REVERSE UNITARITY AND THRESHOLD EXPANSIONS

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MOTIVATION

- Drell-Yan production of lepton pairs

  - Our sanity check at hadron colliders
  - W mass and electroweak couplings, parton densities,…

- Higgs production

  - Major indicator of new physics
  - Important for Higgs coupling extractions

- 1% and 8% precision of NNLO calculations.

- Do we believe these uncertainties? YES, but let's make sure **NNNLO**

EXISTING NNLO METHODS

- Pioneering work by van Neerven et al in Drell-Yan.

  - computing the inclusive cross-section in the soft limit $z \equiv \frac{M_V^2}{\hat{s}} \to 1.$
  - followed by complete calculation for arbitrary partonic energy.

- Additional techniques for Higgs production

  - Soft limit (*Catani, de Florian, Grazzini; Harlander,Kilgore*)
  - Systematic method for threshold expansion and resuming of the series (*Harlander,Kilgore*)

$$\hat{\sigma}_{RR} = (1-z)^{-1-4\epsilon}\left[a_1 + a_2(1-z) + a_3(1-z)^2 + \ldots\right]$$

$$\begin{aligned}\hat{\sigma}_{RV} &= (1-z)^{-1-4\epsilon}\left[b_1 + b_2(1-z) + b_3(1-z)^2 + \ldots\right] \\ &+ (1-z)^{-1-2\epsilon}\left[c_1 + c_2(1-z) + c_3(1-z)^2 + \ldots\right]\end{aligned}$$

REVERSE UNITARITY

Melnikov, CA

- Convert phase-space integrals into loop integrals.

$$\delta\left(p^2 - M^2\right) \to \frac{i}{p^2 - M^2} - c.c.$$

can almost forget about it

- Use IBP identities and the Laporta algorithm to reduce phase-space integrals into master integrals

$$\int d^dk \frac{\partial}{\partial k_\mu} \frac{q^\mu}{k^2 \ldots} = 0 \rightsquigarrow I_1 + I_2 + \ldots = 0$$

- Simplification for cut propagators.

$$\left(\frac{i}{k^2}\right)^n \to 0, \quad n = 0, -1, 2, \ldots$$

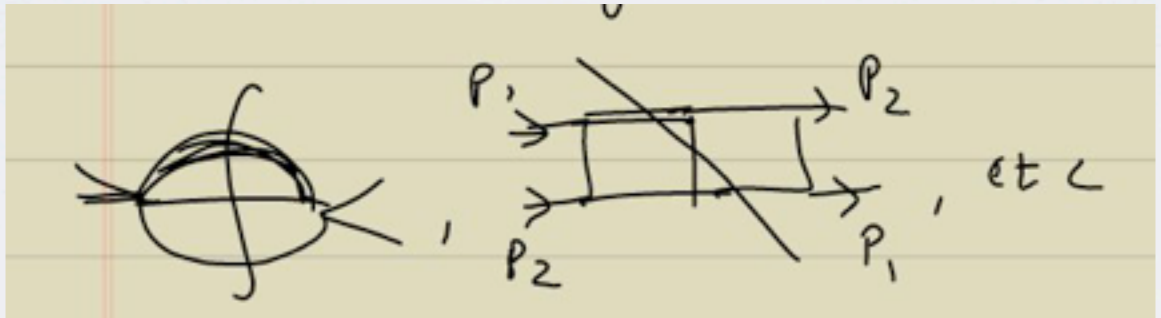
- Few remaining master integrals. Solved using differential equations, derived and solved in the same way as for loop master integrals (*Kotikov; Gehrmann, Remiddi, Smirnov, Veretin, ...*)

REVERSE UNITARITY

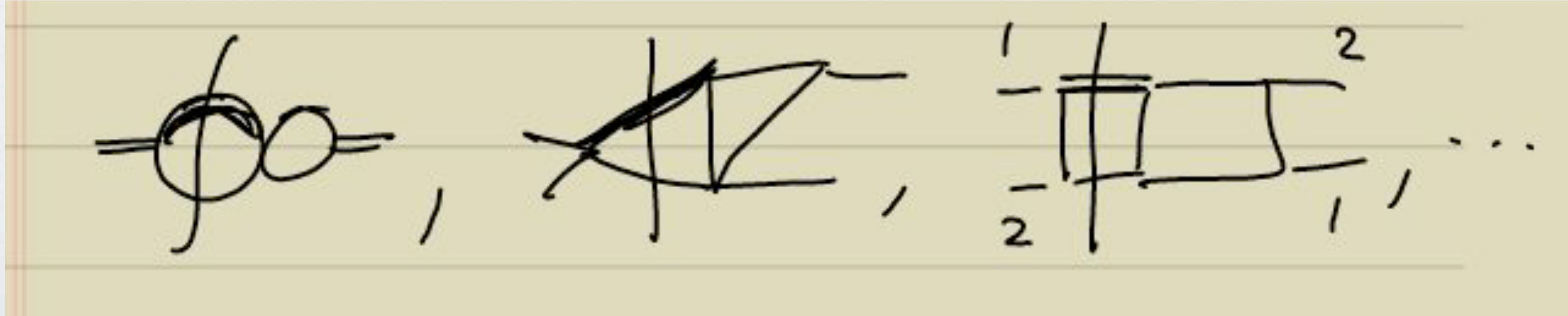
Melnikov, CA

- Solving the differential equations of the master integrals requires a boundary condition: SOFT LIMIT

- 18 double real-radiation master integrals



- 7 real-virtual master integrals



FROM NNLO TO NNNLO

	NLO	NNLO	NNNLO
topologies	1	11	215
master integrals per topology	1	~5	~25
total number of master integrals	1	18	~1000
	integrations over real radiation tree-level graphs		

- *Sheer magnitude of such a calculation is frightening*
- *But, we can hope in sharpening our methods*

IN THIS TALK

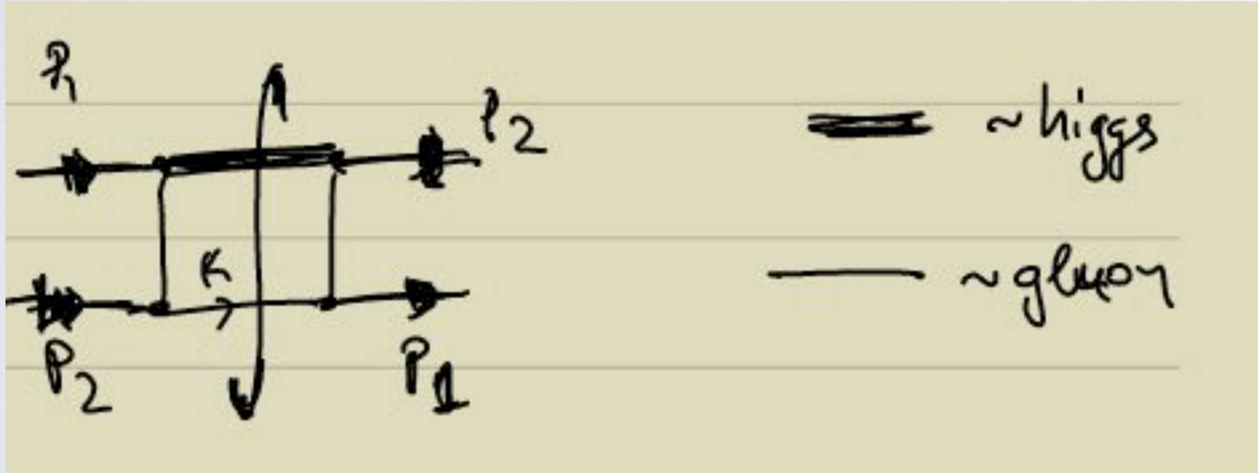
- Threshold series expansion with the "reverse unitarity" method
  - $\mathbf{z=1}$ *limit is extremely useful as a first step towards a complete calculation*
  - *necessary boundary condition for solving master integral differential equations*
  - *important contribution to the cross-section*

- The method allows for a systematic expansion around the soft limit, acquiring as many terms in the series as computer power permits us to do so.

- Enormous simplification permitting the use of IBP identities directly in the soft limit.

THE NLO REAL RADIATION EXAMPLE

Consider the NLO real radiation topology:



$$I\left[\nu_1,\nu_2\right] = \int d^d k \frac{\delta\left((p_{12}-k)^2 - M_V^2\right)\delta\left(k^2\right)}{\left[(k-p_1)^2\right]^{\nu_1}\left[(k-p_2)^2\right]^{\nu_2}}$$

two-scale integral

$$\nu_1,\nu_2 = \ldots,-2,-1,0,1,2,\ldots$$

Scaling of the gluon momentum: $k = \bar{z} \quad l, \quad \bar{z} \equiv 1 - z = 1 \frac{M_V^2}{\hat{s}}$

(no approximation made)

$$I\left[\nu_1,\nu_2\right] = \bar{z}^{1-\nu_{12}-2\epsilon} I_{reg}\left[\nu_1,\nu_2\right]$$

exponent X divergent

= LOGS

$$I_{reg}\left[\nu_1,\nu_2\right] = \int d^d l \frac{\delta\left((l-p_{12})^2\right)\delta\left(l^2\right)}{\left[(l-p_1)^2\right]^{\nu_1}\left[(l-p_2)^2\right]^{\nu_2}}$$

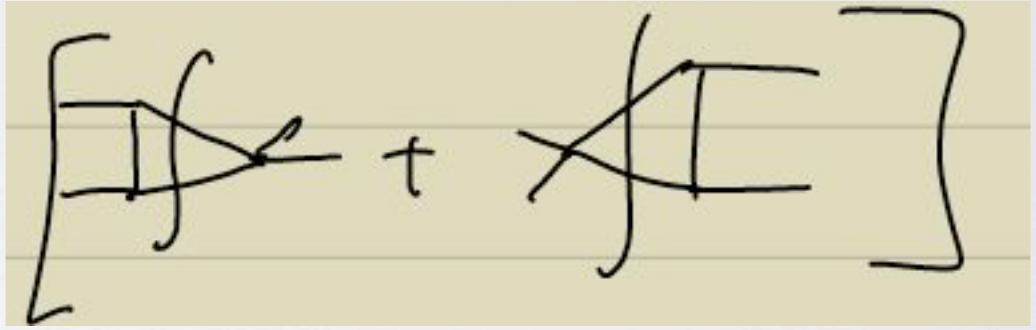
one-scale integral

THE NLO REAL RADIATION EXAMPLE

$$I_{reg}\left[\nu_1,\nu_2\right]=\int d^d l \frac{\delta\left((l-p_{12})^2\right)\delta\left(l^2\right)}{\left[(l-p_1)^2\right]^{\nu_1}\left[(l-p_2)^2\right]^{\nu_2}}$$

Trivial to perform the integration over the rescaled momentum.
But, let's resist the temptation.

$$(l-p_1)^2+(l-p_2)^2=\cancel{(l-p_{12})^2}+\cancel{l^2}-p_{12}^2 \rightsquigarrow$$



Double cut of one-loop form factor integrals

REVERSE UNITARITY:

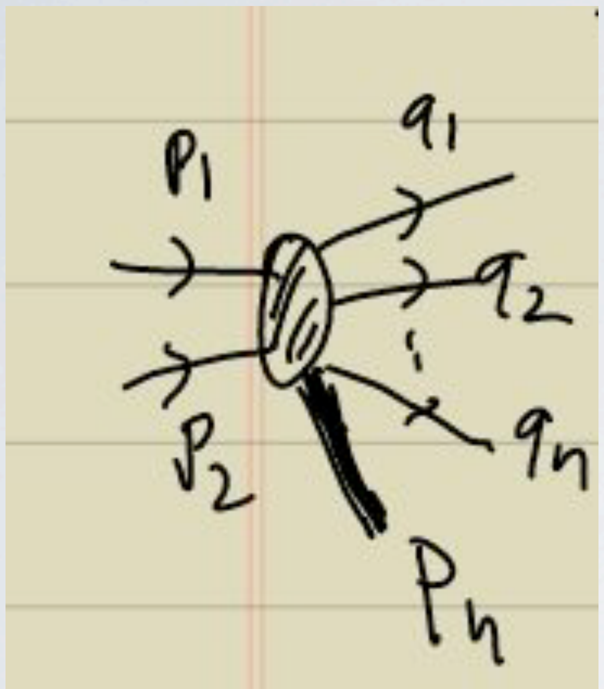
$$\delta\left(l^2\right),\delta\left((l-p_{12})^2\right)\to\cancel{\frac{i}{l^2}},\cancel{\frac{i}{(l-p_{12})^2}} \quad \rightsquigarrow$$

One master integral:
two massless particle phase-space measure

FIRST LESSONS

- Rescaling of gluon momenta which captures their behavior in the soft limit leads to phase-space integrals which depend only on a single kinematic scale (at NLO).
- Reverse unitarity and integration by parts minimize the amount of integrations (down to one integral).
- Calculation is almost entirely algebraic (=algorithmic).

MULTIPLE REAL EMISSION



$$I = \int d^dq_1 \ldots d^dq_N \delta(q_1^2) \ldots \delta(q_N^2) \delta\left((p_{12} - q_{12\ldots N})^2 - M_V^2\right) \left|\mathcal{M}^2\right|^2$$

reverse unitarity

$$I = \int \frac{d^dq_1 \ldots d^dq_N}{q_1^2 \ldots q_N^2 \left((p_{12} - q_{12\ldots N})^2 - M_V^2\right)} \left|\mathcal{M}^2\right|^2$$

SCALING: $q_i \to \bar{z} q_i$ ***(no approximation made yet)***

$$I = \bar{z}^{N(d-2)-1} \int \frac{d^dq_1 \ldots d^dq_N}{q_1^2 \ldots q_N^2 \left((p_{12} - q_{12\ldots N})^2 - z q_{12\ldots N}^2\right)} \left|\mathcal{M}\right|^2 (\bar{z} q_i, p_1, p_2)$$

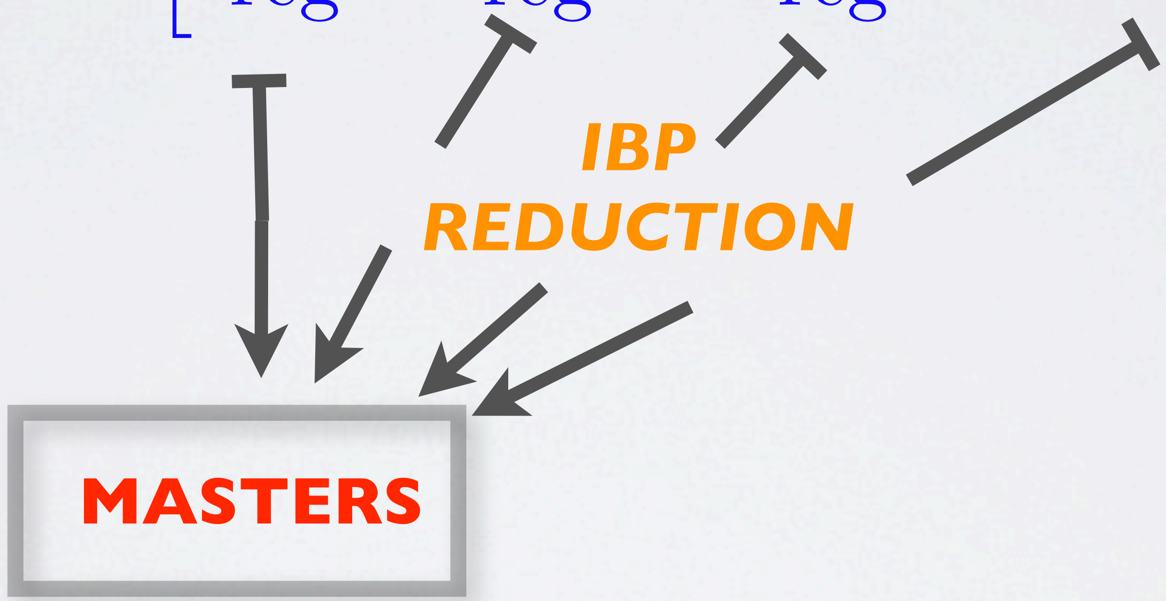
Correct asymptotic behavior

New integral depends on z. But it is regular at z=1.
Can be expanded INSIDE the integration sign.

MULTIPLE REAL RADIATION

Taylor expanding the integrand:

$$I = \bar{z}^{2N\epsilon-1}\left[I_{\mathrm{reg}}^{(0)} + I_{\mathrm{reg}}^{(1)}\bar{z} + I_{\mathrm{reg}}^{(2)}\bar{z}^2 + \ldots\right]$$

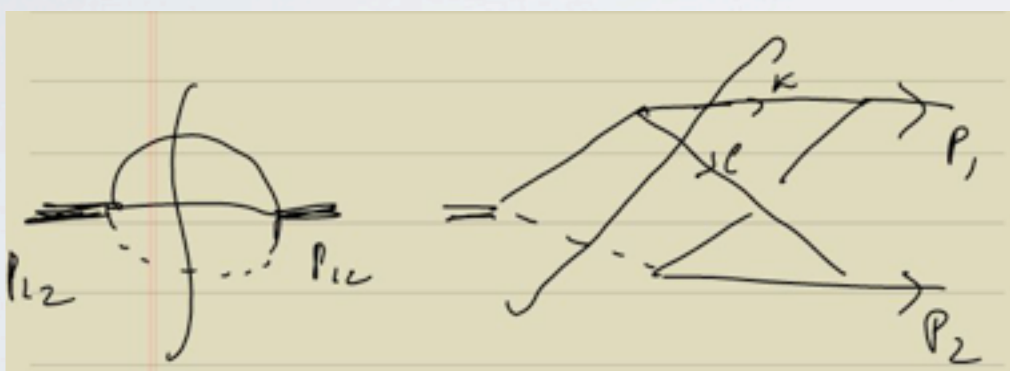


- Integrals of sub-leading terms reduce to the *same master integrals* as the ones making up the strict soft limit!
- Computing more terms in the series expansion is an algebraic problem
- no new master integrals emerge.

DOUBLE REAL RADIATION AT NNLO

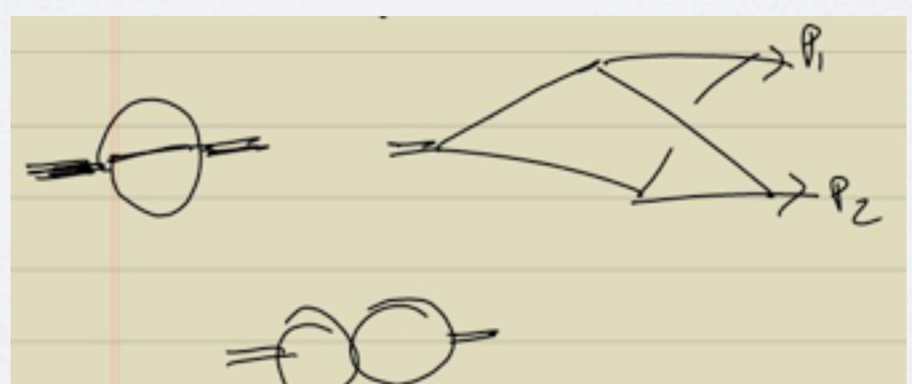
- 18 master integrals for a generic value of z.

- Two master integrals for the expansion around the soft limit:



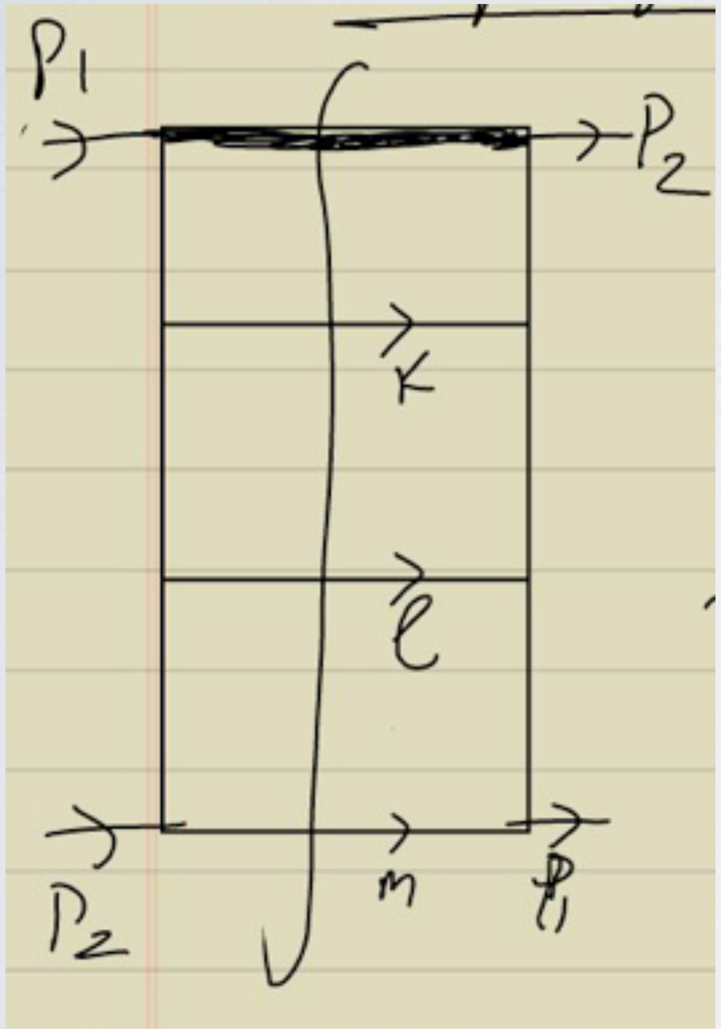
$$- - - - \equiv \left. \frac{1}{(p_{12}-k+l)^2 - z(k+l)^2} \right|_{z=1}$$

- Recall the master integrals for the two-loop form factor:

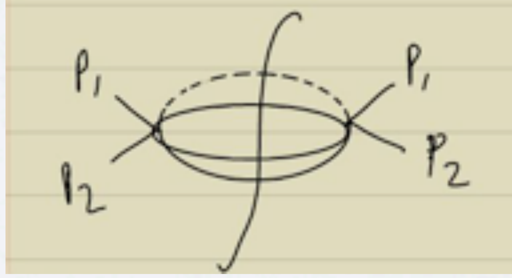


- They are of similar nature (coincide in the "wrong" limit $z=0$).

TRIPLE REAL RADIATION AT NNNLO



- Looked at some of the 215 topologies which appear at NNNLO.
- A verified example of a topology is shown here.
- 23 master integrals for generic z.
- These collapse to one very simple master integral, the phase-space measure, when expanding around threshold.



- Total number of master integrals ~ master integrals for the three-loop form factor with a quadruple cut (< 10).

WORK IN HAPPY PROGRESS

- Identifying and reducing to master integrals all triple real-radiation topologies
- Further steps:
  - extend this method to combinations of real and virtual radiation
  - requires scalings of loop-momenta in the soft limit and it is conceptually harder.
  - success for real-virtual master integrals at NNLO *(Dulat, Mistlberger)*
  - a lot more inventiveness is needed for RVV and RRV at NNNLO, but we hope to get guidance from the two-loop master integral computations for Higgs+1jet production *(Gehrmann,Remiddi)*
- Watch this space (but no promises :))