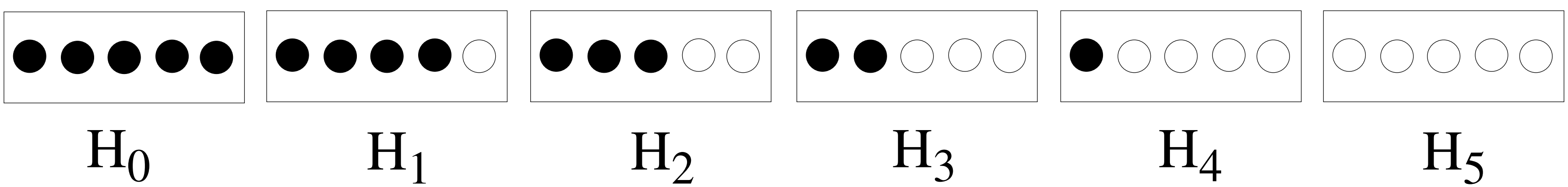
The six box model can help to make the question clear.



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- If, under this assumption, we imagine a great number of trials, we expect a relative frequency of white equal to $P(W\,|\,p)$ [Bernoulli's Theorem]:

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There is no need to adhere to the frequentistic ideology to say this

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Instead, "probability is the limit of frequency for $n \to \infty$" is not more than an empty statement.

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Probability theory (in Laplage's sense) allows to attach probabilities to whatever we feel uncertain about!

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- Other important parameters are related to background, systematics, 'etc.' [arguments not covere here]

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  (Diffidate chi vi promette di far germogliar zecchini nel Campo dei Miracoli! – Collodi docet)

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- It makes little sense to stick to old 'ah hoc' methods that had their *raison d'être* in the computational barrier.
- Mistrust all results that sound as 'confidence', 'probability' etc about physics quantities, if they are obtained by methods that do not contemplate 'beliefs'.

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"The celebrated Monsieur Leibnitz has observed it to be a defect in the common systems of logic, that they are very copious when they explain the operations of the understanding in the forming of demonstrations, but are too concise when they treat of probabilities, and those other measures of evidence on which life and action entirely depend, and which are our guides even in most of our philosophical speculations."

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$\Rightarrow$ still very true after $\approx$ 300 years!

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"The type of critical reasoning which was required for the discovery of this central point was decisively furthered, in my case, especially by the reading of David Hume's and Ernst Mach's philosophical writings."

. . .

[And, in a different writing,]

. . .

"It is to the immortal credit of D. Hume and E. Mach that they, above all others, introduced this critical conception."

(Albert Einstein)