New results on the 3-loop Heavy Flavor Wilson Coefficients in Deep-Inelastic Scattering

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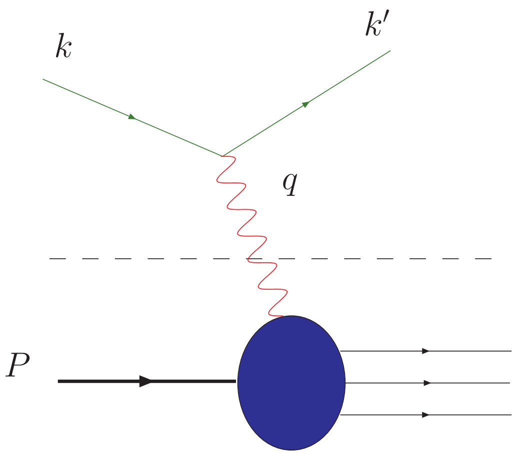
Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow L_{\mu\nu}$

$\longrightarrow W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \ .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.

Factorization of the Structure Functions

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx \, x^{N-1} f(x) \ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1996 Nucl.Phys.B]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j\mid O_i\mid j\rangle\ .$$

$\rightarrow$ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005 Nucl.Phys.B].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Status of OME calculations

Leading Order: [Witten, 1976 Nucl.Phys.B; Babcock, Sivers, 1978 Phys.Rev.D; Shifman, Vainshtein, Zakharov, 1978 Nucl.Phys.B; Leveille, Weiler, 1979 Nucl.Phys.B; Glück, Reya, 1979 Phys.Lett.B; Glück, Hoffmann, Reya, 1982 Z.Phys.C.]

Next-to-Leading Order : [Laenen, van Neerven, Riemersma, Smith, 1993 Nucl. Phys. B] [Large Q^2/m^2: Buza, Matiounine, Smith, Migneron, van Neerven, 1996 Nucl.Phys.B] IBP [Bierenbaum, Blümlein, Klein, 2007 Nucl.Phys.B] via ${}_pF_q$'s, more compact results [Bierenbaum, Blümlein, Klein 2008 Nucl.Phys.B, 2009 Phys.Lett.B]:$O(\alpha_s^2\varepsilon)$ contributions (all N)

NNLO:[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] Moments for F_2: $N = 2...10(14)$ [Blümlein, Klein, Tödtli 2009 Phys. Rev. D] contrib. to transversity: $N = 1...13$

At 3-loop order known:

- $A_{qq,Q}^{\mathrm{PS}}$, $A_{qg,Q}$: complete. [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011 Nucl.Phys.B]
- A_{Qg}, A_{Qq}^{PS}, $A_{qq,Q}^{\mathrm{NS}}$, $A_{gq,Q}$, $A_{gg,Q}$: all terms of $O(n_f T_F^2 C_{A/F})$
- A_{Qq}^{PS}, $A_{qq,Q}^{\mathrm{NS}}$, $A_{Qq}^{\mathrm{NS,trans}}$, $A_{gq,Q}^{\mathrm{NS}}$: all terms of $O(T_F^2 C_{A/F})$
- OMEs with 2 masses: moments 2,4,6, complete, $A_{gg,Q}^{\mathrm{NS}}, O(T_F^2)$, Ladder and Benz topologies with a single massive line: first results this talk.

Calculation Methods

- Generation of diagrams with `QGRAF` [Nogueira 1993 J. Comput. Phys].
- Summation methods based on Zeilberger's algorithm, implemented in the `Mathematica` program `Sigma` [C. Schneider, 2005–].
  - Reduction of the sums to a small number of key sums.
  - Expansion the summands in ε.
  - Simplification by symbolic summation algorithms based on $\Pi\Sigma$-fields [Karr 1981 J. ACM, Schneider 2005–].
  - Harmonic sums are algebraically reduced using shuffle-algebras and structural relations
- In the case of convergent massive 3-loop Feynman integrals, they can be performed in terms of Hyperlogarithms [Generalization of a method by F. Brown, 2008].
- Mellin-Barnes representations.
- Integration by parts identities.
- Fixed Moments: mapping to tadpoles for higher moments `Matad` M. Steinhauser; `qexp` M. Steinhauser, R. Harlander, T. Seidensticker.

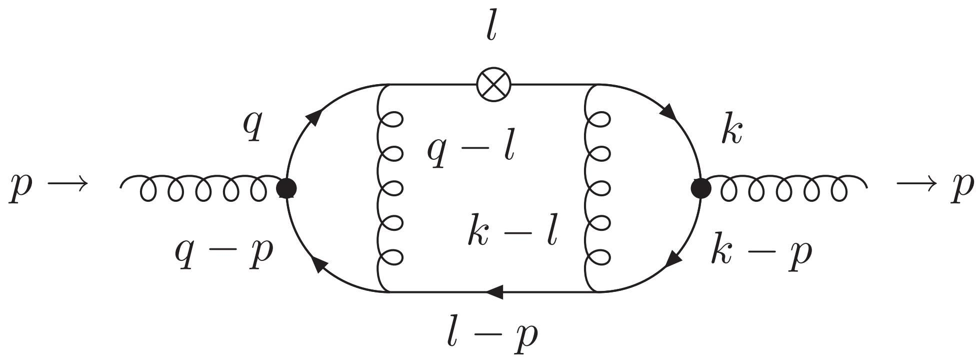
New Functions

- Cyclotomic harmonic sums, polylogarithms and numbers [J. Ablinger, J. Blümlein, C. Schneider, 2011]
- Systematic study of generalized harmonic sums, polylogarithms and numbers [J. Ablinger, J. Blümlein, C. Schneider, 2013]
- Nested generalized cyclotomic harmonic sums weighted by binomials and inverse binomials, polylogarithms and numbers [J. Ablinger, J. Blümlein, C.Raab, C. Schneider, 2013, to appear]
- Computer-algebra implementation in package `HarmonicSums.m`

Ladder Diagrams for Quarkonic OMEs

[Ablinger, Blümlein, Hasselhuhn, Klein, Schneider, Wißbrock; arXiv:1206.2252]

Let's consider the scalar integral with all powers of the propagators equal to one.



After Feynman parameterization, and performing the momentum integrals, we obtain

$$I_{1a} = \frac{i(\Delta.p)^N a_s^3 S_\epsilon^3}{(m^2)^{2-\frac{3}{2}\epsilon}} \hat{I}_{1a} \,,$$

where S_ϵ is the spherical factor $S_\epsilon = \exp\left[\frac{\epsilon}{2}(\gamma_E - \ln(4\pi))\right]$, and

$$\hat{I}_{1a} = -\exp\left(-\frac{3}{2}\epsilon\gamma_E\right)\Gamma(2-3\epsilon/2)\prod_{i=1}^{7}\int_0^1 dw_i \frac{\theta(1-w_1-w_2)w_1^{-\epsilon/2}w_2^{-\epsilon/2}(1-w_1-w_2)}{\left(1+w_1\frac{w_3}{1-w_3}+w_2\frac{w_4}{1-w_4}\right)^{2-3\epsilon/2}}$$
$$\times w_3^{\epsilon/2}(1-w_3)^{-1+\epsilon/2}w_4^{\epsilon/2}(1-w_4)^{-1+\epsilon/2}(1-w_5w_1-w_6w_2-(1-w_1-w_2)w_7)^N$$

We see that by doing a binomial expansion for the polynomial raised to the Nth power (which arises due to the operator insertion), the resulting integrals in w_1 and w_2 can be written in terms of Appell hypergeometric functions:

$$\int_0^1 dw_1 \int_0^1 dw_2 \, \frac{\theta(1-w_1-w_2) w_1^{b-1} w_2^{b'-1}(1-w_1-w_2)^{c-b-b'-1}}{(1-w_1 x - w_2 y)^a}$$
$$= \Gamma\begin{bmatrix} b, b', c-b-b' \\ c \end{bmatrix} F_1\left[a; b, b'; c; x, y\right] .$$

In our case, the parameters x, y correspond to $w_3/(1-w_3)$ and $w_4/(1-w_4)$, respectively. To obtain a series-representation of the integral, we carry out the following analytic continuation:

$$\begin{aligned} F_1\left[a; , b, b'; c; \frac{x}{1-x}, \frac{y}{1-y}\right] &= (1-x)^b (1-y)^{b'} F_1\left[c-a; b, b'; c; x, y\right] \\ &= (1-x)^b (1-y)^{b'} \sum_{m,n}^{\infty} \frac{(c-a)_{m+n} (b)_n (b')_m}{m! n! (c)_{m+n}} x^m y^n \end{aligned}$$

Applying this to our integral $\hat{I}_{1a}$, we obtain

$$\begin{aligned}
\hat{I}_{1a} \quad = \quad & \frac{\exp\left(-\frac{3}{2}\epsilon\gamma_E\right)\Gamma(2-3\epsilon/2)}{(N+1)(N+2)(N+3)} \sum_{m,n=0}^{\infty} \Bigg\{ \\
& \sum_{t=1}^{N+2} \binom{N+3}{t} \frac{(t-\epsilon/2)_m (N+2+\epsilon/2)_{m+n} (N+3-t-\epsilon)_n}{(N+4-\epsilon)_{m+n}} \\
& \times \Gamma\left[\begin{matrix} t, t-\epsilon/2, m+1+\epsilon/2, n+1+\epsilon/2, N+3-t, N+3-t-\epsilon/2 \\ N+4-\epsilon, m+1, n+1, m+t+1+\epsilon/2, N+n-t+4+\epsilon/2 \end{matrix}\right] \\
& - \sum_{s=1}^{N+3} \sum_{r=1}^{s-1} \binom{s}{r} \binom{N+3}{s} (-1)^s \frac{(r-\epsilon/2)_m (s-1+\epsilon/2)_{m+n} (s-r-\epsilon/2)_n}{(s+1-\epsilon)_{m+n}} \\
& \times \Gamma\left[\begin{matrix} r, r-\epsilon/2, s-r, m+1+\epsilon/2, n+1+\epsilon/2, s-r-\epsilon/2 \\ m+1, n+1, m+r+1+\epsilon/2, s-r+n+1+\epsilon/2, s+1-\epsilon \end{matrix}\right] \Bigg\}
\end{aligned}$$

After expanding in ϵ, the summation can be performed using `Sigma`.

The result for this and other integrals can be written in terms of harmonic sums $S_{\vec{a}}$ and their generalizations $S_{\vec{a}}(\vec{\xi};N)$:

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{\mathrm{sign}(b)^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_{\emptyset}(k) = 1$$

$$S_{b,\vec{a}}(\eta,\vec{\xi};N) = \sum_{k=1}^{N} \frac{\eta^k}{k^b} S_{\vec{a}}(\vec{\xi};k), \quad S_{\emptyset} = 1, \quad \eta,\xi \in \Re$$

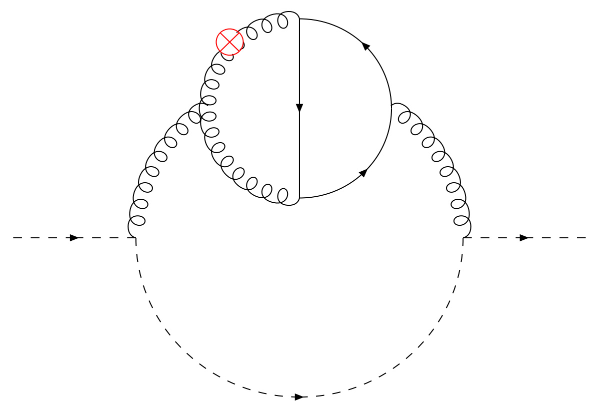
The threefold and fourfold sums in $\hat{I}_{1a}$ give

$$\begin{aligned}
\hat{I}_{1a} \quad = \quad & -\frac{4(N+1)S_1+4}{(N+1)^2(N+2)}\zeta_3 + \frac{2S_{2,1,1}}{(N+2)(N+3)}\frac{1}{(N+1)(N+2)(N+3)}\Bigg\{ \\
& -2(3N+5)S_{3,1} - \frac{S_1^4}{4} + \frac{4(N+1)S_1-4N}{N+1}S_{2,1} + 2\left[(2N+3)S_1 + \frac{5N+6}{N+1}\right]S_3 \\
& +\frac{9+4N}{4}S_2^2 + \left[2\frac{7N+11}{(N+1)(N+2)} + \frac{5N}{N+1}S_1 - \frac{5}{2}S_1^2\right]S_2 + \frac{2(3N+5)S_1^2}{(N+1)(N+2)} \\
& +\frac{N}{N+1}S_1^3 + \frac{4(2N+3)}{(N+1)^2(N+2)}S_1 - \frac{1}{2}(2N+3)S_4 + 8\frac{2N+3}{(N+1)^3(N+2)}\Bigg\}
\end{aligned}$$

This result was checked using `MATAD` for the fixed moments $N = 1\ldots10$.

Mellin-Barnes representations

Consider the Feynman diagram shown below:



The dashed line represents a massless quark and the solid line a massive one. Consider now the scalar integral associated to this diagram with all powers of propagators equal to one:

$$I = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{d^D k_3}{(2\pi)^D} \frac{(\Delta \cdot k_2)^N}{D_1 D_2 D_3 D_4 D_5 D_6 D_7}$$

where

$$D_1 = k_1^2, \quad D_2 = (k_1 - p)^2, \quad D_3 = k_2^2, \quad D_4 = (k_1 - k_2)^2, D_5 = k_3^2 - m^2,$$
$$D_6 = (k_1 - k_3)^2 - m^2, \quad D_7 = (k_2 - k_3)^2 - m^2$$

This integral has poles in ϵ, so Brown's algorithm, in its present form, cannot be applied here. Besides, direct Feynman parameterization of this integral doesn't lead to an expression that can be easily identified with hypergeometric functions:

$$I = i\Gamma\left(1-\frac{3}{2}\epsilon\right)\int_0^1 dx_1\cdots\int_0^1 dx_6\, x_2^{1-\epsilon}(1-x_2)^{-\epsilon}x_4^{-1+\epsilon}(1-x_4)x_5^N x_6^{-1+\epsilon/2}(1-x_6)^{N+1} \times (x_1x_4+x_3(1-x_4))^N\, Q^{-\epsilon/2}$$

where

$$Q = \frac{x_1x_4(1-x_1x_2)}{1-x_2} + x_3(1-x_4) - (x_1x_4+x_3(1-x_4))^2$$

For this reason we use a Mellin-Barnes representation:

$$I = \frac{i}{(2\pi i)^2}\int_{-i\infty}^{i\infty} dz_1 \int_{-i\infty}^{i\infty} dz_2\, \frac{\Gamma(1-3\epsilon/2)\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z1-z2-\epsilon/2)}{\Gamma(3-2\epsilon)\Gamma(-z_1+1)\Gamma(-z_2+1)\Gamma(z_1+z_2+N+2+\epsilon)} \times \Gamma(-z_2+1-\epsilon)\Gamma(-z_1+1+\epsilon/2)\Gamma(z_1+z_2+1) \times \Gamma(z_1+1+\epsilon/2)\Gamma(z_1+z_2+\epsilon)\frac{\Gamma(N+1)}{\Gamma(N+2+\epsilon/2)}$$

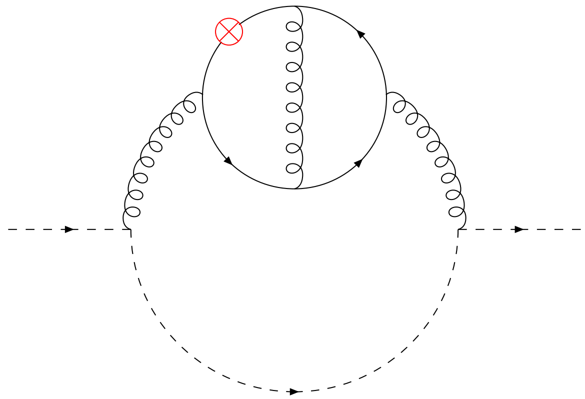
- We use the Mathematica MB.m package by M. Czakon [Comput.Phys.Commun. 175 (2006)] together with the MBresolve.m addition of V. Smirnov et. al. [Eur.Phys.J. C62 (2009)], to resolve the singularities in $\epsilon = D - 4$ for this expression, after which we can expand in ϵ.
- The resulting integrals can all be done using Barnes lemmas, after suitable manipulations. This step is done with the help of barnesroutines.m by D. Kosower [http://mbtools.hepforge.org].

The final result is

$$
\begin{aligned}
I \quad = \quad & \frac{i}{(N+1)^2}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(3 - S_1(N) - \frac{3}{2(N+1)}\right) + 7 - \frac{9}{2(N+1)} - \frac{1}{4(N+1)^2}\right. \\
& \left. + \frac{3}{8}\zeta(2) + \left(\frac{1}{2(N+1)} - 3\right) S_1(N) + \frac{1}{2}S_1(N)^2 - \frac{1}{2}S_2(N)\right]
\end{aligned}
$$

Other integrals associated to the same diagram, but with different operator insertions can be done in a similar way.

We have also used Mellin-Barnes representations for integrals with more than three massive propagators. For example,



In general, these kind of integrals cannot be done solely by means of Barnes lemmas, so in these cases, at some point we take residues, generating sums that are then performed using the package SIGMA. For the diagram above, with all powers of propagators equal to one, we get

$$
\begin{aligned}
I = & \frac{-i}{(2\pi i)^2} \int_{-i\infty}^{i\infty} dz_1 \int_{-i\infty}^{i\infty} dz_2 \int_{-i\infty}^{i\infty} dz_3 \, \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_3+1-3\epsilon/2)}{\Gamma(1+\epsilon)\Gamma(1-z_1)\Gamma(1-z_2)\Gamma(-2z_3+z_1+z_2+N+2+\epsilon)} \\
& \times \frac{\Gamma(z_3-z_2+1-\epsilon)\Gamma(z_3-z_1+1-\epsilon)\Gamma(-z_1-z_2+\epsilon)\Gamma(z_1+z_2+1)\Gamma(N+1)}{\Gamma(2z_3-z_1-z_2-2\epsilon)\Gamma(N+2+\epsilon/2)} \\
& \times \Gamma(-z_3+z_2+N+1+\epsilon/2)\Gamma(-z_3+z_1+1+\epsilon/2)\Gamma(-z_3+z_1+z_2+\epsilon)\Gamma(z_3-z_1-z_2-\epsilon/2)
\end{aligned}
$$

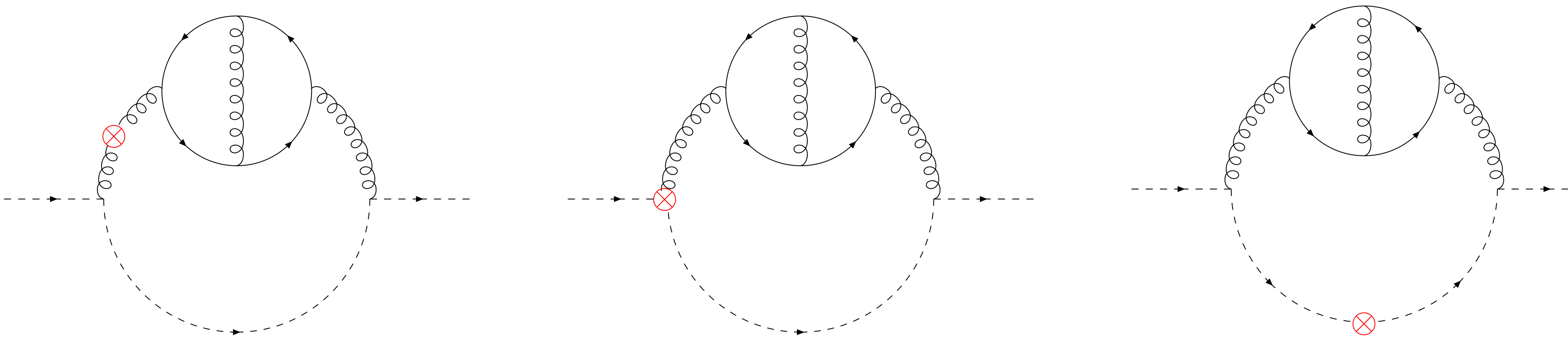
Summing residues is, however, not always straightforward. In the example we are considering, at some point we get the following integral:

$$\int_{-i\infty}^{i\infty} dz_2 \int_{-i\infty}^{i\infty} dz_3 \sum_{k=1}^{\infty} \frac{\Gamma(-z_2)\Gamma(-z_3)\Gamma(1+z_3)^2\Gamma(1+k-z_2)\Gamma(1-k+z_2)\Gamma(-k+z_2-z_3)}{\Gamma(1-z_2)\Gamma(1-k+z_2-z_3)\Gamma(2-k+z_3)}$$
$$\times \Gamma(1-z_2+z_3)\frac{\Gamma(z_2-z_3+N+1)}{\Gamma(-z_3+2+k+N)}$$

It is clear from the last term that if we take residues in z_3, we will end up we a factor of the form $\Gamma(z_2+N+\cdots)$ in the numerator of the integrand, without a corresponding similar "balancing" term in the denominator. After taking residues in z_2, we end up with a non-convergent sum.

In this particular and other similar cases, the problem can be solved by using known gamma function identities to move around the dependence in N, so that we end up with convergent sums.

Fortunately, for some of the other operator insertions with the same topology as the previous example, such as,



the N dependence factorizes. E.g., the 1[rst] diagram with powers of propagators equal to 1 gives,

$$
\begin{aligned}
I \quad = \quad & \frac{-i}{(2\pi i)^2} \frac{\Gamma(N+1)}{\Gamma(N+2+\epsilon/2)} \int_{-i\infty}^{i\infty} dz_1 \int_{-i\infty}^{i\infty} dz_2 \int_{-i\infty}^{i\infty} dz_3 \, \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_3+1-3\epsilon/2)}{\Gamma(1+\epsilon)\Gamma(1-z_1)\Gamma(1-z_2)} \\
& \times \frac{\Gamma(z_3-z_2+1-\epsilon)\Gamma(z_3-z_1+1-\epsilon)\Gamma(-z_1-z_2+\epsilon)\Gamma(z_1+z_2+1)}{\Gamma(2z_3-z_1-z_2-2\epsilon)\Gamma(-2z_3+z_1+z_2+2+\epsilon)} \\
& \times \Gamma(-z_3+z_2+1+\epsilon/2)\Gamma(-z_3+z_1+1+\epsilon/2)\Gamma(-z_3+z_1+z_2+\epsilon)\Gamma(z_3-z_1-z_2-\epsilon/2)
\end{aligned}
$$

The Mellin-Barnes integral is just a number that can be obtained with MATAD.

m_c and m_b Corrections

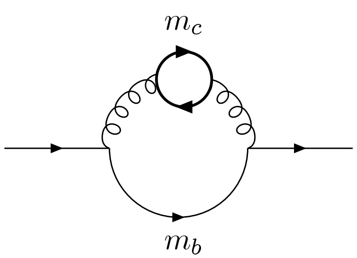
- There are 7 different OMES:

$$\underbrace{A_{Qg},\ A^{\rm NS}_{qq,Q},\ A^{\rm PS}_{Qq},\ A_{gg,Q},\ A_{gq,Q}}_{\text{Contain contributions with b- and c-quarks}}\ \underbrace{A_{qg,Q},\ A^{\rm PS}_{qq,Q}}_{\text{Completely known}}$$

OME	# diagrams
$A_{Qg}^{(3)}$	272
$A_{Qq}^{(3),\rm PS}$	16
$A_{qq,Q}^{(3),\rm NS}$	4
$A_{qq,Q}^{(3),\rm NS,trans}$	4
$A_{gq,Q}^{(3)}$	4
$A_{gg}^{(3)}$	184
Σ	484

The Renormalization proceeds in the following steps:

1. include contributions from reducible diagrams for $A_{gg,Q}$ and A_{Qg}
   - Also at 2-loop-level these OMES receive contributions depending on both masses through reducible diagrams
2. perform on-shell mass renormalization
   - m_c contributes to the renormalization of m_b and v.v.
   - The mass renormalization constant Z_m has been determined for the case of 2 different masses to $O\left(\alpha_s^2\right)$in Broadhurst, Gray, Schilcher,1991.



3. renormalize the coupling in a MOM-scheme, using the background field method
   - New contributions to $Z_g^{\rm MOM}$ in the cases of 2 different masses.
4. remove remaining UV singularities through the Z-factors of the local operators
5. remove collinear singularities via coll. factorization

6. transform coupling constant to $\overline{\mathsf{MS}}$
   - Also scheme transformations differ in the case of 2 masses.
7. choice: m on-shell or $m_{\overline{\mathsf{MS}}}$

Pole structure obtained from renormalization:

General pole structure:

$$A_{ij}^{(3)} = \frac{A_{ij}^{(3),-3}}{\varepsilon^3} + \frac{A_{ij}^{(3),-2}}{\varepsilon^2} + \frac{A_{ij}^{(3),-1}}{\varepsilon} + a_{ij}^{(3)}$$

Different contributions:

- coefficients from the on-shell mass renormalization constant: $\delta m_1^0, \cdots$
- coefficients of the QCD-β-function: $\beta_{0,Q}, \beta_1(nf), \cdots$
- constants parts of $2-loop$ OMEs: $a_{Qg}^{(2)}, \cdots$
- anomalous dimensions: $\gamma_{qq}^{(0)}, \cdots$

- At 3–loop order graphs containing both c- and b- quarks contribute.
- They do neither belong to the pure $c-$ or $b-$ contribution to the structure function.
- Note that:

$$\frac{m_c}{m_b} \simeq \frac{1.3GeV}{4.2GeV} \quad \longrightarrow \quad x^3 := \left(\frac{m_c}{m_b}\right)^6 \simeq 0.0001$$

$\rightarrow$ Expand in m_c/m_b

- for fixed values of N the diagrams can be mapped onto tadpole diagrams by projection operators [Bierenbaum, Blümlein, Klein 2009.]
- e.g. $N = 2$

$$\Pi_{\mu\nu} \quad = \quad \frac{1}{d-1}\left(\frac{-g_{\mu\nu}}{p^2} + d\,\frac{p(\mu)p(\nu)}{p^4}\right)$$

- more complex structures occur for higher Moments
- expansion in masses was performed using EXP [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]
- Computation time: ~ 1 CPU-year

$$
\begin{aligned}
a_{Qg}^{(3)}(N=6) \quad = \quad & T_F^2 C_A \Biggl\{ \frac{69882273800453}{367569090000} - \frac{395296}{19845}\zeta_3 + \frac{1316809}{39690}\zeta_2 + \frac{832369820129}{14586075000}x + \frac{1511074426112}{624023544375}x^2 - \frac{84840004938801319}{690973782403905000}x^3 \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{11771644229}{194481000} + \frac{78496}{2205}\zeta_2 - \frac{1406143531}{69457500}x - \frac{105157957}{180093375}x^2 + \frac{2287164970759}{7669816654500}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\Bigr] + \ln^3\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\frac{324148}{19845} + \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{156992}{6615} \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{128234}{3969} - \frac{112669}{330750}x + \frac{98746}{51975}x^2 + \frac{31340489}{17027010}x^3\Bigr] + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{68332}{6615} \\
& + \ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{83755534727}{583443000} + \frac{78496}{2205}\zeta_2 + \frac{1406143531}{69457500}x + \frac{105157957}{180093375}x^2 - \frac{2287164970759}{7669816654500}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{2668087}{79380} + \frac{112669}{661500}x - \frac{49373}{51975}x^2 - \frac{31340489}{34054020}x^3\Bigr] + \ln^3\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{412808}{19845}\Biggr\} \\
& + T_F^2 C_F \Biggl\{ - \frac{3161811182177}{71471767500} + \frac{447392}{19845}\zeta_3 + \frac{9568018}{4862025}\zeta_2 - \frac{64855635472}{2552563125}x + \frac{1048702178522}{97070329125}x^2 + \frac{1980566069882672}{2467763508585375}x^3 \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{1786067629}{204205050} - \frac{111848}{15435}\zeta_2 - \frac{128543024}{24310125}x - \frac{22957168}{3361743}x^2 - \frac{2511536080}{2191376187}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\Bigl[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\Bigr] - \ln^3\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\frac{111848}{19845} - \ln^2\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{223696}{46305} \\
& + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{22238456}{4862025} - \frac{1504864}{231525}x - \frac{355888}{40425}x^2 - \frac{255717856}{42567525}x^3\Bigr] + \ln\Bigl(\frac{m_2^2}{\mu^2}\Bigr)\ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{223696}{46305} \\
& + \ln\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[-\frac{24797875607}{1021025250} - \frac{111848}{15435}\zeta_2 + \frac{128543024}{24310125}x + \frac{22957168}{3361743}x^2 + \frac{2511536080}{2191376187}x^3\Bigr] \\
& + \ln^2\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\Bigl[\frac{3232799}{4862025} + \frac{752432}{231525}x + \frac{177944}{40425}x^2 + \frac{127858928}{42567525}x^3\Bigr] - \ln^3\Bigl(\frac{m_1^2}{\mu^2}\Bigr)\frac{1230328}{138915}\Biggr\} + O\left(x^4 \ln^3(x)\right)
\end{aligned}
$$

Calculation of Convergent Massive 3-Loop Graphs

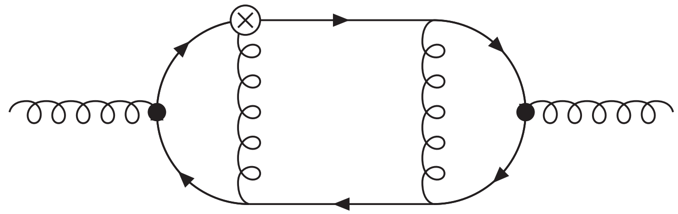
Many of the Feynman integrals appearing in the calculation of the massive 3-loop operator matrix elements are finite.

We have generalized a method originally proposed by F. Brown [Comm. Math. Phys. 2008] to the case where we have masses and operator insertions in order to find general N representations for all convergent 3-loop topologies.

Here we work in the α-representation to calculate the integrals.

The corresponding graph polynomials of a graph G are given by

- $U = \sum_T \prod_{l \notin T} \alpha_l$, where T denotes the spanning trees of G
- $V = \sum_{l \in massive} \alpha_l$
- Dodgson polynomials arise from the operator insertions.



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$$
\begin{aligned}
I_4(N) &= \int \cdots \int d\alpha_1\ d\alpha_2\ d\alpha_3\ d\alpha_4\ d\alpha_5\ d\alpha_6\ d\alpha_7\ d\alpha_8\ \frac{\sum_{j=0}^{N} T_{4\alpha}^{N-j} T_{4b}^{j}}{U^2 V^2} \\
T_{4\alpha} &= \alpha_5\alpha_7\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_2\alpha_5\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_5\alpha_7\alpha_8 + \alpha_2\alpha_3\alpha_8 \\
&\quad +\alpha_7\alpha_2\alpha_8 + \alpha_6\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_2\alpha_3\alpha_6 + \alpha_4\alpha_2\alpha_8 + \alpha_2\alpha_6\alpha_4 + \alpha_4\alpha_7\alpha_2 \\
T_{4b} &= +\alpha_2\alpha_5\alpha_4 + \alpha_4\alpha_2\alpha_8 + \alpha_4\alpha_7\alpha_2 + \alpha_2\alpha_5\alpha_8 + \alpha_2\alpha_3\alpha_5 + \alpha_7\alpha_2\alpha_8 + \alpha_3\alpha_7\alpha_2 + \alpha_8\alpha_5\alpha_4 \\
&\quad +\alpha_5\alpha_7\alpha_4 + \alpha_4\alpha_1\alpha_8 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_1\alpha_7 + \alpha_1\alpha_3\alpha_7 \\
U &= \alpha_2\alpha_5\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_3\alpha_5 + \alpha_5\alpha_7\alpha_4 + \alpha_1\alpha_6\alpha_4 + \alpha_1\alpha_3\alpha_6 + \alpha_2\alpha_3\alpha_6 + \alpha_2\alpha_6\alpha_4 \\
&\quad +\alpha_5\alpha_6\alpha_4 + \alpha_1\alpha_5\alpha_4 + \alpha_3\alpha_5\alpha_7 + \alpha_1\alpha_3\alpha_7 + \alpha_1\alpha_7\alpha_4 + \alpha_3\alpha_7\alpha_2 + \alpha_4\alpha_7\alpha_2 + \alpha_3\alpha_5\alpha_6 \\
&\quad +\alpha_2\alpha_3\alpha_8 + \alpha_2\alpha_5\alpha_8 + \alpha_5\alpha_7\alpha_8 + \alpha_8\alpha_5\alpha_4 + \alpha_8\alpha_5\alpha_6 + \alpha_5\alpha_3\alpha_8 + \alpha_1\alpha_8\alpha_5 + \alpha_1\alpha_8\alpha_6 \\
&\quad +\alpha_6\alpha_2\alpha_8 + \alpha_1\alpha_8\alpha_3 + \alpha_4\alpha_1\alpha_8 + \alpha_4\alpha_2\alpha_8 + \alpha_7\alpha_2\alpha_8 + \alpha_8\alpha_1\alpha_7 \\
V &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6 + \alpha_7
\end{aligned}
$$

- The integral above is a projective integral, one α-parameter may be set 1
- The operators sit on on-shell diagrams which obey specific symmetries. These are generally not obeyed by the operator insertion.
- For the above example : after applying symmetry transformations $\alpha_1 \to x_1 - \alpha_2$, $\alpha_3 \to x_2 - \alpha_4$, $\alpha_5 \to x_5 - \alpha_6$ $\alpha_2, \alpha_4, \alpha_6$ are only contained in the operator polynomials and may be integrated out at this stage.

- Feynman parameter integrals are performed in terms of Hyperlogarithms,
  [Brown 2008 Comm. Math. Phys.]
  $L(\overrightarrow{w}, z) : \mathbb{C}\backslash\Sigma \rightarrow \mathbb{C}$, where
  - $\Sigma = \{\sigma_0, \sigma_1, ..., \sigma_N\}$ are distinct points in $\mathbb{C}$ which may contain variables
  - $\overrightarrow{w}$ is a word over the alphabet $\mathfrak{A} = \{a_0, a_1, ..., a_N\}$ where each letter a_i corresponds to a point σ_i
- $L(\overrightarrow{w}, z)$ is uniquely defined by the following properties

1. $L(\{\}, z) = 1$, and $L(0^n, z,) = \frac{1}{n!}\log^n(z)$ for $n \geq 1$

2. $\frac{\partial}{\partial z} L(\{a_i \overrightarrow{w}\}, z) = \frac{1}{z-\sigma_i} L(\overrightarrow{w}, z)$ for $z \in \mathbb{C}\backslash\Sigma$

3. If $\overrightarrow{w}$ is not of the form $w = (0, 0, \cdots, 0)$, then $\lim_{z\rightarrow 0} L(\overrightarrow{w}, z) = 0$.

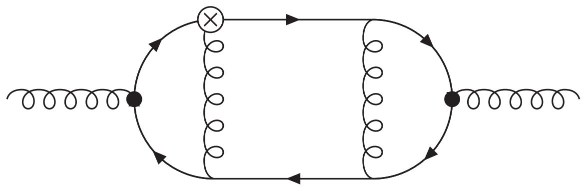
- e.g. $L(a_i, z) = \log(z - \sigma_i) - \log(\sigma_i)$

- The hyperlogarithms satisfy shuffle relations $L(\overrightarrow{w_1}, z)\, L(\overrightarrow{w_2}, z) = L(\overrightarrow{w_1} \sqcup\!\sqcup \overrightarrow{w_2}, z)$, e.g.: $L(\{a_1, a_2\}, z)\, L(\{a_3\}, z) = L(\{a_3, a_1, a_2\}, z) + L(\{a_1, a_3, a_2\}, z) + L(\{a_1, a_2, a_3\}, z)$
- The indices a_i contain further integration variables.
- Using these properties after partial fractioning and integration by parts, one can express any primitive for expressions consisting of rational and hyperlogarithmic functions in terms of different hyperlogarthmic functions. These primitives have to be evaluated at the respective integration limits
- Due to the operator-insertions leading to power-type functions, the integrals do not fit directly into the framework of the algorithm for general values of N.
- In order to use the algorithm also on integrals with general values of N, a generating function is constructed e.g. by the mapping

$$p(\alpha_1, \cdots, \alpha_n)^N \to \frac{1}{1 - x\, p(\alpha_1, \cdots, \alpha_n)} \,.$$

- Performing the Feynman-parameter integrations then leads to an expression which contains hyperlogarithms $L_w(x)$ in the variable x.
- Finally the Nth coefficient of this expression in x has to be extracted analytically. This has been done with the package HarmonicSums by J.Ablinger. [Ablinger, Blümlein, Schneider; 2013]

Six Massive Lines and Vertex Insertion



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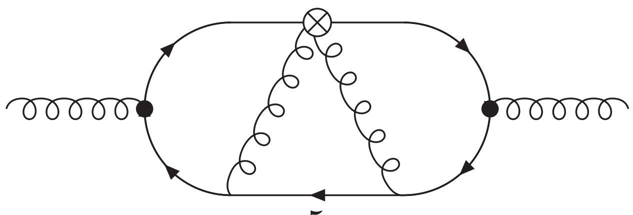
$$
\begin{aligned}
\hat{I}_4 \;=\;& \frac{Q_1(N)}{2(1+N)^5(2+N)^5(3+N)^5} + \frac{Q_2(N)}{(1+N)^2(2+N)^2(3+N)^2}\zeta_3 + \frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{2(1+N)^2(2+N)^2(3+N)^2}S_{-3} \\
&+\frac{\left(-24-5N+2N^2\right)}{12(2+N)^2(3+N)^2}S_1^3 - \frac{1}{2(1+N)(2+N)(3+N)}S_2^2 + \frac{1}{(2+N)(3+N)}S_1^2S_2 \\
&+\frac{Q_4(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_1^2 - \frac{3}{2}S_5 - \frac{Q_5(N)}{6(1+N)^2(2+N)^2(3+N)^2}S_3 - 2S_{-2,-3} - 2\zeta_3S_{-2} - S_{-2,1}S_{-2} \\
&+\frac{(-1)^N\left(65+101N+56N^2+13N^3+N^4\right)}{(1+N)^2(2+N)^2(3+N)^2}S_{-2,1} + \frac{\left(59+42N+6N^2\right)}{2(1+N)(2+N)(3+N)}S_4 + \frac{(5+N)}{(1+N)(3+N)}\zeta_3S_1(2) \\
&-\frac{Q_6(N)}{4(1+N)^3(2+N)^2(3+N)^2}S_2 - \zeta_3S_2 - \frac{3}{2}S_3S_2 - 2S_{2,1}S_2 + \frac{\left(99+225N+190N^2+65N^3+7N^4\right)}{2(1+N)^2(2+N)^2(3+N)}S_{2,1} \\
&+\frac{Q_3(N)}{(1+N)^4(2+N)^4(3+N)^4}S_1 - \frac{(11+5N)}{(1+N)(2+N)(3+N)}\zeta_3S_1 - \frac{Q_7(N)}{4(1+N)^2(2+N)^2(3+N)^2}S_2S_1 - S_{2,3} \\
&+\frac{(53+29N)}{2(1+N)(2+N)(3+N)}S_3S_1 - \frac{3(3+2N)}{(1+N)(2+N)(3+N)}S_1S_{2,1} + \frac{\left(-79-40N+N^2\right)}{2(1+N)(2+N)(3+N)}S_{3,1} - 3S_{4,1} \\
&+S_{-2,1,-2} + \frac{\mathbf{2^{N+1}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,2}\left(\frac{1}{2},1\right) - \frac{\left(-7+2N^2\right)}{(1+N)(2+N)(3+N)}S_{2,1,1} \\
&+5S_{2,2,1} + 6S_{3,1,1} + \frac{\mathbf{2^{N}}\left(-28-25N-4N^2+N^3\right)}{(1+N)^2(2+N)(3+N)^2}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
&-\frac{(5+N)}{(1+N)(3+N)}S_{1,1,2}\left(2,\frac{1}{2},1\right) - \frac{(5+N)}{2(1+N)(3+N)}S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)
\end{aligned}
$$

The 2^N factors cancel in the large N limit:

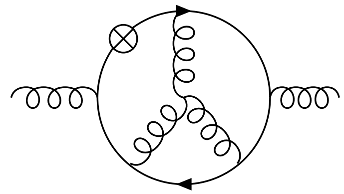
$$
\begin{aligned}
\hat{I}_4 \approx\ & \zeta_2^2\left[\frac{1115231}{20N^{10}}-\frac{74121}{4N^9}+\frac{122951}{20N^8}-\frac{40677}{20N^7}+\frac{13391}{20N^6}-\frac{873}{4N^5}+\frac{1391}{20N^4}-\frac{417}{20N^3}+\frac{101}{20\,N^2}\right]\\
&+\zeta_3\Bigg[\left(-\frac{95855}{2N^{10}}+\frac{31525}{2N^9}-\frac{10295}{2N^8}+\frac{3325}{2N^7}-\frac{1055}{2N^6}+\frac{325}{2N^5}-\frac{95}{2N^4}+\frac{25}{2N^3}-\frac{5}{2\,N^2}\right)\ln(N)\\
&-\frac{23280115}{2016N^{10}}+\frac{2093041}{1008N^9}-\frac{177251}{1008N^8}-\frac{25843}{336N^7}+\frac{2569}{48N^6}-\frac{155}{8N^5}+\frac{91}{24N^4}+\frac{2}{3N^3}-\frac{11}{12\,N^2}\Bigg]\\
&+\zeta_2\Bigg[\left(\frac{19171}{N^{10}}-\frac{6305}{N^9}+\frac{2059}{N^8}-\frac{665}{N^7}+\frac{211}{N^6}-\frac{65}{N^5}+\frac{19}{N^4}-\frac{5}{N^3}+\frac{1}{N^2}\right)\ln^2(N)\\
&+\left(\frac{103016863}{2520N^{10}}-\frac{3091261}{315N^9}+\frac{2571839}{1260N^8}-\frac{6215}{21N^7}-\frac{293}{20N^6}+\frac{2071}{60N^5}-\frac{103}{6N^4}+\frac{67}{12N^3}-\frac{1}{N^2}\right)\ln(N)\\
&+\frac{292993001621}{302400N^{10}}-\frac{4402272031}{30240N^9}+\frac{22261739}{840N^8}-\frac{78507473}{14112N^7}+\frac{180961}{144N^6}-\frac{111807}{400N^5}+\frac{629}{12N^4}-\frac{319}{72N^3}-\frac{7}{4\,N^2}\Bigg]\\
&+\left(\frac{249223}{6N^{10}}-\frac{145015}{12N^9}+\frac{10295}{3N^8}-\frac{11305}{12N^7}+\frac{1477}{6N^6}-\frac{715}{12N^5}+\frac{38}{3N^4}-\frac{25}{12N^3}+\frac{1}{6\,N^2}\right)\mathbf{ln^3(N)}\\
&+\left(\frac{193493767}{10080N^{10}}+\frac{210658237}{10080N^9}-\frac{21541697}{2520N^8}+\frac{243269}{96N^7}-\frac{30539}{48N^6}+\frac{2123}{16N^5}-\frac{59}{3N^4}+\frac{5}{8N^3}+\frac{1}{2\,N^2}\right)\ln^2(N)\\
&+\left(-\frac{2207364771673}{4233600N^{10}}+\frac{1390655509}{352800N^9}+\frac{285594061}{22050N^8}-\frac{67234111}{14400N^7}+\frac{8617073}{7200N^6}-\frac{35209}{144N^5}+\frac{116}{3N^4}-\frac{119}{24N^3}+\frac{1}{N^2}\right)\ln(N)\\
&+\frac{1344226725047831}{889056000N^{10}}-\frac{165849841805771}{889056000N^9}+\frac{808151260279}{27783000N^8}-\frac{708430537}{120960N^7}+\frac{304474703}{216000N^6}\\
&-\frac{606811}{1728N^5}+\frac{1867}{24N^4}-\frac{1813}{144N^3}+\frac{1}{N^2}+O(N^{-11})
\end{aligned}
$$

V-Topology



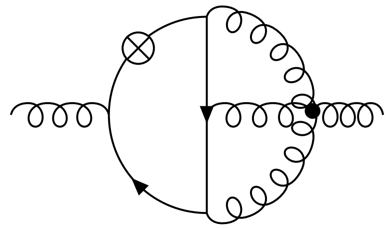
- This graph receives a natural and a more difficult contribution. The latter one led to a new function class : nested generalized cyclotomic sums, weighted with binomials and inverse binomials of the type $\binom{2i}{i}$.
- At the side of the iterated integrals many root-valued letters appear (around 30).
- The scalar diagram exhibits terms growing like $8^N, 4^N, 2^N, N \to \infty$. The growth 2^N survives in the scalar case.
- Asymptotic representations can be constructed analytically to arbitrary precision.
- Various special new numbers appear, the simplest of which is π, through which ζ_2 is no longer and elementary constant here.

General Values of N: Higher Topologies



$$I(x) = \frac{1}{(1+N)(2+N)x}\Big\{\zeta_3\Big[2L(\{-1\},x) - 2(-1+2x)L(\{1\},x) - 4L(\{1,1\},x)\Big] - 3L(\{-1,0,0,1\},x)$$
$$+2L(\{-1,0,1,1\},x) - 2xL(\{0,0,1,1\},x) + 3xL(\{0,1,0,1\},x) - xL(\{0,1,1,1\},x)$$
$$+(-3+2x)L(\{1,0,0,1\},x) + 2xL(\{1,0,1,1\},x) - (-1+5x)L(\{1,1,0,1\},x) + xL(\{1,1,1,1\},x)$$
$$-2L(\{1,0,0,1,1\},x) + 3L(\{1,0,1,0,1\},x) - L(\{1,0,1,1,1\},x) + 2L(\{1,1,0,0,1\},x)$$
$$+2L(\{1,1,0,1,1\},x) - 5L(\{1,1,1,0,1\},x) + L(\{1,1,1,1,1\},x)\Big\}$$

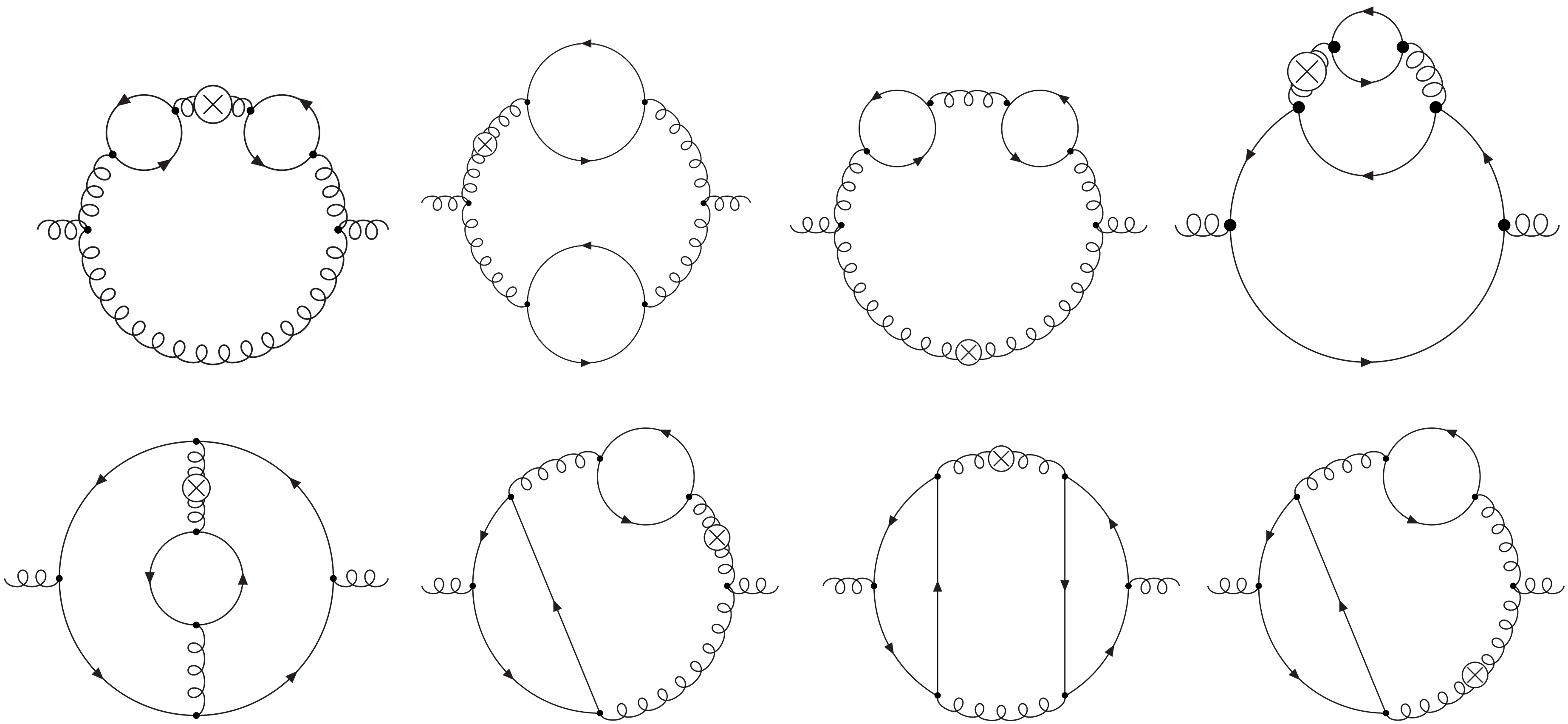
$$I(N) = \frac{1}{(N+1)(N+2)(N+3)}\Big\{\frac{648+1512N+1458N^2+744N^3+212N^4+32N^5+2N^6}{(1+N)^3(2+N)^3(3+N)^3}$$
$$-\frac{2\left(-1+(-1)^N+N+(-1)^N N\right)}{(1+N)}\zeta_3 - (-1)^N S_{-3} - \frac{N}{6(1+N)}S_1^3 + \frac{1}{24}S_1^4$$
$$-\frac{\left(7+22N+10N^2\right)}{2(1+N)^2(2+N)}S_2 - \frac{19}{8}S_2^2 - \frac{1+4N+2N^2}{2(1+N)^2(2+N)}S_1^2 + \frac{9}{4}S_2 - \frac{(-9+4N)}{3(1+N)}S_3$$
$$-\frac{1}{4}S_4 - 2(-1)^N S_{-2,1} + \frac{(-1+6N)}{(1+N)}S_{2,1} + \frac{54+207N+246N^2+130N^3+32N^4+3N^5}{(1+N)^3(2+N)^2(3+N)^2}S_1$$
$$+4\zeta_3 S_1 - \frac{(-2+7N)}{2(1+N)}S_2 S_1 + \frac{13}{3}S_3 S_1 - 7S_{2,1}S_1 - 7S_{3,1} + 10S_{2,1,1}\Big\}$$

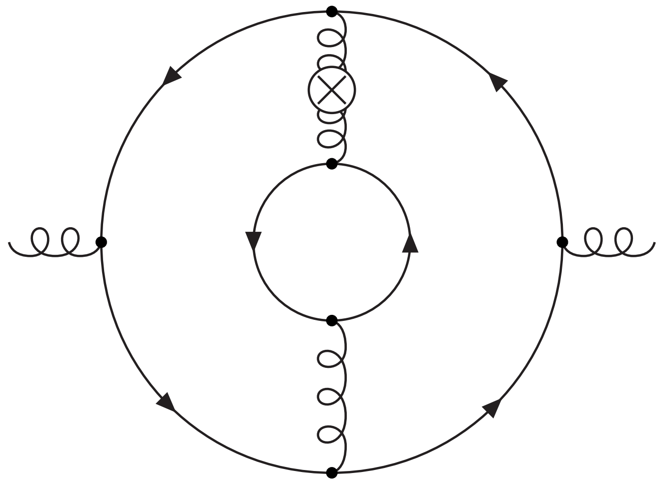


$$
\begin{aligned}
I(N) \quad = \quad & \frac{1}{(N+1)(N+2)}\Biggl\{\frac{2\left(1-13(-1)^N+(-1)^N 2^{3+N}+N-7(-1)^N N+3(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}\zeta_3 \\
& +\frac{1}{(2+N)}S_3+\frac{(-1)^N}{2(2+N)}S_1^3-\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_2+\frac{5(-1)^N}{2}S_2^2 \\
& +\frac{(-1)^N(3+2N)}{2(1+N)^2(2+N)}S_1^2-\frac{(-1)^N}{2}S_2S_1^2+\frac{3(-1)^N(4+3N)}{(1+N)(2+N)}S_3+3(-1)^N S_4+\frac{2}{(2+N)}S_{-2,1} \\
& +2(-1)^N\zeta_3 S_1\,(2)+\frac{2(-1)^N(3+N)}{(1+N)(2+N)}S_{2,1}-12(-1)^N S_1\zeta_3 \\
& +\frac{(-1)^N(5+7N)}{2(1+N)(2+N)}S_1S_2+3(-1)^N S_1S_3+4(-1)^N S_{2,1}S_1-4(-1)^N S_{3,1} \\
& -\frac{4\left((-1)^N 2^{2+N}-3(-2)^N N+3(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}S_{1,2}\left(\frac{1}{2},1\right)-5(-1)^N S_{2,1,1} \\
& +\frac{2\left(-(-1)^N 2^{2+N}-13(-2)^N N+5(-1)^N 2^{1+N}N\right)}{(1+N)(2+N)}S_{1,1,1}\left(\frac{1}{2},1,1\right) \\
& -2(-1)^N S_{1,1,2}\left(2,\frac{1}{2},1\right)-(-1)^N S_{1,1,1,1}\left(2,\frac{1}{2},1,1\right)\Biggr\}
\end{aligned}
$$

Diagrams with two massive lines

Using the different methods here discussed, we have been able to calculate the scalar integrals associated with complicated diagrams involving two separate massive loops (of equal mass):



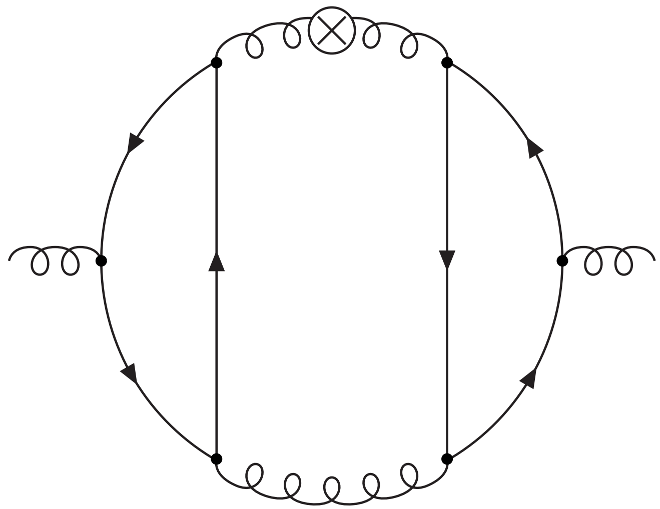


$$
\frac{(-1)^N+1}{2}\Biggl\{-\frac{1}{\varepsilon}\frac{4}{15(N-1)N(N+1)^2(N+2)}+\frac{(N-5)(3N+8)}{96(N+1)^2(N+2)(2N-3)(2N-1)}S_1(N)
$$

$$
+\frac{N^2-3N+6}{64(N+1)(N+2)(2N-3)(2N-1)4^N}\binom{2N}{N}\left[\sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j}j^2}S_1(j)-\sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j}j^3}-7\zeta_3\right]
$$

$$
+\frac{P_3}{3600(N-1)^2N^2(N+1)^3(N+2)(2N-3)(2N-1)}\Biggr\}
$$

$$
P_3=225N^7-775N^6+7702N^5-4194N^4-16783N^3+13129N^2+1176N-1440
$$



$$\frac{(-1)^N+1}{2}\Bigg\{\frac{27N^2+49N+38}{2880(N+1)^2(N+2)4^N}\binom{2N}{N}\left[\sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j}j^2}S_1(j)-\sum_{j=1}^{N}\frac{4^j}{\binom{2j}{j}j^3}-7\zeta_3\right]$$
$$+\frac{1}{90(N+1)}\Big[S_3(N)-S_{2,1}(N)+7\zeta_3\Big]+\frac{1}{90N(N+1)^2(N+2)}\Big[S_2(N)-S_1(N)^2\Big]$$
$$+\frac{60N^2+191N+120}{1440(N+1)^2(N+2)}S_1(N)-\frac{81N^3+194N^2+83N+60}{720N(N+1)^2(N+2)}-\frac{1}{\varepsilon}\frac{1}{12(N+1)}\Bigg\}$$

Integration by parts

We use Reduze [A. von Manteuffel, C. Studerus, 2012] to express all scalar integrals required in the calculation in terms of a small(er) set of master integrals.

Reduze is a C++ program based on Laporta's algorithm. It is somewhat difficult to adapt this algorithm to the case where we have operator insertions, due to the dependence on the arbitrary parameter N. For this reason we apply the same trick we used before in the case of Brown's algorithm, and resum the insertions introducing a new parameter x:

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

This can be then treated as an additional propagator, and Laporta's algorithm can be applied without further modification.

If we denote the master integrals by M_i, then the reduction algorithm will allow us to express any given integral I as

$$I = \sum_i c_i(x) M_i(x)$$

The coefficients $c_i(x)$ will be functions of the form

$$c_i(x) = \sum_{j=1}^{n_1} \frac{a_i}{(1 - x\Delta \cdot p)^j} + \sum_{j=-n_2}^{n_3} b_j (x\Delta \cdot p)^j$$

where a_j and b_j are constants, and n_1, n_2 and n_3 are non-negative integers. We then have two alternatives:

- Calculate the M_i's as functions of x, introduce them in $\sum_i c_i(x) M_i(x)$, and then extract the Nth term of the Taylor expansion in x of this expression, to obtain I as a function of N.
- Undo the introduction of x in the M_i's directly, as well as in the coefficients $c_i(x)$, noticing, for example, that

$$(x\Delta \cdot p)^j M_i(N) \to M_i(N - j) \quad \text{and} \quad \frac{1}{1 - x\Delta \cdot p} M_i(N) \to \sum_{j=1}^{N} M_i(j)$$

The algorithm to calculate $A_{qq,Q}^{\mathrm{NS},(\mathrm{trans})}$ is now fully available and running. The next four complete OMEs will be calculated this way during the coming months.

Conclusions

- All $O(n_f T_F^2)$ contributions have been completed.
- The gluonic $O(T_F^2)$ terms are currently calculated, after all principal topologies have been solved.
- The renormaliztion in the 2-mass case has been performed and for all OMEs the moments N = 2,4,6 were calculated. The general N result in this case for A_{gg} is nearly complete. Also the setup for a VFNS in case both charm and bottom is becoming massless, has been derived, since no hierarchy exists for these terms individually. This scheme is different from the former single mass VFNS.
- Ladder, V-Graph and Benz-topologies for graphs, with no singularities in ε can be systematically calculated.
- Here new functions occur (including a larger number of root-letters in iterated integrals)
- Progress has been made in the calculation of the remaining pure-singlet and non-singlet graphs for all contributing topologies and the calculation of the next four massive Wilson coefficients is underway.