Monte Carlo Techniques in small-x Physics: Formal Studies and Phenomenology

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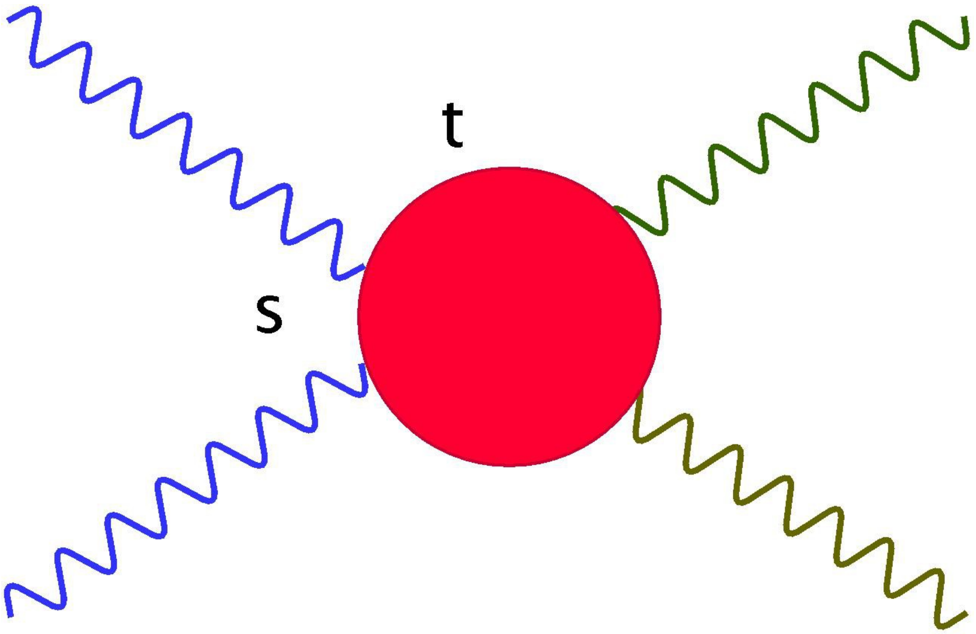
Outline

- Brief introduction to BFKL dynamics
- The LO BFKL equation in the adjoint representation (color octet)
- The NLO BFKL equation in the adjoint representation (color octet)
- The BKP equation – The Odderon
- Using a Monte Carlo approach to solve the BFKL equation in the color octet
- Outlook

High energy limit in QCD

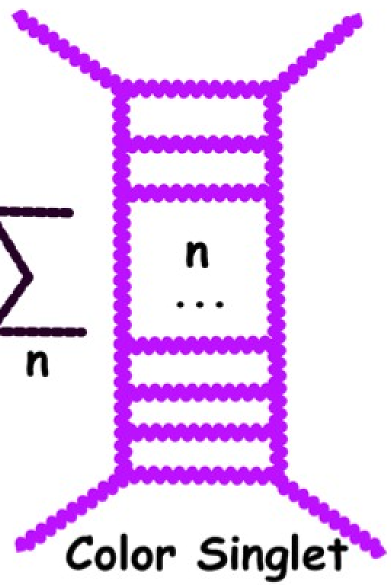
- We want the elastic amplitude of the process with Mandelstam variables s, t in pQCD
- We need to have a hard scale $Q^2 \gg \Lambda_{QCD}^2$
- We want the amplitude in the high energy (Regge) limit where $s \gg |t|, Q^2$
- The hard scale ensures that $\alpha_s(Q^2) \ll 1$
- The problem then becomes a problem of resumming terms of the form $(\alpha_s \ln s)^n$



Ladder diagrams

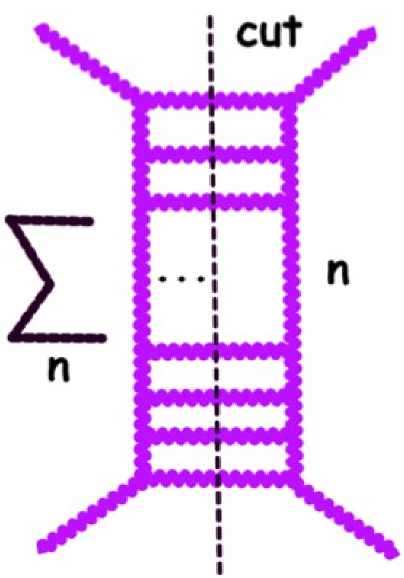
All-orders resummation of $\alpha_s(Q^2)\log\left(\frac{s}{Q^2}\right)$ terms: How? Ladder structure

$A_{\text{elastic}}(s,t) = \sum_n$



Optical Theorem :

$\sigma_{\text{TOT}} \simeq \frac{1}{s}\text{Im}\mathcal{A}_{\text{elastic}}(s,t=0) = \frac{1}{s}\sum_n$



$= \frac{1}{s}\sum_n |A_n(s,t)|^2$

The BFKL equation and Multi-Regge kinematics I

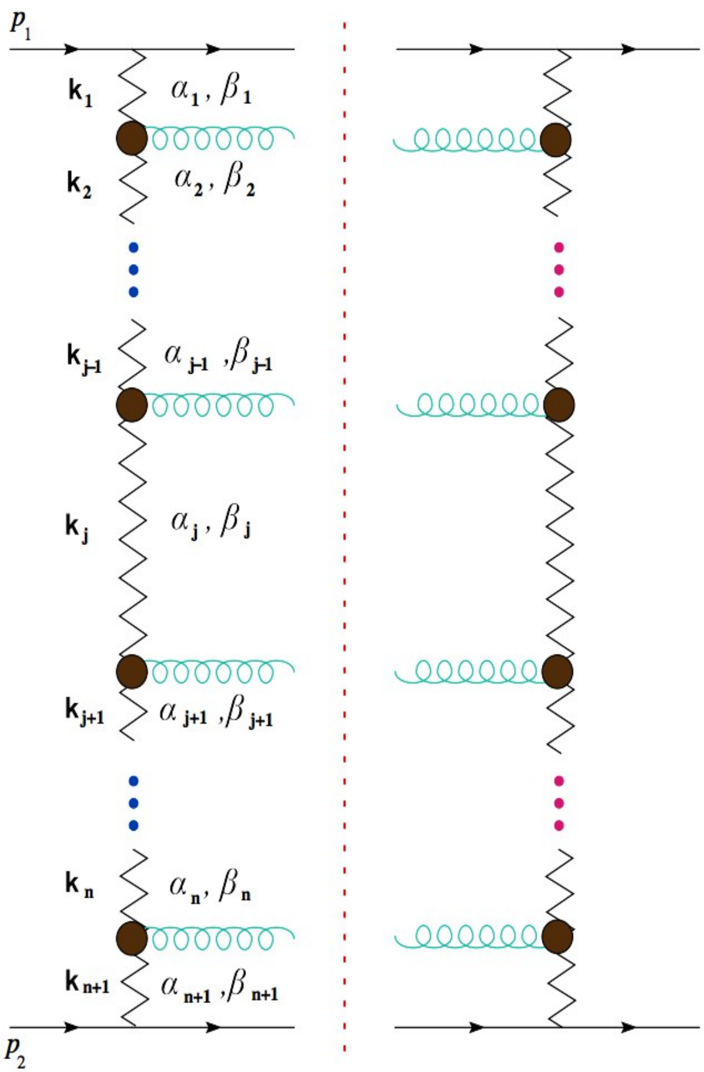
Decompose into Sudakov variables, e.g.

$$k_1 = \alpha_1 p_1 + \beta_1 p_2 + k_{1\perp}$$

Tracking leading logarithms only suggests a restriction of the kinematical conf guration to the so-called
Multi-Regge kinematics (MRK):

$$\mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \ldots \mathbf{k}_i^2 \simeq \mathbf{k}_{i+1}^2 \ldots \simeq \mathbf{k}_n^2 \simeq \mathbf{k}_{n+1}^2 \gg \mathbf{q}^2 \simeq s_0,$$
$$1 \gg \alpha_1 \gg \alpha_2 \gg \ldots \alpha_i \gg \alpha_{i+1} \gg \alpha_{n+1} \gg \frac{s_0}{s},$$
$$1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \ldots \gg |\beta_2| \gg |\beta_1| \gg \frac{s_0}{s}.$$

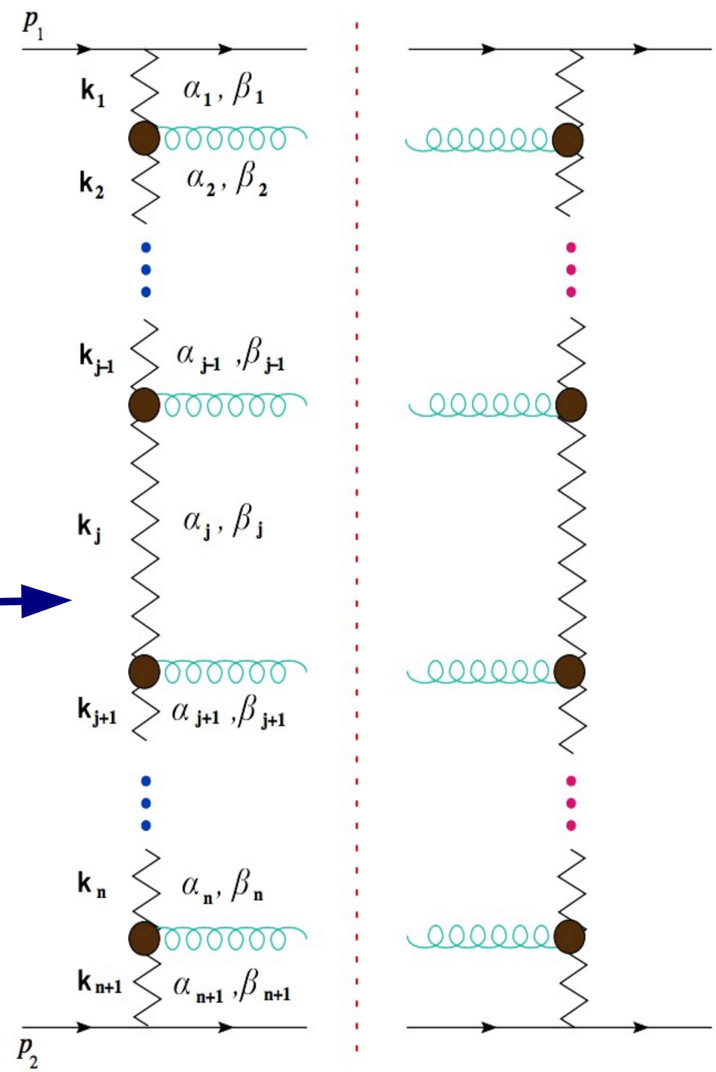
s_0 is a typical normalization scale for the BFKL equation



The BFKL equation and Multi-Regge kinematics II

There are also virtual (loop) corrections which are encoded into modifying the gluon propagators in the t-channel such that they become the so-called Reggeized gluon propagators.

Reggeized gluon



The propagator of a reggeized gluon is:

$$D_{\mu\nu}(s, q^2) = -i\frac{g_{\mu\nu}}{q^2}\left(\frac{s}{\mathbf{k}^2}\right)^{\epsilon(q^2)}$$

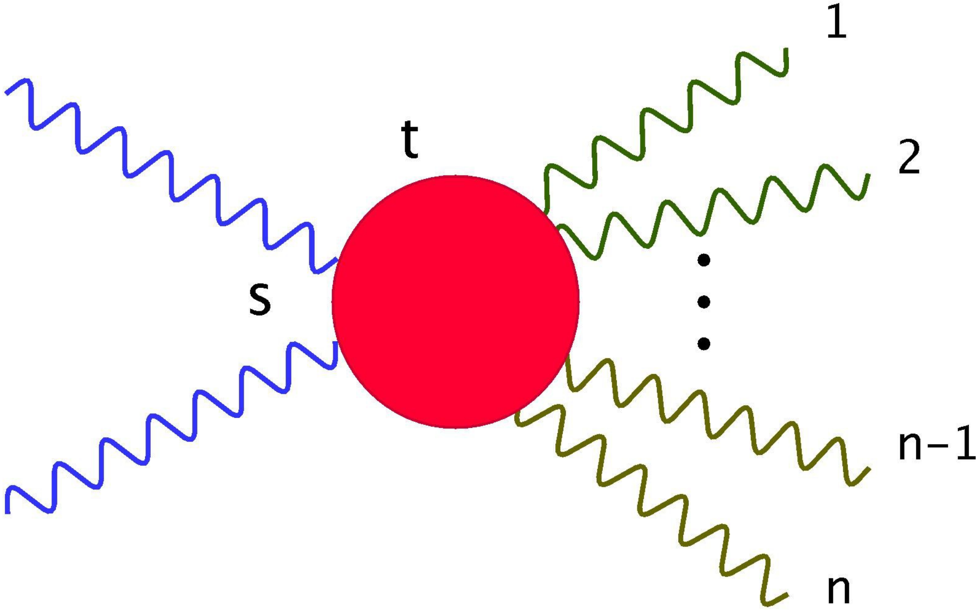
$$\epsilon(t) = \frac{N_c\alpha_s}{4\pi^2}\int -\mathbf{q}^2 \frac{d^2\mathbf{k}}{\mathbf{k}^2(\mathbf{k}-\mathbf{q})^2}$$

with $\mathbf{k^2} \ll s$ a typical momentum scale and $1+\epsilon(q^2)$ the gluon trajectory

The LL BFKL equation and the Multi-Regge kinematics schematically

$$\mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \dots \mathbf{k}_i^2 \simeq \mathbf{k}_{i+1}^2 \dots \simeq \mathbf{k}_n^2 \simeq \mathbf{k}_{n+1}^2 \gg \mathbf{q}^2 \simeq s_0,$$
$$1 \gg \alpha_1 \gg \alpha_2 \gg \dots \alpha_i \gg \alpha_{i+1} \gg \alpha_{n+1} \gg \frac{s_0}{s},$$
$$1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \dots \gg |\beta_2| \gg |\beta_1| \gg \frac{s_0}{s}.$$

Multi-Regge limit:
Regge limit in all sub-channels, $s \gg s_i \gg t_i \sim Q^2$
strong ordering in rapidity:
$Y \sim \ln(s),\ y_i \sim \ln(s_i),\ y_i \gg y_{i-1}$



LO BFKL:
Fadin, Kuraev, Lipatov (1977),
Balitsky, Lipatov (1978)

Quasi-Multi-Regge kinematics I

To have the BFKL equation to NNL accuracy, resum term of the form: $\alpha_s(\alpha_s \ln s)^n$

NLO BFKL:

Fadin, Lipatov (1998)
Ciafaloni, Gamici (1998)

- The ways to obtain a term of the type above is by either losing a logarithm of s starting from an amplitude at LL or by including loop corrections to the vertices.
- For the real emission corrections, the key feature that generates these logarithmic terms is the strong ordering in rapidity.
- Thus, if we allow for a state where two of the emitted particles are close to each other, we are in the Quasi-Multi-Regge-kinematics (QMRK):

$$\mathbf{k}_1^2 \simeq \mathbf{k}_2^2 \simeq \ldots \mathbf{k}_i^2 \simeq \mathbf{k}_{i+1}^2 \ldots \simeq \mathbf{k}_n^2 \simeq \mathbf{k}_{n+1}^2 \gg \mathbf{q}^2 \simeq s_0,$$
$$1 \gg \alpha_1 \gg \alpha_2 \gg \ldots \alpha_i \gg \alpha_{i+1} \gg \alpha_{n+1} \gg \frac{s_0}{s},$$
$$1 \gg |\beta_{n+1}| \gg |\beta_n| \gg \ldots \gg |\beta_2| \gg |\beta_1| \gg \frac{s_0}{s}.$$

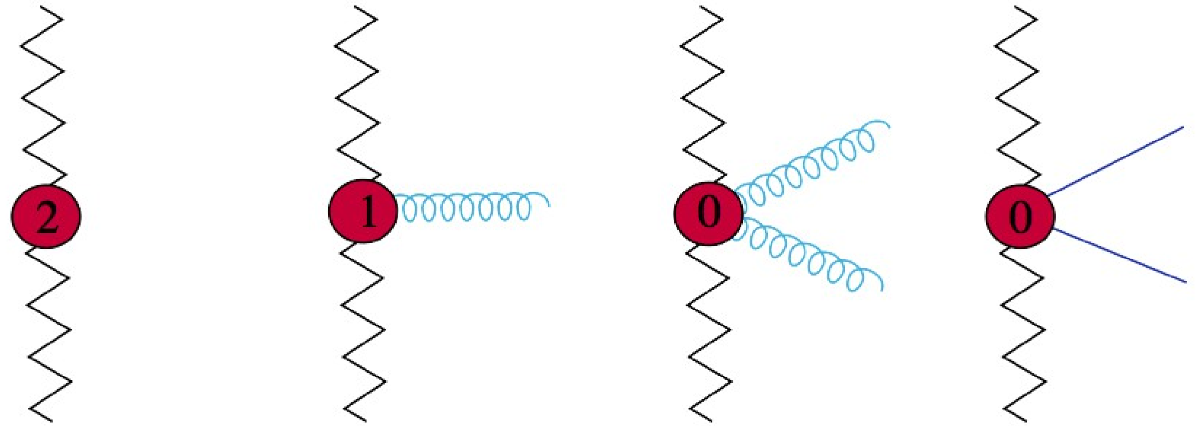
- The relations above still hold with the exception of a pair of particles. The pair can be a pair of gluons or a quark anti-quark pair.

Quasi-Multi-Regge kinematics II

To have the BFKL equation to NNL accuracy, resum term of the form: $\alpha_s(\alpha_s \ln s)^n$

NLL BFKL:

Fadin, Lipatov (1998)
Ciafaloni, Gamici (1998)



Some generic statements on the BFKL dynamics

- Usually one has in mind the BFKL equation for the case of forward scattering (momentum transfer t = 0 and vacuum quantum numbers exchanged in the t-channel (color singlet, Pomeron)
- The BFKL equation though, was from the beginning developed for arbitrary t and for all possible t-channel color states. The BFKL kernel for the latter case is know to NLO

Fadin, Gorbachev (2000)

Fadin, Fiore (2005)

A few words on color

This -Nc is a color factor, assuming that the color state of the two gluons in the graph is the color singlet. If this is the case, then the kernel is IR finite!

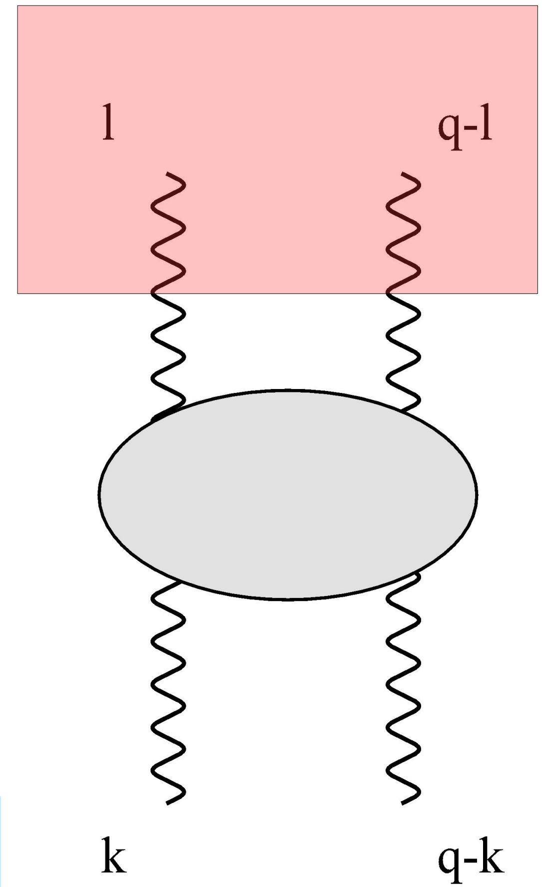
Color state of the two gluons

For QCD, the possible states are:

$$\mathbf{1},\ \mathbf{8_A},\ \mathbf{8_S},\ \mathbf{10}+\overline{\mathbf{10}},\ \mathbf{27}$$

with color factors:

$$-3,\ -\frac{3}{2},\ -\frac{3}{2},\ 0,\ 1$$



$$K_{\mathrm{BFKL}}(\mathbf{l},\mathbf{q}-\mathbf{l};\mathbf{k},\mathbf{q}-\mathbf{k}) = -N_c g^2\left[\mathbf{q}^2-\frac{\mathbf{k}^2(\mathbf{q}-\mathbf{l})^2}{(\mathbf{k}-\mathbf{l})^2}-\frac{(\mathbf{q}-\mathbf{k})^2\mathbf{l}^2}{(\mathbf{k}-\mathbf{l})^2}\right]$$
$$+(2\pi)^3\mathbf{k}^2(\mathbf{q}-\mathbf{k})^2\left[\beta(\mathbf{k})+\beta(\mathbf{q}-\mathbf{k})\right]\delta^{(2)}(\mathbf{k}-\mathbf{l})\,.$$

Where $\beta(\mathbf{k}^2) = -\frac{N_c}{2}g^2\int\frac{d^2\mathbf{l}}{(2\pi)^3}\frac{\mathbf{k}^2}{\mathbf{l}^2(\mathbf{l}-\mathbf{k})^2}$

is again the gluon Regge trajectory named now as $\beta(k^2)$

Why the color octet representation is important

Symmetric octet

- It was in a generalized leading logarithmic approximation, and by iterating the BFKL kernel in the s-channel, where the Bartels-Kwiecinski-Praszalowicz (BKP) equation was proposed

Bartels (1980)

Kwiecinski, Praszalowicz (1980)

- BKP was found to have a hidden integrability being equivalent to a periodic spin chain of a XXX Heisenberg ferromagnet. This was the first example of the existence of integrable systems in QCD

Lipatov (1986, 1990, 1993)

- **It will be directly connected to any numerical solution of the BKP, if any such work is to be done with the aim to perform phenomenological studies for the Odderon**

Antisymmetric octet

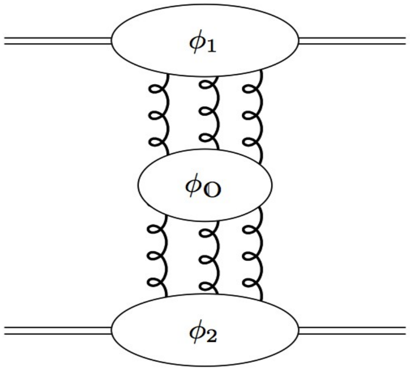
- Corrections to the Bern-Dixon-Smirnov (BDS) iterative ansatz (Bern, Dixon, Smirnov, 2005) for the n-point maximally helicity violating (MHV) and planar amplitudes were found in MRK in the six-point amplitude at two loops

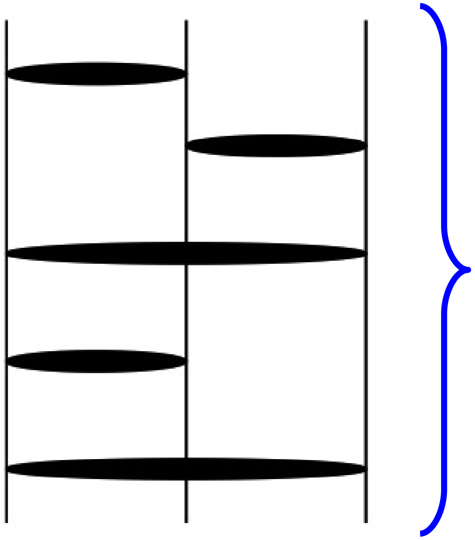
Bartels, Lipatov, Sabio Vera (2009, 2010)

in other words, it is a fundamental ingredient of the finite remainder of scattering amplitudes with arbitrary number of external legs and internal loops

BKP – The Odderon

- Pomeron is the state of two interacting reggeized gluons in the t-channel in the color singlet. It has the quantum numbers of the vacuum
- Odderon is the state of three interacting gluons exchanged in the t-channel in the color singlet but with C =-1 and P=-1
- Any pair of two gluons in the Odderon forms symmetric color octet subsystems





Ladder structure of the Odderon. BKP resums term of the form $\alpha_s(\alpha_s \log s)^n$

NLO corrections recently available

Bartels, Fadin, Lipatov, Vacca (2012)

The Odderon is nowhere to be seen so far

Solving BFKL with Monte Carlo integration techniques

- Many people have worked on it, the origin goes back to the late 90's:

Kwiecinski, Lewis, Martin (1996), Schmidt (1996), Orr, Stirling (1998)

Effective Feynman Rules:
simplest case, $t = 0$, leading order

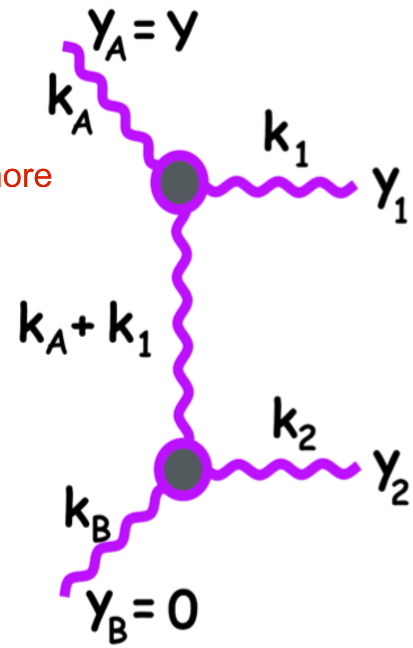
Note the change in the naming of the gluon Regge trajectory once more

Gluon Regge trajectory:

$$\omega\left(\vec{q}\right) = -\frac{\alpha_s N_c}{\pi} \log \frac{q^2}{\lambda^2}$$

Modified t-channel propagators:

$$\left(\frac{s_i}{s_0}\right)^{\omega(t_i)} = e^{\omega(t_i)(y_i - y_{i+1})}$$



$$\left(\frac{\alpha_s N_c}{\pi}\right)^2 \int d^2\vec{k}_1 \frac{\theta\left(k_1^2 - \lambda^2\right)}{\pi k_1^2} \int d^2\vec{k}_2 \frac{\theta\left(k_2^2 - \lambda^2\right)}{\pi k_2^2} \delta^{(2)}\left(\vec{k}_A + \vec{k}_1 + \vec{k}_2 - \vec{k}_B\right)$$

$$\times \int_0^Y dy_1 \int_0^{y_1} dy_2 e^{\omega\left(\vec{k}_A\right)(Y - y_1)} e^{\omega\left(\vec{k}_A + \vec{k}_1\right)(y_1 - y_2)} e^{\omega\left(\vec{k}_A + \vec{k}_1 + \vec{k}_2\right) y_2}$$

Very simplified -pictorial- view, the main elements and ideas are here though

The LO BFKL equation in the color octet

$$\left\{\omega + (c_{\mathcal{R}} - 1)\frac{\bar{\alpha}_s}{2}\left[\frac{2}{\epsilon} - \log\left(\frac{\mathbf{q}_1^2}{\mu^2}\right) - \log\left(\frac{\mathbf{q}_1'^2}{\mu^2}\right)\right]\right.$$
$$\left. + c_{\mathcal{R}}\frac{\bar{\alpha}_s}{2}\left[\log\left(\frac{\mathbf{q}_1^2}{\lambda^2}\right) + \log\left(\frac{\mathbf{q}_1'^2}{\lambda^2}\right)\right]\right\}\mathcal{G}_\omega\left(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}\right) = \delta^{(2)}\left(\mathbf{q}_1 - \mathbf{q}_2\right)$$
$$+ c_{\mathcal{R}}\int \frac{d^2\mathbf{k}}{\pi\mathbf{k}^2}\theta\left(\mathbf{k}^2 - \lambda^2\right)\frac{\bar{\alpha}_s}{2}\left[1 + \frac{\mathbf{q}_1'^2(\mathbf{q}_1 + \mathbf{k})^2 - \mathbf{q}^2\mathbf{k}^2}{(\mathbf{q}_1' + \mathbf{k})^2\mathbf{q}_1^2}\right]\mathcal{G}_\omega\left(\mathbf{q}_1 + \mathbf{k}, \mathbf{q}_2; \mathbf{q}\right)$$

C_R = ½ for octet

C_R = 1 for singlet

This can now be iterated and, performing the Mellin transform,

Gluon Green's Function →

$$\mathcal{F}\left(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}; \mathrm{Y}\right) = \int \frac{d\omega}{2\pi i} e^{\omega \mathrm{Y}} \mathcal{G}_\omega\left(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}\right)$$

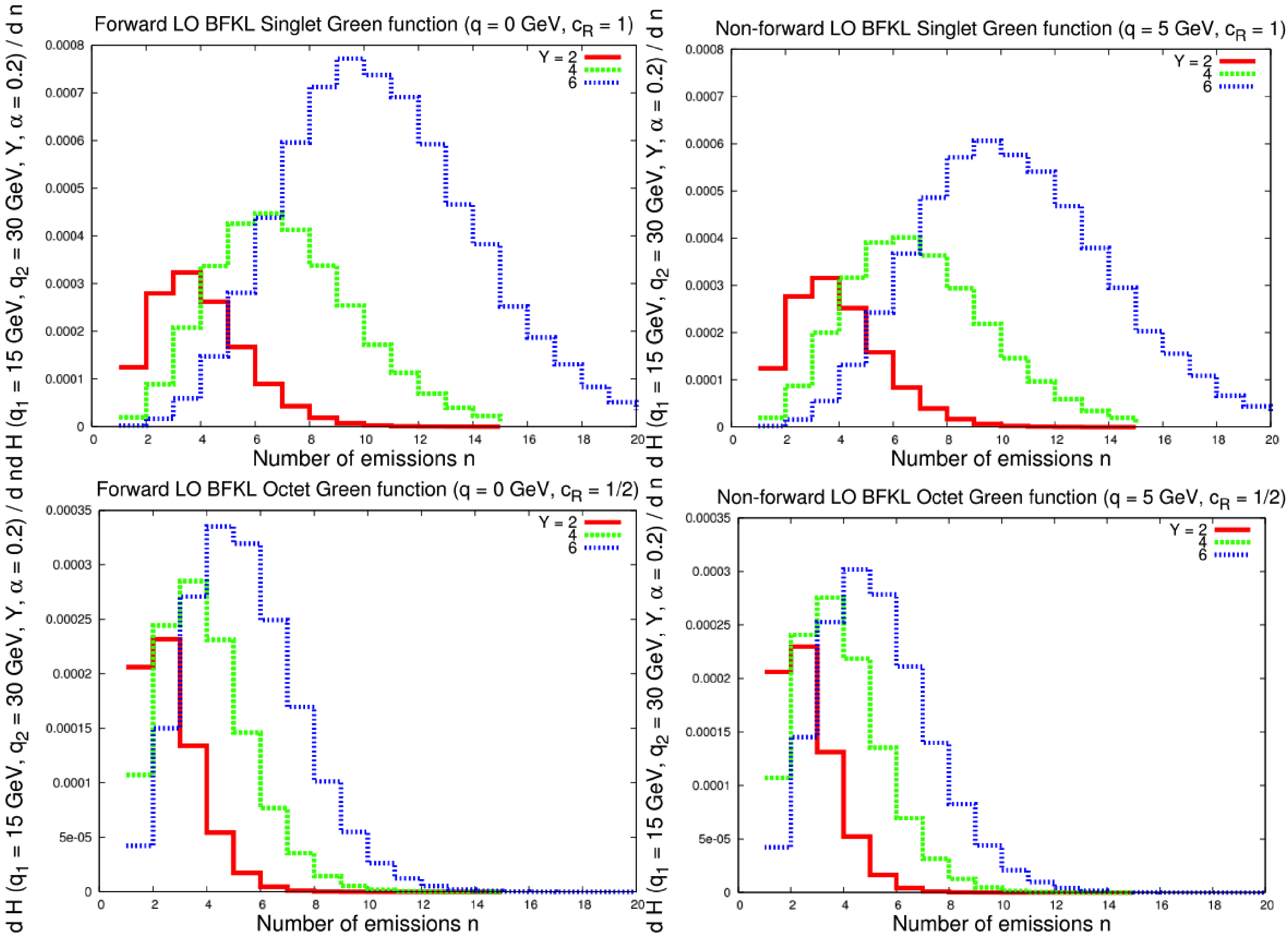
we finally obtain

$$\mathcal{F}\left(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}; \mathrm{Y}\right) = \exp\left\{\omega^{(\epsilon;\lambda)}(\mathbf{q}_1; \mathbf{q})\mathrm{Y}\right\}\left\{\delta^{(2)}\left(\mathbf{q}_1 - \mathbf{q}_2\right)\right.$$
$$+ \sum_{n=1}^{\infty}\prod_{i=1}^{n} c_{\mathcal{R}}\int \frac{d^2\mathbf{k}_i}{\pi\mathbf{k}_i^2}\theta(\mathbf{k}_i^2 - \lambda^2)\,\xi\left(\mathbf{q}_1 + \sum_{l=1}^{i-1}\mathbf{k}_l, \mathbf{k}_i; \mathbf{q}\right)\delta^{(2)}\left(\mathbf{q}_1 + \sum_{l=1}^{n}\mathbf{k}_l - \mathbf{q}_2\right)$$
$$\left.\times \int_0^{y_{i-1}} dy_i \exp\left\{\left[\omega^{(\epsilon;\lambda)}\left(\mathbf{q}_1 + \sum_{l=1}^{i}\mathbf{k}_l; \mathbf{q}\right) - \omega^{(\epsilon;\lambda)}\left(\mathbf{q}_1 + \sum_{l=1}^{i-1}\mathbf{k}_l; \mathbf{q}\right)\right] y_i\right\}\right\},$$

} Monte Carlo

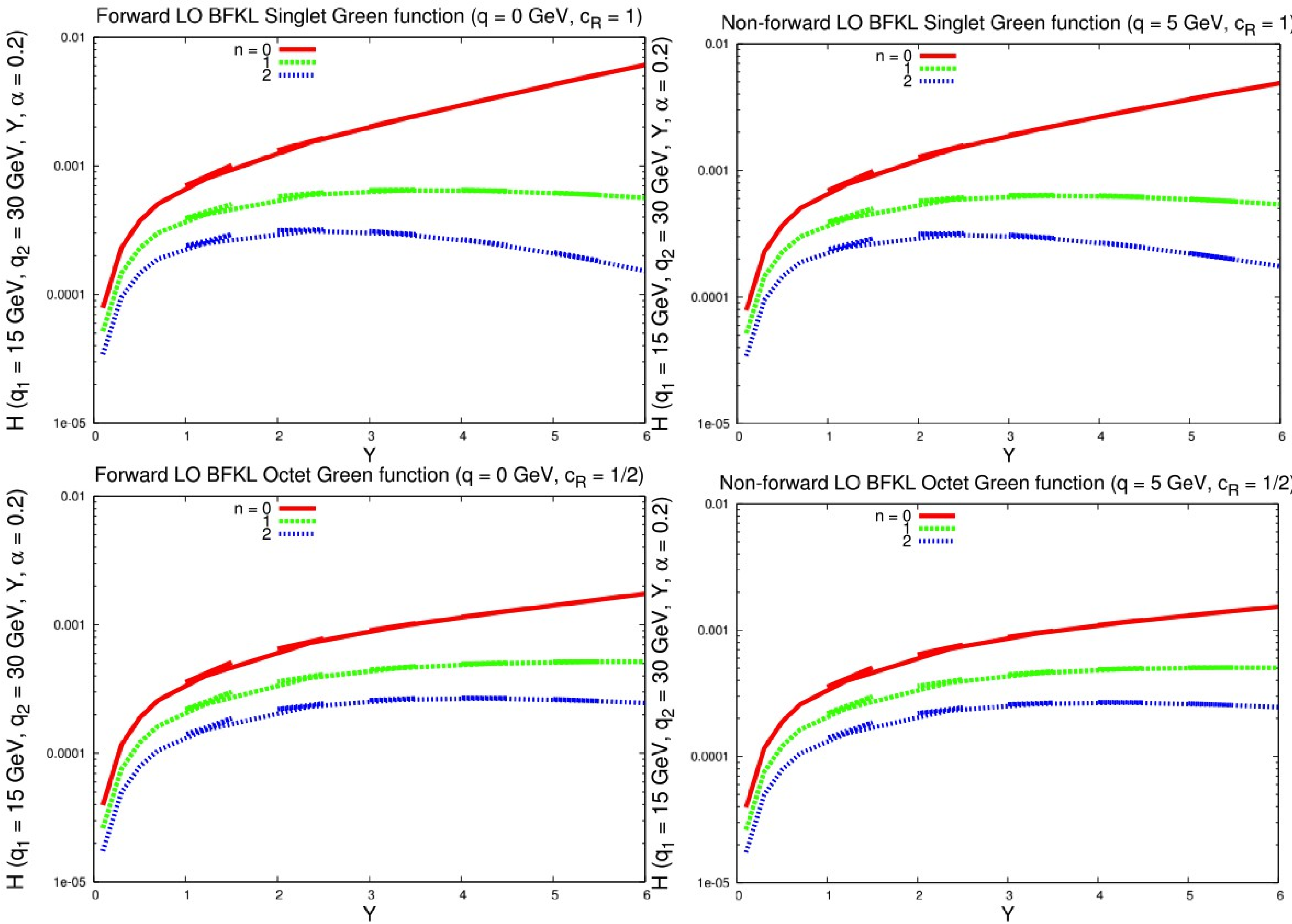
Numerical results

$$\mathcal{H}\left(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}; \mathrm{Y}\right) \equiv \mathcal{F}\left(\mathbf{q}_1, \mathbf{q}_2; \mathbf{q}; \mathrm{Y}\right) \left(\frac{e^{\frac{1}{\epsilon}} \mu^2}{\sqrt{\mathbf{q}_1^2 \mathbf{q}_1'^2}} \right)^{\bar{\alpha}_s (c_{\mathcal{R}} - 1) \mathrm{Y}}$$



Numerical results

$$\mathcal{H}(\mathbf{q}_1,\mathbf{q}_2;\mathbf{q};\mathrm{Y}) \equiv \mathcal{F}(\mathbf{q}_1,\mathbf{q}_2;\mathbf{q};\mathrm{Y})\left(\frac{e^{\frac{1}{\epsilon}}\mu^2}{\sqrt{\mathbf{q}_1^2\mathbf{q}_1'^2}}\right)^{\bar{\alpha}_s(c_{\mathcal{R}}-1)\mathrm{Y}}$$



The NLO BFKL equation in the color octet

Fadin, Lipatov (2012)

$$\mathcal{F}\left(\mathbf{q}_1,\mathbf{q}_2;\mathbf{q};\mathrm{Y}\right) = \left(\frac{\mathbf{q}^2\lambda^2}{\mathbf{q}_1^2\mathbf{q}_1'^2}\right)^{\frac{\bar\alpha}{2}\left(1-\frac{\zeta_2}{2}\bar\alpha\right)\mathrm{Y}} e^{\frac{3}{4}\zeta_3\bar\alpha^2\mathrm{Y}}\Bigg\{\delta^{(2)}\left(\mathbf{q}_1-\mathbf{q}_2\right)$$

$$+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left[\int d^2\mathbf{k}_i\frac{\bar\alpha}{4}\left(1-\frac{\zeta_2}{2}\bar\alpha\right)\frac{\theta\left(\mathbf{k}_i^2-\lambda^2\right)}{\pi\mathbf{k}_i^2}\left(1+\frac{\left(\mathbf{q}_1'+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2\left(\mathbf{q}_1+\sum_{l=1}^{i}\mathbf{k}_l\right)^2-\mathbf{q}^2\mathbf{k}_i^2}{\left(\mathbf{q}_1'+\sum_{l=1}^{i}\mathbf{k}_l\right)^2\left(\mathbf{q}_1+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2}\right)\right.$$

$$\left.+\Phi\left(\mathbf{q}_1+\sum_{l=1}^{i-1}\mathbf{k}_l,\mathbf{q}_1+\sum_{l=1}^{i}\mathbf{k}_l\right)\right]\delta^{(2)}\left(\mathbf{q}_1+\sum_{l=1}^{n}\mathbf{k}_l-\mathbf{q}_2\right)$$

$$\times\int_0^{y_{i-1}}dy_i\left(\frac{\left(\mathbf{q}_1+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2}{\left(\mathbf{q}_1+\sum_{l=1}^{i}\mathbf{k}_l\right)^2}\right)^{1+\frac{\bar\alpha y_i}{2}\left(1-\frac{\zeta_2}{2}\bar\alpha\right)}\left(\frac{\left(\mathbf{q}_1'+\sum_{l=1}^{i-1}\mathbf{k}_l\right)^2}{\left(\mathbf{q}_1'+\sum_{l=1}^{i}\mathbf{k}_l\right)^2}\right)^{\frac{\bar\alpha y_i}{2}\left(1-\frac{\zeta_2}{2}\bar\alpha\right)}\Bigg\}$$

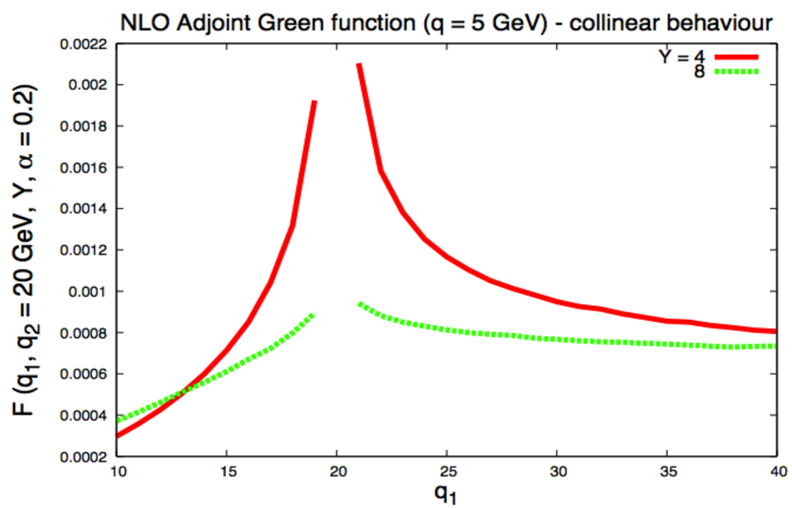
where

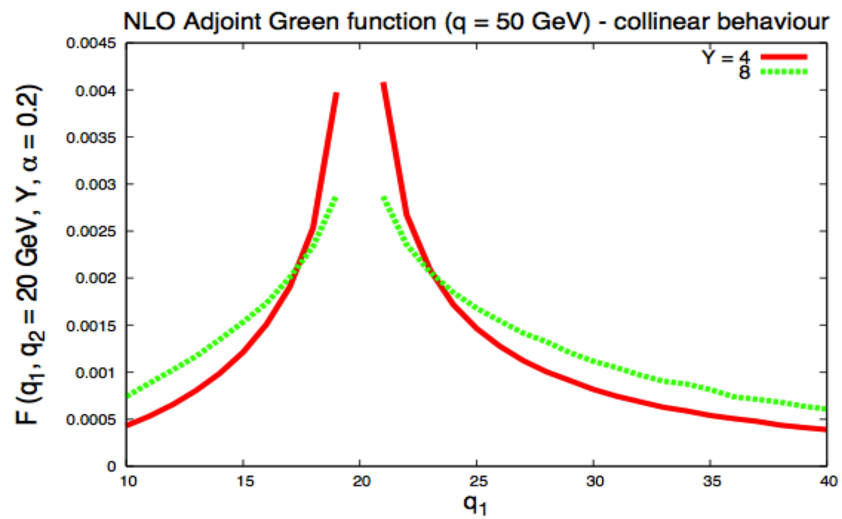
$$\begin{aligned}
\Phi\left(\mathbf{q}_1, \mathbf{q}_1+\mathbf{k}\right) &= \\
&\frac{\bar{\alpha}^2}{32\pi}\frac{1}{\mathbf{q}_1^2(\mathbf{k}+\mathbf{q}_1')^2}\left\{\mathbf{q}^2\left[\ln\left(\frac{\mathbf{q}_1^2}{\mathbf{q}^2}\right)\ln\left(\frac{\mathbf{q}_1'^{\,2}}{\mathbf{q}^2}\right)+\ln\left(\frac{(\mathbf{q}_1+\mathbf{k})^2}{\mathbf{q}^2}\right)\ln\left(\frac{(\mathbf{q}_1'+\mathbf{k})^2}{\mathbf{q}^2}\right)\right.\right. \\
&+\frac{1}{2}\ln^2\left(\frac{\mathbf{q}_1^2}{(\mathbf{q}_1+\mathbf{k})^2}\right)+\frac{1}{2}\ln^2\left(\left.\frac{\mathbf{q}_1'^2}{(\mathbf{q}_1'+\mathbf{k})^2}\right)\right]+\frac{1}{2}\frac{\left(\mathbf{q}_1^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2-\mathbf{q}_1'^2\left(\mathbf{q}_1+\mathbf{k}\right)^2\right)}{\mathbf{k}^2} \\
&\quad\times\left[\ln\left(\frac{\mathbf{q}_1'^2}{(\mathbf{q}_1'+\mathbf{k})^2}\right)\ln\left(\frac{\mathbf{q}_1'^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2}{\mathbf{k}^4}\right)-\ln\left(\frac{\mathbf{q}_1^2}{(\mathbf{q}_1+\mathbf{k})^2}\right)\ln\left(\frac{\mathbf{q}_1^2\left(\mathbf{q}_1+\mathbf{k}\right)^2}{\mathbf{k}^4}\right)\right] \\
&-\frac{\left(\mathbf{q}_1^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2+\mathbf{q}_1'^2\left(\mathbf{q}_1+\mathbf{k}\right)^2\right)}{\mathbf{k}^2}\left[\ln^2\left(\frac{\mathbf{q}_1^2}{(\mathbf{q}_1+\mathbf{k})^2}\right)+\ln^2\left(\frac{\mathbf{q}_1'^2}{(\mathbf{q}_1'+\mathbf{k})^2}\right)\right] \\
&+\left[\mathbf{q}^2\left(\mathbf{k}^2-\mathbf{q}_1^2-\left(\mathbf{q}_1+\mathbf{k}\right)^2\right)+2\mathbf{q}_1^2\left(\mathbf{q}_1+\mathbf{k}\right)^2-\mathbf{q}_1^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2-\mathbf{q}_1'^2\left(\mathbf{q}_1+\mathbf{k}\right)^2\right. \\
&\quad\left.+\frac{\left(\mathbf{q}_1^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2-\mathbf{q}_1'^2\left(\mathbf{q}_1+\mathbf{k}\right)^2\right)}{\mathbf{k}^2}\left(\mathbf{q}_1^2-\left(\mathbf{q}_1+\mathbf{k}\right)^2\right)\right]\mathcal{I}\left(\mathbf{q}_1^2,\left(\mathbf{q}_1+\mathbf{k}\right)^2,\mathbf{k}^2\right) \\
&+\left[\mathbf{q}^2\left(\mathbf{k}^2-\mathbf{q}_1'^2-\left(\mathbf{q}_1'+\mathbf{k}\right)^2\right)+2\mathbf{q}_1'^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2-\mathbf{q}_1'^2\left(\mathbf{q}_1+\mathbf{k}\right)^2-\mathbf{q}_1^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2\right. \\
&\quad\left.\left.+\frac{\left(\mathbf{q}_1'^2\left(\mathbf{q}_1+\mathbf{k}\right)^2-\mathbf{q}_1^2\left(\mathbf{q}_1'+\mathbf{k}\right)^2\right)}{\mathbf{k}^2}\left(\mathbf{q}_1'^2-\left(\mathbf{q}_1'+\mathbf{k}\right)^2\right)\right]\mathcal{I}\left(\mathbf{q}_1'^2,\left(\mathbf{q}_1'+\mathbf{k}\right)^2,\mathbf{k}^2\right)\right\}
\end{aligned}$$

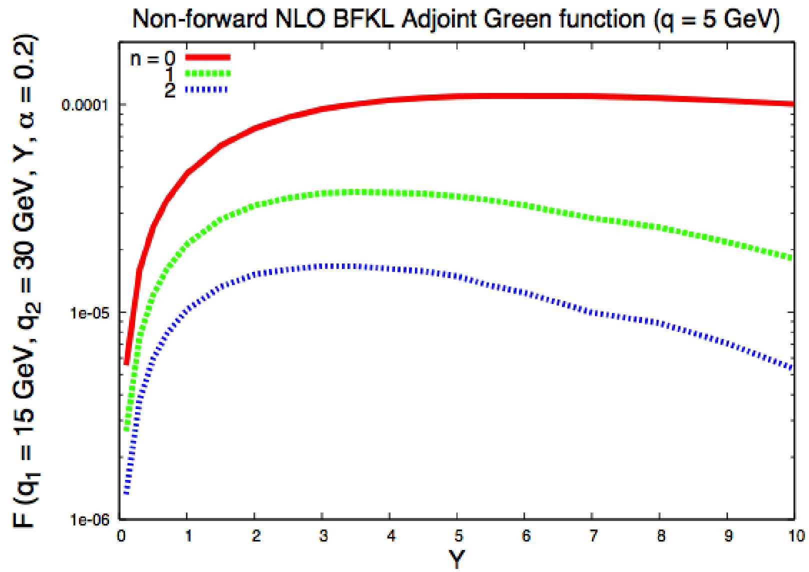
with

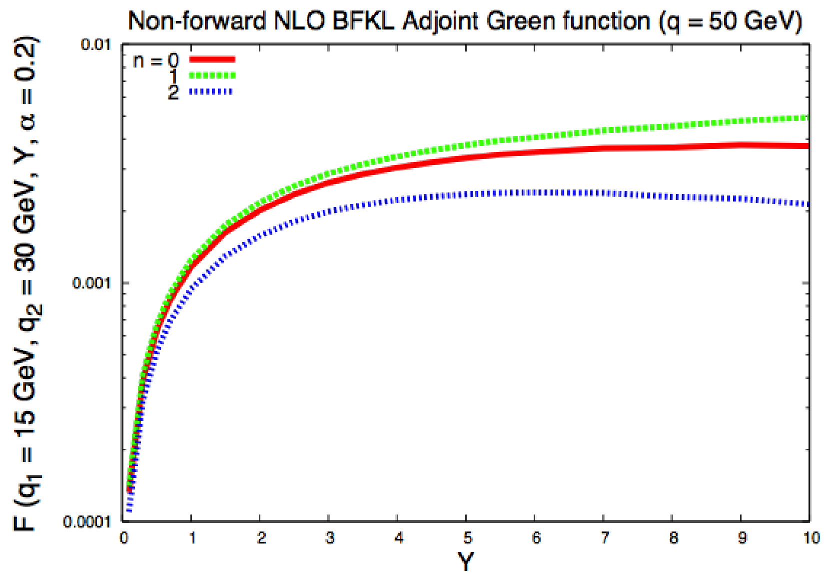
$$\mathcal{I}\left(\mathbf{p}^2,\mathbf{q}^2,\mathbf{r}^2\right) = \int_0^1 \frac{dx}{\mathbf{p}^2(1-x)+\mathbf{q}^2x-\mathbf{r}^2x(1-x)}\ln\left(\frac{\mathbf{p}^2(1-x)+\mathbf{q}^2x}{\mathbf{r}^2x(1-x)}\right).$$

Numerical results









Outlook

- We have studied the LO and NLO BFKL equation in the adjoint representation using Monte Carlo techniques
- By using numerical methods, it is possible to probe regions that analytic work cannot access
- The experience gained will be used for having a numerical solution of the BKP, the LO BKP project is currently underway.
- We would like to have a solid phenomenological study program for Odderon searches

Backup slides

$$f\left(\vec{k}_a, \vec{k}_b, \mathrm{Y}\right) = \sum_{n=-\infty}^{\infty} f_n\left(|\vec{k}_a|, |\vec{k}_b|, \mathrm{Y}\right) e^{in\theta}$$

$$f_n\left(|\vec{k}_a|, |\vec{k}_b|, \mathrm{Y}\right) = \frac{1}{\pi|\vec{k}_a||\vec{k}_b|} \int \frac{d\gamma}{2\pi i} \left(\frac{\vec{k}_a^2}{\vec{k}_b^2}\right)^{\gamma-\frac{1}{2}} e^{\omega_n(a,\gamma)\mathrm{Y}}$$

$$f_n\left(|\vec{k}_a|, |\vec{k}_b|, \mathrm{Y}\right) = \int_0^{2\pi} \frac{d\theta}{2\pi} f\left(\vec{k}_a, \vec{k}_b, \mathrm{Y}\right) \cos\left(n\theta\right)$$

$$\chi(n,\gamma) = 2\Psi(1) - \Psi\left(\gamma + \frac{n}{2}\right) - \Psi\left(1 - \gamma + \frac{n}{2}\right)$$

Backup slides

$$\int_{virtual} + \int_{real} = \int_{virtual+real,unres.} + \int_{real,res}$$