Standard and quasi-conformal BFKL kernels

V.S Fadin

Budker Institute of Nuclear Physics
Novosibirsk

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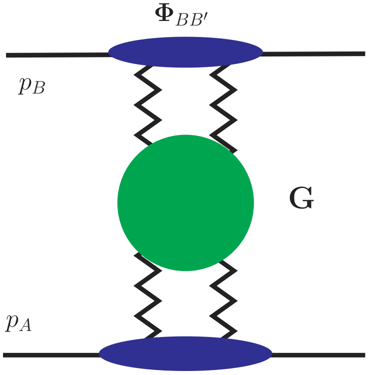
Contents

The BFKL (Balitsky-Fadin-Kuraev-Lipatov) approach is based on the remarkable property of QCD – gluon reggeization. The scattering amplitudes are represented by the convolution

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B}$$



Introduction

The universal (process independent) Greens's function G can be presented as

$$\hat{\mathcal{G}} = e^{Y\hat{\mathcal{K}}},$$

$\hat{\mathcal{K}}$ is the BFKL kernel, Y is the total rapidity $(Y = \ln(s/s_0))$. Talking about the BFKL approach, one usually means BFKL Pomeron, that is, a colourless state in the t -channel. But the approach is applicable for any colour state, which two gluons can form. For QCD, that is for tree colours, there are 6 irreducible representations:

$$\underline{1}, \underline{8_a}, \underline{8_s}, \underline{10}, \underline{\overline{10}}, \underline{27}.$$

For $N_c > 3$ there are 7 possible representations.
Now the kernel is known in the NLO both for forward scattering, i.e. for $t = 0$ and the colour singlet in the t–channel,
V.S. F., L.N. Lipatov, 1998
M. Ciafaloni, G. Camici, 1998

and for arbitrary t and any possible colour state in the t–channel
V. S. F., D. A. Gorbachev, 2000
V. S. F., R. Fiore, 2005
For phenomenological applications, the most interesting is the Pomeron. But from theoretical point of view the gluon channel (antisymmetric colour octet, or adjoint representation, in the t-channel) is even more important, first of all because of the gluon reggeization. The idea of the gluon reggeization appeared as the result of the fixed order calculations. Evidently it must be proved. It was done in using bootstrap relations, which follow from the requirement of compatibility of the multi-Regge form of amplitudes with the s-channel unitarity. Now fulfillment of these relations is proved in the NLO
V.S. F., M.G. Kozlov, A.V. Reznichenko, 2012

But there are at least two other reasons for significance of the kernel of the BFKL equation for the adjoint representations. One is related to the BKP equation
J. Bartels,1980
J. Kwiecinski, M. Praszalowicz, 1980
-the generalization of the BFKL equation to bound states consisting of three and more reggeized gluons, in particular the C-odd three gluon system — Odderon. The BFKL kernel for symmetric adjoint representation appears in the BKP equation for the odderon because any pair of the three reggeized gluons are in the colour octet state.
Recently, another application of the BFKL approach, related to the BDS ansatz
Z. Bern, L. J. Dixon and V. A. Smirnov, 2005
was extensively developed

The approach was used for verification of the BDS ansatz for the inelastic amplitudes in $N = 4$ SUSY and calculation of the remainder factor
J. Bartels, L. N. Lipatov, A. Sabio Vera, 2009
L. N. Lipatov and A. Prygarin, 2011
It was demonstrated that the BDS amplitude $M_{2\to 4}^{BDS}$ should be multiplied by the factor containing the contribution of the Mandelstam cuts, and this contribution was found in the LLA and in the NLA
V.S. F. and L. N. Lipatov, 2011
At large N_c, when only planar diagrams are taken into account, there is degeneracy in signature, i.e. no difference between symmetric ant antisymmetric adjoint representations.

Möbius representation

For scattering of colourless objects the BFKL equation can be written in the Möbius invariant form

L. N. Lipatov, 1989

The Möbius invariance can be made evident by transformation from the transverse momentum to the transverse coordinate representation

V.S. F, R. Fiore, A. Papa, 2007

For scattering of colourless objects, one can use gauge invariance of the colour singlet impact factors and the colour singlet BFKL kernel and omit the terms in the kernel proportional to $\delta(\vec{r}_{1'2'})$, as well as change the terms independent either of $\vec{r}_1$ or of $\vec{r}_2$ in such a way that the resulting kernel provides vanishing of cross-sections for scattering of zero-size dipoles.

The kernel obtained in such a way dipole or in the impact parameter (transverse coordinate) space is called Möbius form of the BFKL kernel.

(Quasi)conformal kernel

The Möbius form of the NLO kernel can be written as

$$\langle \vec r_1 \vec r_2|\hat{\mathcal{K}}_M|\vec r_1{}' \vec r_2{}'\rangle = \delta(\vec r_{11'})\delta(\vec r_{22'}) \int d\vec r_0\, g_0(\vec r_1, \vec r_2; \vec r_0)$$

$$+\delta(\vec r_{11'})g_1(\vec r_1, \vec r_2; \vec r_2{}') + \delta(\vec r_{22'})g_1(\vec r_2, \vec r_1; \vec r_1{}') + \frac{1}{\pi} g_2(\vec r_1, \vec r_2; \vec r_1{}', \vec r_2{}')$$

where $\vec r_{ij'} = \vec r_i - \vec r_j{}'$, with the functions $g_{1,2}$ turning into zero when their first two arguments coincide. The first three terms contain ultraviolet singularities which cancel in their sum, as well as in the LO, with account of the "dipole" property of the "target" impact factors. The coefficient of $\delta(\vec r_{11'})\delta(\vec r_{22'})$ is written in the integral form in order to make the cancellation evident. The term $g_2(\vec r_1, \vec r_2; \vec r_1{}', \vec r_2{}')$ is absent in the LO because the LO kernel in the momentum space does not contain terms depending on all three independent momenta simultaneously.

(Quasi)conformal kernel

The Möbius form obtained by transformation of the ”standard” kernel defined in momentum space according to the prescription (called standard)
V.S. F., R. Fiore, 1998
into impact parameter space turned out not (quasi)-conformal. But in the NLO kernel there is an ambiguity, analogous to the well known ambiguity of the NLO anomalous dimensions, because it is possible to redistribute radiative corrections between the kernel and the impact factors. The ambiguity permits to make transformations

$$\hat{\mathcal{K}} \to \hat{\mathcal{K}} - \alpha_s[\hat{\mathcal{K}}^{(B)}, \hat{U}]$$

conserving the LO kernel $\hat{\mathcal{K}}^{(B)}$ (which is fixed in our case by the requirement of conformal invariance of its Möbius form) and changing the NLO part of the kernel. Using this transformation it turns out possible to make the Möbius form quasi-conformal.

(Quasi)conformal kernel

$$g_0^{QC}(\vec{r}_1,\vec{r}_2;\vec{r}_0) = 6\pi\zeta(3)\,\delta(\vec{r}_0) - g_1^{QC}(\vec{r}_1,\vec{r}_2;\vec{r}_0)\;,$$

$$g_1^{QC}(\vec{r}_1,\vec{r}_2;\vec{r}_2{}') = \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{22'}^{\,2}\vec{r}_{12'}^{\,2}}\Bigg[\frac{\beta_0}{2N_c}\Bigg(\ln\left(\frac{\vec{r}_{12}^{\,2}\mu^2}{4e^{2\psi(1)}}\right)+\frac{\vec{r}_{12'}^{\,2}-\vec{r}_{22'}^{\,2}}{\vec{r}_{12}^{\,2}}\ln\left(\frac{\vec{r}_{22'}^{\,2}}{\vec{r}_{12'}^{\,2}}\right)\Bigg)$$

$$+\frac{67}{18}-\zeta(2)-\frac{5a_f+2a_s}{9}\Bigg],\quad \beta_0=\frac{11N_c}{3}-\frac{2a_f}{3}-\frac{a_s}{6},$$

$$g_2^{QC}(\vec{r}_1,\vec{r}_2;\vec{r}_1{}',\vec{r}_2{}') = \frac{1}{\vec{r}_{1'2'}^{\,4}}\Bigg(\frac{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}-2\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}{d}\ln\left(\frac{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}\right)-1\Bigg)\Big(1-b$$

$$+\frac{b_s}{2}\Big)+\Bigg(\frac{(2b_s-3b_f)}{2\vec{r}_{1'2'}^{\,2}}\frac{\vec{r}_{12}^{\,2}}{d}+\frac{1}{2\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}\left(\frac{\vec{r}_{12}^{\,4}}{d}-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{1'2'}^{\,2}}\right)\Bigg)\ln\left(\frac{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}}\right)$$

$$+\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\vec{r}_{1'2'}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}\vec{r}_{1'2'}^{\,2}}{\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}}\right),\quad d=\vec{r}_{12'}^{\,2}\vec{r}_{21'}^{\,2}-\vec{r}_{11'}^{\,2}\vec{r}_{22'}^{\,2}\;.$$

Here $a_j = 2\kappa_j n_j T_j$, $j = f, s$, T_j are defined by the relations

(Quasi)conformal kernel

$$\mathrm{Tr}\left(T_j^a T_j^b\right) = T_j \delta^{ab},$$

$$b_j = \frac{4 n_j \kappa_j}{N_c^2 - 1} \mathrm{Tr}\left(\frac{C_j^2}{N_c^2} - \frac{C_j}{2N_c}\right), \quad C_j = T_j^a T_j^a,$$

T_j^a are the colour group generators, κ_f (κ_s) is equal to $1/2$ for Majorana fermions (neutral scalars) in self-conjugated representations and 1 otherwise.

This form of the NLO kernel is immeasurably simple compared with the kernel in the momentum space. In fact, there are three reasons for the simplicity:

— Möbius representation (i.e. limitation of space of functions),

— transformation $\emptyset K \to \emptyset K - [\emptyset K^B \hat{U}]$ with the operator $\hat{U} = \hat{U}_1 + \hat{U}_2$,

— use of impact parameter space.

Difference between standard and quasi-conformal kernels

The simplicity of the Möbius form of the quasi-conformal NLO BFKL kernel suggested to use just this form for finding the kernel in the momentum space. The way to do that was not evident, and even the possibility to do it seemed doubtful, because the Möbius form is defined on a special class of functions in the coordinate space. However, it was shown
V.S. F., R. Fiore, A.V. Grabovsky, A. Papa, 2011
that such possibility exists due to the gauge invariance of the kernel and the way to obtain the kernel in the momentum space from its Möbius form was elaborated. But technically obtaining it turned out to be not easy.

Difference between standard and quasi-conformal kernels

An explicit form of the operator $\hat{U}$ in the momentum space
V.S. F., R. Fiore, A.V. Grabovsky, A. Papa, 2011

$$\langle \vec{q}_1, \vec{q}_2|\alpha_s \hat{U}|\vec{q}_1{}', \vec{q}_2{}'\rangle = \delta(\vec{q}_{11'} + \vec{q}_{22'})\frac{\alpha_s N_c}{4\pi^2} R_u(\vec{q}_1, \vec{q}_2; \vec{k})$$

$$-\frac{\alpha_s \beta_0}{8\pi} \ln\left(\vec{q}_1{}^2 \vec{q}_2{}^2\right) \delta(\vec{q}_{11'})\delta(\vec{q}_{22'}) \ ,$$

where β_0 is the first coefficient of the Gell-Mann–Low function,

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f$$

and

$$R_u(\vec{q}_1, \vec{q}_2; \vec{k}) = \frac{1}{\vec{q}_1{}^2} \ln\left(\frac{\vec{q}_1{}'^2 \vec{q}_2{}^2}{\vec{k}^2 \vec{q}^{\,2}}\right) + \frac{1}{\vec{q}_2{}^2} \ln\left(\frac{\vec{q}_2{}'^2 \vec{q}_1{}^2}{\vec{k}^2 \vec{q}^{\,2}}\right) + \frac{1}{\vec{k}^2} \ln\left(\frac{\vec{q}_1{}'^2 \vec{q}_2{}'^2}{\vec{q}_1{}^2 \vec{q}_2{}^2}\right)$$

$$-2\frac{\vec{q}_1 \vec{k}}{\vec{k}^2 \vec{q}_1{}^2} \ln\left(\frac{\vec{q}_1{}'^2}{\vec{k}^2}\right) + 2\frac{\vec{q}_2 \vec{k}}{\vec{k}^2 \vec{q}_2{}^2} \ln\left(\frac{\vec{q}_2{}'^2}{\vec{k}^2}\right) - 2\frac{\vec{q}_1 \vec{q}_2}{\vec{q}_1{}^2 \vec{q}_2{}^2} \ln\left(\frac{\vec{q}^{\,2}}{\vec{k}^2}\right) \ .$$

Difference between standard and quasi-conformal kernels

Note that R_u has the gauge invariance properties:

$$R_u(\vec{q}_1, \vec{q}_2; \vec{q}_1) = R_u(\vec{q}_1, \vec{q}_2; -\vec{q}_2) = 0,$$

$$(\vec{q}_1^{\,2}\vec{q}_2^{\,2}R_u(\vec{q}_1, \vec{q}_2; \vec{k}))|_{\vec{q}_1=0} = (\vec{q}_1^{\,2}\vec{q}_2^{\,2}R_u(\vec{q}_1, \vec{q}_2; \vec{k}))|_{\vec{q}_2=0} = 0 \ .$$

Indeed, these properties are required to conserve the gauge invariance.
The difference between the standard BFKL kernel, defined according to the prescriptions
V.S. F., R. Fiore,1998
and the quasi-conformal BFKL kernel turned out to be rather simple
V.S. F., R. Fiore, A. Papa, 2012

Difference between standard and quasi-conformal kernels

$$D(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s^2 N_c^2}{8\pi^3}\Bigg[-\frac{\beta_0}{2N_c}\left(\frac{2}{\vec{k}^{\,2}} - 2\frac{\vec{q}_1\vec{k}}{\vec{k}^{\,2}\vec{q}_1^{\,2}} + 2\frac{\vec{q}_2\vec{k}}{\vec{k}^{\,2}\vec{q}_2^{\,2}} - 2\frac{\vec{q}_1\vec{q}_2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right)$$

$$\times \ln\left(\frac{\vec{q}_1{}'^2\vec{q}_2{}'^2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\right) + \frac{\vec{q}_1{}'^2}{\vec{q}_1^{\,2}\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2{}'^2}{\vec{q}_2^{\,2}\vec{q}_1{}'^2}\right)\ln\left(\frac{\vec{q}_2^{\,2}\vec{q}_1{}'^2}{\vec{q}^{\,2}\vec{k}^{\,2}}\right) + \frac{\vec{q}_2{}'^2}{\vec{q}_2^{\,2}\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_2^{\,2}\vec{q}_1{}'^2}{\vec{q}_1^{\,2}\vec{q}_2{}'^2}\right)$$

$$\times \ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2{}'^2}{\vec{q}^{\,2}\vec{k}^{\,2}}\right) - 4\left(\frac{[\vec{q}_1\times\vec{q}_2]}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}} + \frac{[\vec{q}_1\times\vec{k}]}{\vec{q}_1^{\,2}\vec{k}^{\,2}} + \frac{[\vec{q}_2\times\vec{k}]}{\vec{q}_2^{\,2}\vec{k}^{\,2}}\right)$$

$$\left([\vec{q}_1\times\vec{q}_2] I_{\vec{q}_1,\vec{q}_2} - [\vec{q}_1{}'\times\vec{q}_2{}'] I_{\vec{q}_1{}',\vec{q}_2{}'}\right)\Bigg].$$

The most natural conclusion is that the simplicity of the Möbius form of the quasi-conformal kernel is caused mainly by using the impact parameter space. The other possibility is that the quasi-conformal kernel can be written in simple form also in the transverse momentum space.

Standard and Möbius invariant forms

The modified (with subtracted gluon trajectory depending on total t-channel momenta) kernel in the antisymmetric adjoint representtion can be written as follows:

$$K(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = K^B(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \left(1 - \frac{\alpha_s N_c}{2\pi}\zeta(2)\right)$$

$$+\delta^{(2)}(\vec{q}_1 - \vec{q}_1{}') \frac{\vec{q}_1{}^2 \vec{q}_2{}^2}{\vec{q}^{\,2}} \frac{\alpha_s^2 N_c^2}{4\pi^2} 3\zeta(3) + \frac{\alpha_s^2 N_c^2}{32\pi^3} R(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \,,$$

K^B is the leading order kernel, which can be written in the explicitly Möbius invariant form:

$$K^B(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = -\delta^{(2)}(\vec{q}_1 - \vec{q}_1{}') \frac{\vec{q}_1{}^2 \vec{q}_2{}^2}{\vec{q}^{\,2}} \frac{\alpha_s N_c}{4\pi^2} \int \frac{\vec{q}^{\,2} d^2 l}{(\vec{q}_1 - \vec{l})^2 (\vec{q}_2 + \vec{l})^2}$$

$$\left(\frac{\vec{q}_1^{\,2}(\vec{q}_2 + \vec{l})^2 + \vec{q}_2^{\,2}(\vec{q}_1 - \vec{l})^2}{\vec{q}^{\,2} \vec{l}^{\,2}} - 1\right) + \frac{\alpha_s N_c}{4\pi^2} \left(\frac{\vec{q}_1{}^2 \vec{q}_2{}'^{\,2} + \vec{q}_1{}'^{\,2} \vec{q}_2^{\,2}}{\vec{q}^{\,2} \vec{k}^{\,2}} - 1\right) .$$

$$R(\vec{q}_1, \vec{q}_1{}', \vec{q}) = \frac{1}{2}\left(\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\right) + \ln\left(\frac{\vec{q}_1{}'^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2{}'^{2}}{\vec{q}^{\,2}}\right)\right.$$

$$\left. + \ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1{}'^{\,2}}\right)\right) - \frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} + \vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}}{\vec{q}^{\,2}\vec{k}^{\,2}}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1{}'^{\,2}}\right) - \frac{\vec{q}_1^{\,2}\vec{q}_2{}'^{\,2} - \vec{q}_2^{\,2}\vec{q}_1{}'^{\,2}}{2\vec{q}^{\,2}\vec{k}^{\,2}}$$

$$\times \ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1{}'^{\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_1{}'^{\,2}}{\vec{k}^{\,4}}\right) + 4\frac{(\vec{k}\times\vec{q}_1)}{\vec{q}^{\,2}\vec{k}^{\,2}}\left(\vec{k}^{\,2}(\vec{q}_1\times\vec{q}_2)\right.$$

$$\left. - \vec{q}_1^{\;2}(\vec{k}\times\vec{q}_2) - \vec{q}_2^{\;2}(\vec{k}\times\vec{q}_1)\right) I_{\vec{q}_1,-\vec{k}} + (\vec{q}_1 \leftrightarrow -\vec{q}_2,\ \ \vec{q}_1{}' \leftrightarrow -\vec{q}_2{}')\ .$$

$$\vec{k} = \vec{q}_1 - \vec{q}_1{}' = \vec{q}_2{}' - \vec{q}_2,\ \ (\vec{a}\times\vec{b}) = a_x b_y - a_y b_x$$

$$I_{\vec{p},\vec{q}} = \int_0^1 \frac{dx}{(\vec{p}+x\vec{q})^2}\ln\left(\frac{\vec{p}^{\,2}}{x^2\vec{q}^{\,2}}\right)\ .$$

Standard and Möbius invariant forms

The contribution $R(\vec{q}_1, \vec{q}_1{}', \vec{q})$ violates the Möbius invariance. In the paper
V.S. F., L.N. Lipatov, 2012
it was assumed that there is a conformal invariant representation of the kernel. Since its eigenvalues do not depend on the representation and on the total momentum transfer, they were found using the limit

$$|\vec{q}_1| \sim |\vec{q}_1{}'| \ll |\vec{q}| \approx |\vec{q}_2| \approx |\vec{q}_2{}'| \,.$$

In this limit

$$K(z) = K^B(z)\left(1 - \frac{\alpha_s N_c}{2\pi}\zeta(2)\right) + \delta^{(2)}(1-z)\frac{\alpha_s^2 N_c^2}{4\pi^2}3\zeta(3) + \frac{\alpha_s^2 N_c^2}{32\pi^3}R(z)\,,$$

where $z = q_1/q_1'$,

$$K^B(z) = \frac{\alpha_s N_c}{8\pi^2}\left(\frac{z+z^*}{|1-z|^2} - \delta^{(2)}(1-z)\int \frac{d\vec{l}}{|l|^2}\frac{l+l^*}{|1-l|^2}\right)\,,$$

Standard and Möbius invariant forms

$$R(z)=\left(\frac{1}{2}-\frac{1+|z|^2}{|1-z|^2}\right)\ln^2|z|^2-\frac{1-|z|^2}{2|1-z|^2}\ln|z|^2\ln\frac{|1-z|^4}{|z|^2|}$$

$$+\left(\frac{1}{1-z}-\frac{1}{1-z^*}\right)(z-z^*)\int_0^1\frac{dx}{|x-z|^2}\ln\frac{|z|^2}{x^2}\,.$$

$p=p_x+ip_y$ and $p^*=p_x-ip_y$ for the two-dimensional vectors $\vec{p}=(p_x,ip_y)$.Vice versa, two complex numbers z and z^* are equivalent to the vector $\vec{z}$ with the components $(z+z^*)/2$ and $(z-z^*)/(2i)$.

Standard and Möbius invariant forms

Due to the Möbius invariance, the kernel $K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ can be written as $K(z)$ with the argument $z = q_1 q_2'/(q_2 q_1')$. If we denote

$$K(\vec{q}_1, \vec{q}_1{}'; \vec{q}) - K_c(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = \frac{\alpha_s^2 N_c^2}{32\pi^3} \Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) \,,$$

then

$$\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q}) = R(\vec{q}_1, \vec{q}_1{}'; \vec{q}) - R(z) \,,$$

Since $R(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ is not conformal invariant, $\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q})$ cannot be written using the single variable z. Using relations between dilogarithms it can be written in the form

Standard and Möbius invariant forms

$$\Delta(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = \ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}} + \ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}$$

$$-2\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}+\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\ln\frac{\vec{q}_2^{\,2}}{\vec{q}_2^{\,\prime\,2}}+\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2}-\vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}\vec{q}^{\,2}}\left(\ln\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_2^{\,\prime\,2}}{\vec{q}^{\,2}}\right.$$

$$\left.-\ln\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\ln\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right)+\frac{4}{\vec{q}^{\,2}\vec{k}^{\,2}}\left(\vec{k}^{\,2}[\vec{q}_1\times\vec{q}_2]-\vec{q}_1^{\,2}[\vec{k}\times\vec{q}_2]-\vec{q}_2^{\,2}[\vec{k}\times\vec{q}_1]\right)$$

$$\times\left([\vec{q}_1\times\vec{q}_2]I_{\vec{q}_1,\vec{q}_2}-[\vec{q}_1^{\,\prime}\times\vec{q}_2^{\,\prime}]I_{\vec{q}_1^{\,\prime},\vec{q}_2^{\,\prime}}\right)\,.$$

Important properties of Δ are its symmetries with respect to the exchanges $\vec{q}_1 \leftrightarrow -\vec{q}_2$, $\vec{q}_1^{\,\prime} \leftrightarrow -\vec{q}_2^{\,\prime}$ and $\vec{q}_i \leftrightarrow -\vec{q}_i$, as well as the gauge invariance (vanishing at zero momentum of each reggeon), which are easily seen from this representation.

Similarity transformation

If the kernels $\hat{\mathcal{K}}$ and $\hat{\mathcal{K}}_c$ coincide in the leading order and are related by a similarity transformation, there must exist an operator $\hat{\mathcal{O}}$ satisfying the commutation relation

$$[\hat{\mathcal{K}}^B, \hat{O}] = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi} \hat{\Delta} \,.$$

One can find a formal expression for this operator allowing to construct the similarity transformation in perturbation theory. Indeed, it is enough to calculate the matrix element of the above commutation relation between the eigenfunctions of the Born kernel with the corresponding eigenvalues $\omega^B_{\nu n}$ in the form

$$\left(\omega^B_{\nu' n'} - \omega^B_{\nu n}\right) \langle \nu' n' | \hat{\mathcal{O}} | \nu n \rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi} \langle \nu' n' | \hat{\Delta} | \nu n \rangle .$$

It can be seen from this equation that the solution $\hat{\mathcal{O}}$ exists only if the operator $\hat{\Delta}$ has vanishing matrix elements between states with the same eigenvalues. In this case

Similarity transformation

$$\hat{O} = \frac{\alpha_s^2}{32\pi^3}\sum_{n,n'}\int d\nu d\nu' \frac{|\nu' n'\rangle\langle\nu' n'|\hat{\Delta}|\nu n\rangle\langle\nu n|}{\omega^B_{\nu' n'} - \omega^B_{\nu n}}$$

$$\langle\vec{q}_1, q_2|\hat{\mathcal{O}}|\vec{q}_1{}', \vec{q}_2{}'\rangle = \left(\frac{\alpha_s}{2\pi}\right)^2 \frac{1}{8\pi}\sum_{n,n'}\int d\nu \int d\nu'$$

$$\frac{\langle\vec{q}_1, q_2|\nu' n'\rangle\langle\nu' n'|\hat{\Delta}|\nu n\rangle\langle\nu n|\vec{q}_1{}', \vec{q}_2{}'\rangle}{\omega^B_{\nu' n'} - \omega^B_{\nu n}} .$$

Since the kernel $\hat{\Delta}$ is known in the momentum space, we can transform it into the (n, ν) representation,

$$\langle\nu n|\hat{\Delta}|\nu' n'\rangle = \int \frac{\vec{q}^{\,2} d\vec{q}_1}{\vec{q}_1^{\,2}(\vec{q}-\vec{q}_1)^2}\int \frac{\vec{q}^{\,2} d\vec{q}_1{}'}{\vec{q}_1{}'^{\,2}(\vec{q}-\vec{q}_1{}')^2}\langle\nu' n'|\vec{q}_1{}', \vec{q}_2{}'\rangle$$

$$\times\langle\Delta(\vec{q}_1, \vec{q}_1{}'; \vec{q})\rangle\langle\vec{q}_1, \vec{q}_2|\nu n\rangle$$

using the known eigenfunctions in the momentum space, which allows to find the matrix element $\langle\vec{q}_1, \vec{q}_2|\hat{\mathcal{O}}|\vec{q}_1{}', \vec{q}_2{}'\rangle$.

Similarity transformation

But the final expression for $\hat{\mathcal{O}}$ obtained by this method is rather complicated. It turned out more simple to guess the form of the operator $\hat{O}$. We supposed that the conformal invariant kernel can be obtained using the substraction procedure different from the standard one. If it is so, then the operator $\hat{\mathcal{O}}$ in the momentum representation must be proportional $K^B(\vec{q}_1, \vec{q}_1\,'; \vec{q})$. Then, from the symmetry arguments it follows that the most attractive candidate for $\hat{\mathcal{O}}$ is

$$\hat{\mathcal{O}}_t = C\left[\ln\left(\hat{\vec{q}}_1^{\,2}\hat{\vec{q}}_2^{\,2}\right), \hat{\mathcal{K}}_r^B\right] ,$$

where C is some coefficient. We checked this asumption and found

$$O(\vec{q}_1, \vec{q}_1\,'; \vec{q}) \;=\; \frac{\alpha_s N_c}{16\pi^2}\left(\frac{\vec{q}_1^{\,2}\vec{q}_2\,'^{\,2} + \vec{q}_1\,'^{\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}_1\,'^{\,2}\vec{q}_2\,'^{\,2}}\right)$$

Summary

- It is proved that complete colour singlet BFKL kernel can be restored from its Möbius form
- The difference between quasi-conformal and standard colour singlet BFKL kernels in the momentum space is found
- This difference turned out to be rather simple
- It is proved that in the adjoint representation of the colour group quasi-conformal and standard BFKL kernels are connected by simularity transformation.
- The simularity transformation is found explicitly