Local subtraction at NNLO

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The message

Q: How to compute NNLO cross sections?

A: Like you always thought you would:

1. Derive all relevant QCD factorization formulae
2. Stare at them and write down approximate cross section(s)
3. Integrate approximate cross section(s) over unresolved emission(s)

The message

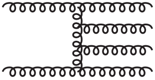
This can be done and it works and has nice properties.

NNLO cross section

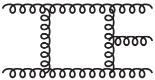
The NNLO correction to a generic *m*-jet cross section

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$$

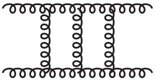
- Doubly-real: $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
  - tree level, $m+2$ parton kinematics
  - implicit IR poles from PS integration



- Real-virtual: $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
  - one-loop, $m+1$ parton kinematics
  - explicit/implicit IR poles form loop/PS integration



- Doubly-virtual: $\mathrm{d}\sigma^{\mathrm{VV}}_{m}$
  - two-loops, m parton kinematics
  - explicit IR poles form loop integration



Need to deal with implicit IR poles!

Basics of subtraction

Stragegy: rearrange IR singularities between various contributions by subtracting and adding back suitable approximate cross sections

Structure of subtractions: governed by jet functions

$$
\begin{aligned}
\sigma^{\rm NNLO} = & \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\} \\
& + \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\} \\
& + \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m
\end{aligned}
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$$+ \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$
$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

1. ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$

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$$+ \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$
$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

1. ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
2. ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$

Basics of subtraction

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\begin{aligned}
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&+ \int_{m+1} \Big\{ \Big[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \Big] J_{m+1} - \Big[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \Big(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \Big)^{\mathrm{A_1}} \Big] J_m \Big\} \\
&+ \int_m \Big\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \Big[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \Big] + \int_1 \Big[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \Big(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \Big)^{\mathrm{A_1}} \Big] \Big\} J_m
\end{aligned}
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1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
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3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$

Basics of subtraction

Stragegy: rearrange IR singularities between various contributions by subtracting and adding back suitable approximate cross sections

Structure of subtractions: governed by jet functions

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m \right] \right\}$$
$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$
$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2}$
4. $\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
5. $\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)^{\mathrm{A}_1}$ regularizes the singly-unresolved limit of $\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$

Basics of local subtraction

Goal: devise a subtraction scheme that is

- general: applicable to any process, any observable
- explicit: expressions include color, should be possible to implement a general version
- local: all color and spin correlations taken into account, mathematically rigorous
- efficient: option to constrain subtractions to near singular regions, also important check

Use full squared matrix elements, including color

Construct subtraction terms based on

- factorization of squared matrix elements in IR limits as embodied by standard QCD factorization formulae
- exact factorization of phase space via momentum remappings

Constructing the subtraction terms - I

Factorization of squared matrix elements in IR limits

$$\mathbf{X}_R |\mathcal{M}_{m+p}(\{p\}_{m+p})|^2 \simeq \mathrm{Sing}_R \otimes |\mathcal{M}_m(\{p\}_m)|^2$$

- $\mathbf{X}_R$ is a symbolic operator that takes some specific limit R in which p patrons are unresolved ($p = 1, 2$ at NNLO)
- Sing_R is the singular factor in this limit (Altarelli-Parisi splitting function, soft current and generalizations)

Issues

- Singular regions overlap in phase space $\Rightarrow$ matching of limits to avoid multiple subtraction. E.g. at NLO: collinear + soft - collinear limit of soft

$$\mathbf{A}_1 |\mathcal{M}_{m+1}^{(0)}|^2 = \sum_i \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i \neq r} \mathbf{C}_{ir} \mathbf{S}_r \right] |\mathcal{M}_{m+1}^{(0)}|^2$$

- cumbersome if done in a brute force way, however efficient solution is known
- candidate subtraction terms obtained are only defined in strict IR limits $\Rightarrow$ extension of IR factorization formulae over full phase space

Constructing the subtraction terms - II

Factorization of phase space via momentum remapping: $\{p\}_{m+p} \xrightarrow{X_R} \{\tilde{p}\}_m^{(R)}$

- implements momentum conservation
- $\tilde{p}$ are on-shell
- recoil is taken away by whole event
- leads to exact factorization of phase space

$$\mathrm{d}\phi_{m+p}(\{p\}_{m+p}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m^{(R)}; Q)[\mathrm{d}p_{p,m}^{(R)}]$$

Issues

- extension of IR limits defined by these mappings must respect delicate structure of cancellations in all limits. [E.g. counterterms for single unresolved real radiation (unintegrated and integrated) must have universal IR limits. This is not guaranteed by QCD factorization.]
- different collinear and soft type mappings used
- obtained counterterms are hard to integrate over unresolved phase space

Subtraction terms - general features

Given completely explicitly for any process (with colorless initial state)

Building blocks are IR limit formulae and momentum remappings

- Altarelli-Parisi splitting functions, soft currents (tree level and one-loop, also triple AP functions)
- extension away from limits based on momentum mappings that generalize to any number of unresolved partons
- phase space plays no essential role in construction

Fully local in color $\otimes$ spin space

- no need to consider the color decomposition of real emission matrix elements
- azimuthal correlations correctly taken into account in gluon splitting

Straightforward to constrain subtractions to near singular regions

- more efficient implementation
- independence of physical results on phase space cut is a strong check

Integrating the subtraction terms - I

Subtraction terms inherit the structure of limit formulae:

$$\mathcal{X}_R = \mathrm{Sing}_R \otimes |\mathcal{M}_m(\{\tilde{p}\}_m^{(R)})|^2$$

- $\mathcal{X}_R$ is the subtraction term corresponding to limit R
- it is a product in color and spin space of the factorized matrix element depending on momenta $\{\tilde{p}\}_m^{(R)}$, but independent of $[\mathrm{d}p_{p,n}^{(R)}]$
- and the singular factor Sing_R that depends on variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- recall exact phase space factorization

$$\mathrm{d}\phi_{m+p}(\{p\}_{m+p}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m^{(R)}; Q)[\mathrm{d}p_{p,m}^{(R)}]$$

Integrated subtraction terms computed once and for all

$$X_R = \int \mathcal{X}_R = \left(\int [\mathrm{d}p_{p,n}^{(R)}]\,\mathrm{Sing}_R\right) \otimes |\mathcal{M}_m(\{p\}_m)|^2$$

- explicit parametrization of $[\mathrm{d}p_{p,n}^{(R)}]$ $\Rightarrow$ parametric integral representations
- compute the parametric integrals

Integrating the subtraction terms - II

Computing the integrated subtraction terms is cumbersome

- Many ($\mathcal{O}(100)$) difficult higher-dimensional integrals to compute. E.g.

$$\mathcal{I}_{3\mathcal{C},5}(x_{\widetilde{irs}},\epsilon;\alpha_0,d_0) = 2^{1-4\epsilon}\frac{\Gamma^2(1-\epsilon)}{\pi\Gamma(1-2\epsilon)}x_{\widetilde{irs}}\int_0^{\alpha_0}\mathrm{d}\alpha\int_0^1\mathrm{d}t_{is}\,\mathrm{d}\tau_{rs}\,\mathrm{d}v_r\,\mathrm{d}w_s\,\alpha^{-1-2\epsilon}$$
$$\times(1-\alpha)^{2d_0-3+2\epsilon}(\alpha+(1-\alpha)x_{\widetilde{irs}})^{-1-2\epsilon}(2\alpha+(1-\alpha)x_{\widetilde{irs}})^2 t_{is}^{-1-2\epsilon}(1-t_{is})^{-2\epsilon}$$
$$\times\tau_{rs}^{-1-\epsilon}(1-\tau_{rs})^{-\epsilon}v_r^{-\epsilon}(1-v_r)^{-\epsilon}w_s^{-1/2-\epsilon}(1-w_s)^{-1/2-\epsilon}(1-\tau_{rs}+\tau_{rs}t_{is})^{2\epsilon}$$
$$\times[(1+t_{is})\alpha+(1-\alpha)x_{\widetilde{irs}}(1-(1-t_{is})v_r)]^{-1}\Big\{(1-\tau_{rs}+\tau_{rs}t_{is})(2\alpha+(1-\alpha)x_{\widetilde{irs}})$$
$$+t_{is}\Big[\alpha+(1-\alpha)x_{\widetilde{irs}}[(1-\tau_{rs})v_r+\tau_{rs}(1-v_r)-2\sqrt{\tau_{rs}(1-\tau_{rs})v_r(1-v_r)}(1-2w_s)]\Big]\Big\}^{-1}$$

- Needed as an ϵ-expansion to $\mathrm{O}(\epsilon^0)$

$$\mathcal{I}_{3\mathcal{C},5}(x_{\widetilde{irs}},\epsilon;\alpha_0,d_0) = \frac{1}{4\epsilon^4}-\frac{\ln x_{\widetilde{irs}}}{\epsilon^3}+\mathrm{O}(\epsilon^{-2})$$

  rest of the expansion coefficients obtained numerically
- Method of computation: sector decomposition (in this case)

Performing the integrals

Have explored several different methods to compute integrals which arise

- ➠ use of IBPs to reduce to master integrals + solution of MIs by differential equations
- ➠ use of MB representations to extract pole structure + summation of nested series
- ➠ use of sector decomposition

Analytical vs. numerical

- ✘ Complete analytic integration does not seem practical/feasible.
- ✔ However, whenever available, analytical results show that the integrals are smooth functions (in the colloquial sense) of kinematical variables.
- ✔ Hence, for practical purposes, numerical forms are sufficient. E.g. suitable approximating functions may be obtained by fitting or interpolation.

Performing the integrals

Method	Analytical	Numerical
IBP	✔ singly-unresolved integrals ✘ bottleneck is the proliferation of denominators	✔ by evaluating the analytic expressions ✘ no numbers without full analytical results
MB	✔ iterated singly-unresolved integrals ✘ bottleneck is the evaluation of sums	✔ direct numerical evaluation of MB integrals possible ✔ fast and accurate
SD	✔ easy to automate ✘ only in principle, except for lowest order poles	✘ numerical behavior is generally worse than MB method (speed, accuracy)

Integrated approximate cross sections

$$\sigma^{\rm NNLO} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$
$$+ \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$
$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

Structure

- after summing over unobserved flavors, they are products (in color space) of various insertion operators with lower point cross sections

Insertion operators

- color and flavor structure known
- first two leading poles known analytically
- remaining poles and finite parts to be computed numerically

Integrated approximate cross sections

$$\sigma^{\rm NNLO} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$
$$+ \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$
$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

Structure

- after summing over unobserved flavors, they are products (in color space) of various insertion operators with lower point cross sections

Insertion operators

- color and flavor structure known
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Last missing piece: $\int_2 {\rm d}\sigma^{\rm RR,A_2}_{m+2}$

Integrated doubly-unresolved approximate cross section

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_2^{(0)}(\{p\}_m;\epsilon)$$

- structure of insertion operator in color $\otimes$ flavor space

$$\mathbf{I}_2^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ \sum_i \Big[\mathrm{C}_{12,f_i}^{(0)} \mathbf{T}_i^2 + \sum_k \mathrm{C}_{2,f_i f_k}^{(0)} \mathbf{T}_k^2 \Big] \mathbf{T}_i^2$$
$$+ \sum_{j,l} \Big[\mathrm{S}_2^{(0),(j,l)} C_{\mathrm{A}} + \sum_i \mathrm{CS}_{2,f_i}^{(0),(j,l)} \mathbf{T}_i^2 \Big] \mathbf{T}_j \mathbf{T}_l$$
$$+ \sum_{i,k,j,l} \mathrm{S}_2^{(0),(i,k)(j,l)} \{\mathbf{T}_i \mathbf{T}_k, \mathbf{T}_j \mathbf{T}_l\} \Bigg\}$$

- $\mathrm{C}_{2,f_i}^{(0)}$, $\mathrm{C}_{2,f_i f_k}^{(0)}$, $\mathrm{S}_2^{(0),(j,l)}$, $\mathrm{CS}_{2,f_i}^{(0),(j,l)}$ and $\mathrm{S}_2^{(0),(i,k)(j,l)}$ are functions with ϵ-poles up to $1/\epsilon^{-4}$ that depend on event kinematics and also on phase space cut parameters
- kinematical dependence enteres through variables

$$x_i = y_{iQ} \equiv \frac{2p_i \cdot Q}{Q^2} \qquad \text{and} \qquad Y_{ik,Q} = \frac{y_{ik}}{y_{iQ} y_{kQ}}$$

Integrated doubly-unresolved approximate cross section

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \mathbf{I}_2^{(0)}(\{p\}_m;\epsilon)$$

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$\mathbf{I}_2^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_\mathrm{s}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2\Bigg\{\frac{3C_\mathrm{A}^2+10C_\mathrm{A}C_\mathrm{F}+8C_\mathrm{F}^2}{4\epsilon^4}+\Bigg[\frac{77C_\mathrm{A}^2}{24}$$
$$+\frac{109C_\mathrm{A}C_\mathrm{F}}{12}+6C_\mathrm{F}^2-\frac{C_\mathrm{A}T_\mathrm{R}n_\mathrm{f}}{2}-\frac{7C_\mathrm{F}T_\mathrm{R}n_\mathrm{f}}{3C_\mathrm{F}}+\left(\frac{3C_\mathrm{A}^2}{2}-C_\mathrm{A}C_\mathrm{F}-4C_\mathrm{F}^2\right)\ln y_{12}$$
$$-\frac{C_\mathrm{A}(3C_\mathrm{A}+4C_\mathrm{F})}{2}(\ln y_{13}+\ln y_{23})\Bigg]\frac{1}{\epsilon^3}+\mathrm{O}(\epsilon^{-2})\Bigg\}$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected
- remaining expansion coefficients computed numerically

Status

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$
$$+ \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$
$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- ✔ unintegrated RR counterterms
- ✔ unintegrated RV counterterms
- ✔ RV counterterms integrated
- ✔ RR counterterms integrated (numerics is work in progress)

Cancellation of IR singularities I - kinematical

$$\sigma^{\rm NNLO} = \int_{m+2} \Big\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \Big[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \Big] \Big\}$$
$$+ \int_{m+1} \Big\{ \Big[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big] J_{m+1} - \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] J_m \Big\}$$
$$+ \int_m \Big\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \Big[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \Big] + \int_1 \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] \Big\} J_m$$

Cancellation of kinematical singularities for $e^+e^- \to 2, 3$ jets $(m = 2, 3)$

- in all singly- and doubly-unresolved limits

$$\frac{{\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m + \Big[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \Big]}{{\rm d}\sigma^{\rm RR}_{m+2} J_{m+2}} \to 1$$

- in all singly-unresolved limits

$$\frac{- \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} + \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] J_m}{{\rm d}\sigma^{\rm RV}_{m+1} J_{m+1}} \to 1$$

Cancellation of IR singularities II - poles

$$\sigma^{\rm NNLO} = \int_{m+2} \Big\{ d\sigma^{\rm RR}_{m+2} J_{m+2} - d\sigma^{\rm RR,A_2}_{m+2} J_m - \Big[d\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - d\sigma^{\rm RR,A_{12}}_{m+2} J_m \Big] \Big\}$$
$$+ \int_{m+1} \Big\{ \Big[d\sigma^{\rm RV}_{m+1} + \int_1 d\sigma^{\rm RR,A_1}_{m+2} \Big] J_{m+1} - \Big[d\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 d\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] J_m \Big\}$$
$$+ \int_m \Big\{ d\sigma^{\rm VV}_m + \int_2 \Big[d\sigma^{\rm RR,A_2}_{m+2} - d\sigma^{\rm RR,A_{12}}_{m+2} \Big] + \int_1 \Big[d\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 d\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] \Big\} J_m$$

Cancellation of poles for $e^+e^- \to 2,3$ jets ($m = 2,3$) up to $\mathrm{O}(\epsilon^{-2})$ analytically

$$d\sigma^{\rm VV}_m + \int_2 \Big[d\sigma^{\rm RR,A_2}_{m+2} - d\sigma^{\rm RR,A_{12}}_{m+2} \Big] + \int_1 \Big[d\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 d\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] = \mathrm{O}(\epsilon^{-2})$$

- cancellation of PS cut parameter dependence in sum of integrated approximate cross sections is highly nontrivial: strong check
- nontrivial interplay of kinematical dependence in various terms
- cancellation of lower order poles to be checked numerically (work in progress)

Conclusions and outlook

Local subtraction at NNLO

- general, explicit, local subtraction scheme for computing NNLO QCD jet cross sections
- construction based on IR limit formulae
- fully worked out for processes with colorless initial state
- integration of all approximate cross sections finished

Immediate next steps

- consolidate numerics for integrated subtraction terms
- physical applications to assess overall performance