# Reduction at the integrand level beyond NLO

Costas G. Papadopoulos

NCSR "Demokritos", Athens, Greece
&
CERN, Geneva, Switzerland

CERN 2012, October 3, 2012

Over the last few years very important activity to extend unitarity and integrand level reduction ideas beyond one loop

J. Gluza, K. Kajda and D. A. Kosower, “Towards a Basis for Planar Two-Loop Integrals,” Phys. Rev. D **83** (2011) 045012 [arXiv:1009.0472 [hep-th]].

D. A. Kosower and K. J. Larsen, “Maximal Unitarity at Two Loops,” Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

P. Mastrolia and G. Ossola, “On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes,” JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, “Hepta-Cuts of Two-Loop Scattering Amplitudes,” JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

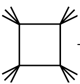
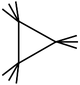
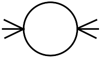
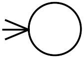
Y. Zhang, “Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods,” JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, “Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division,” arXiv:1209.4319 [hep-ph].

basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \to$ cut-constructible part $\qquad R \to$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

OPP integrand level:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0,i_1,i_2,i_3) + \tilde{d}(q;i_0,i_1,i_2,i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0,i_1,i_2) + \tilde{c}(q;i_0,i_1,i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0,i_1) + \tilde{b}(q;i_0,i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q;i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

$\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are "spurious" terms (vanish upon integration). Their $q$-dependence is known

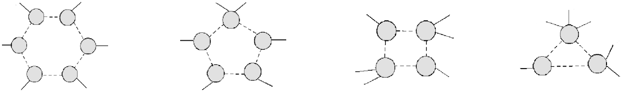
Ossola, Papadopoulos and Pittau, Nucl. Phys. B 763, 147 (2007)

Can be solved either using cuts or simply by polynomial fitting

# The one-loop calculation in a nutshell

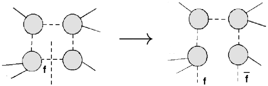
The computation of $pp(p\bar{p}) \to e^{+}\nu_e\mu^{-}\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, `HELAC` evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$ with the values of the loop momentum $q$ provided by `CutTools`

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop ($q$ is fixed) to get a $n+2$ *tree-like* process




The $R_2$ contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

# Two-loop amplitudes

- Reduction at the integrand level
- Evaluation of Master Integrals

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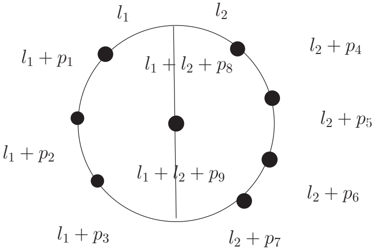
- Generic two-loop graph: iGraph

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, arXiv:1206.4180 [hep-ph].




$$D(l_1 + p_i)\ ,\ D(l_2 + p_j)\ ,\ D(l_1 + l_2 + p_k)$$

# Two-loop amplitudes

The general strategy consists in finding function $x_j \equiv x_j(l_1, l_2)$

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1 \ .$$

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Let us go a step back at one loop

$$1 = T_1(q)D_1 + T_2(q)D_2 + \cdots + T_n(q)D_n$$

W. L. van Neerven and J. A. M. Vermaseren, Phys. Lett. B **137** (1984) 241.

## Hilbert's Nullstellensatz theorem

Hilbert's Nullstellensatz (German for "theorem of zeros," or more literally, "zero-locus-theorem" see Satz) is a theorem which establishes a fundamental relationship between geometry and algebra. This relationship is the basis of algebraic geometry, an important branch of mathematics. It relates algebraic sets to ideals in polynomial rings over algebraically closed fields. This relationship was discovered by David Hilbert who proved Nullstellensatz and several other important related theorems named after him (like Hilbert's basis theorem).

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$$q^2 \sum_{j=1}^{n} x_j + 2q_\mu \sum_{j=1}^{n} x_j {p_j}^\mu + \sum_{j=1}^{n} x_j \mu_j = 1 \quad .$$

$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j {p_j}^\mu = 0 \quad , \sum_{j=1}^{n} x_j \mu_j = 1$$

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- solution exists for $n = 6$ $d = 4$

# Two-loop amplitudes

Linear terms $T(q) = P_1(q)$, count tensor structures:

$$1 \ , \ q^{\mu} \ , \ q^{\mu} q^{\nu} \ , \ q^2 q^{\mu} \ .$$

There are, for $d = 4$, therefore $1+4+10+4 = 19$ independent tensor structures.

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In $d$ dimensions, tensor up to rank $k$, $N(d,k)$ number of independent tensor structures

$$N(d,k) = \begin{pmatrix} d-1+k \\ k \end{pmatrix} + \sum_{p=0}^{k+1} \begin{pmatrix} d-1+p \\ p \end{pmatrix} \ . \tag{1}$$

In the table below we give the results for various ranks and dimensionalities.

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $d$ =1 | 3 | 4 | 5 | 6 | 7 |
| 2 | 4 | 8 | 13 | 19 | 26 |
| 3 | 5 | 13 | 26 | 45 | 71 |
| 4 | 6 | 19 | 45 | 90 | 161 |
| 5 | 7 | 26 | 71 | 161 | 322 |
| 6 | 8 | 34 | 105 | 266 | 588 |

Values of $N(d,k)$

The OPP-"miracle" is that the OPP equation works with only 10(6) different coefficients

$$1 = \sum_{i=1}^{5} D_i(q)(c_i^{(0)} + c_i^{(1)}\epsilon_i(q))$$

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Back to two loops: iGraphs can be denoted by the triplet $(n_1, n_2, n_3)$, $n = n_1 + n_2 + n_3$

$$n_{1,2,3} \leq 4 \ (= d) \ , \quad n_1 + n_2 + n_3 \leq 11 \ (= 2d + 3) \ .$$

# Linear terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j)$$

$$T(d) = (4d^2 + 18d + 2)/2$$

| $n$ | $d=6$ | $d=5$ | $d=4$ | $d=3$ | $d=2$ | $d=1$ |
|---|---|---|---|---|---|---|
| 3 | 39-0 | 33-0 | 27-0 | 21-0 | 15-0 | 9-0 |
| 4 | 52-0 | 44-0 | 36-0 | 28-0 | 20-0 | 12-2 |
| 5 | 65-1 | 55-1 | 45-1 | 35-1 | 25-1 | 15-5 |
| 6 | 78-3 | 66-3 | 54-3 | 42-3 | 30-3 | |
| 7 | 91-6 | 77-6 | 63-6 | 49-6 | 35-8 | |
| 8 | 104-10 | 88-10 | 72-10 | 56-10 | | |
| 9 | 111-15 | 99-15 | 81-15 | 63-17 | | |
| 10 | 130-21 | 110-21 | 90-21 | | | |
| 11 | 143-28 | 121-28 | 99-30 | | | |
| 12 | 156-36 | 132-36 | | | | |
| 13 | 169-45 | 143-47 | | | | |
| 14 | 182-55 | | | | | |
| 15 | 195-55 | | | | | |
| $T(d)$ | 127 | 96 | 69 | 46 | 27 | 10 |

# Quadratic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) + \sum_{j \le k} d_{ijk}(l_1 \cdot t_j)(l_1 \cdot t_k) + \cdots$$

$$T(d) = 4d^3/3 + 10d^2 + 20d/3 - 2 \quad (2)$$

| $n$ | $d=4$ | $d=3$ | $d=2$ |
|---|---|---|---|
| 3 | 135-4 | 84-3 | 45-3 |
| 4 | 180-6 | 128-6 | 60-6 |
| 5 | 225-18 | 140-16 | 75-15 |
| 6 | 270-38 | 168-32 | 90-30 |
| 7 | 315-65 | 196-53 | |
| 8 | 360-98 | 224-80 | |
| 9 | 405-136 | 252-108 | |
| 10 | 450-180 | | |
| 11 | 495-225 | | |
| $T(d)$ | 270 | 144 | 60 |

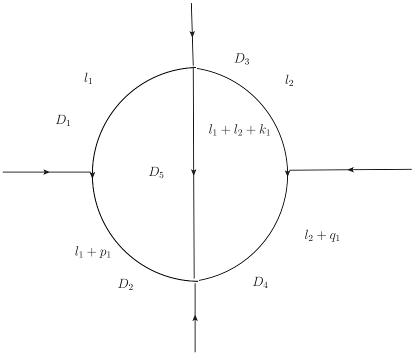
# Cubic terms

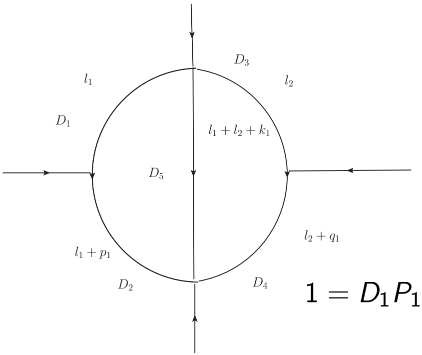
$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \cdots + \sum_{j \leq k} g_{ijkl}(l_1 \cdot t_j)(l_1 \cdot t_k)(l_1 \cdot t_l) + \cdots$$

$$T(d) = 2d^4/3 + 22d^3/3 + 71d^2/6 + d/6 + 1$$

| $n$ | $d = 6$ | $d = 5$ | $d = 4$ | $d = 3$ |
|---|---|---|---|---|
| 5 | | | | 420/332 |
| 6 | | | | 504/352 |
| 7 | | | 1155/803 | 588/360 |
| 8 | | | 1320/823 | 672/360 |
| 9 | | 2574/1603 | 1485/831 | |
| 10 | | 2860/1623 | 1650/831 | |
| 11 | 5005/2848 | 3146/1631 | | |
| 12 | 5460/2868 | 3432/1631 | | |
| 13 | 5915/2876 | | | |
| 14 | 6370/2876 | | | |
| $T(d)$ | 2876 | 1631 | 831 | 360 |

# OPP at two loops



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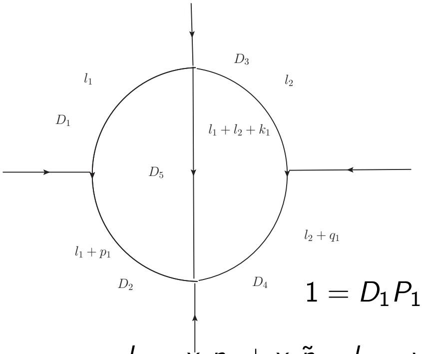


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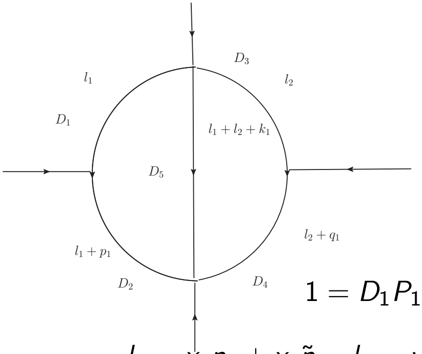
$$l_1 = x_1 p_1 + x_2 \tilde{p}_1 \quad l_2 = y_1 q_1 + y_2 \tilde{q}_1 \quad x_1 = \frac{l_1 \cdot p_1}{p_1^2} \quad \ldots$$

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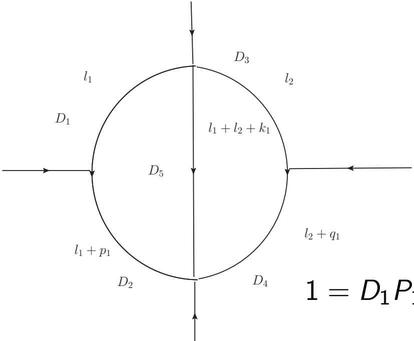
$$P_i = \sum a_i[n_1, n_2, n_3, n_4] x_1^{n_1} x_2^{n_2} y_1^{n_3} y_2^{n_4}$$

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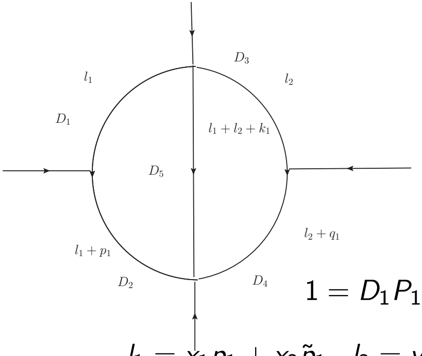


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$$1 = \sum D_i T_i + \sum D_i D_j T_{ij} + \sum D_i D_j D_k T_{ijk}$$

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- Rational terms $R_1 + R_2$ ?

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- A HELAC-NNLO framework ?