Alexander von Humboldt
Stiftung/Foundation
Sofja Kowaleskaja Award

Integrand Reduction for Multi-Loop Scattering Amplitudes

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- arXiv:1107.6041 [hep-ph], JHEP 1111 (2011) 014, with ***Ossola***
- arXiv:1205.7087 [hep-ph], to appear in PLB, with ***Ossola, Mirabella & Peraro***
- arXiv:1209.4319 [hep-ph], with ***Ossola, Mirabella & Peraro***

CERN, 3.10.2012

Multi-Loop Integrand Decomposition

Generic amplitude

$$\mathcal{A}_n = \int d^d\bar{q}_1 \ldots \int d^d\bar{q}_\ell \quad \mathcal{I}_{i_1\cdots i_n}(\bar{q}_1,\ldots,\bar{q}_\ell)$$

$$\equiv \int d^d\bar{q}_1 \ldots \int d^d\bar{q}_\ell \, \frac{\mathcal{N}_{i_1\cdots i_n}(\bar{q}_1,\ldots,\bar{q}_\ell)}{D_{i_1}(\bar{q}_1,\ldots,\bar{q}_\ell)\cdots D_{i_n}(\bar{q}_1,\ldots,\bar{q}_\ell)},$$

$$D_i = \left(\sum_a \alpha_{i,a}\bar{q}_a + p_i\right)^2 - m_i^2 \qquad \alpha_{i,a} \in \{0,\pm 1\}.$$

Multi-Loop Integrand Decomposition

☑ Integrand Reduction Formula

$$
\begin{aligned}
\mathcal{N}_{i_1\cdots i_n} = & \sum_{1=i_1<<i_{\max}}^{n} \Delta_{i_1 i_2 \ldots i_{\max}} \prod_{h \neq i_1 i_2 \ldots i_{\max}}^{n} D_h \\
& + \sum_{1=i_1<<(i_{\max}-1)}^{n} \Delta_{i_1 i_2 \ldots (i_{\max}-1)} \prod_{h \neq i_1 i_2 \ldots (i_{\max}-1)}^{n} D_h \\
& + \sum_{1=i_1<<(i_{\max}-2)}^{n} \Delta_{i_1 i_2 \ldots (i_{\max}-2)} \prod_{h \neq i_1 i_2 \ldots (i_{\max}-2)}^{n} D_h \\
& + \ldots \quad \ldots \quad \ldots \\
& + \sum_{1=i_1<i_2}^{n} \Delta_{i_1 i_2} \prod_{h \neq i_1 i_2}^{n} D_h \\
& + \sum_{1=i_1}^{n} \Delta_{i_1} \prod_{h \neq i_1}^{n} D_h \\
& + Q_{\emptyset} \prod_{h=1}^{n} D_h \ ,
\end{aligned}
$$

Multi-(particle)-pole decomposition

$$\mathcal{I}_{i_1\cdots i_n} = \frac{\mathcal{N}_{i_1\cdots i_n}}{D_{i_1} D_{i_2}\cdots D_{i_n}}$$

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2}\cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2}\cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2}\cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Parametric ***form of the residues*** is ***process independent***

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max-1}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max-1}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max-1}}}$$

$$+ \sum_{1=i_1<<i_{\max-2}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max-2}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max-2}}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

The actual ***values of the coefficients*** in the residues are are ***process dependent***

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}}$$

$$+ \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

☑ Parametric form of the residues is process independent.

Knowing the parametric form of residues is ***mandatory***!!!

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{1=i_1<<i_{\max}}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}}} + \sum_{1=i_1<<i_{\max}-1}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-1}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-1}} + \sum_{1=i_1<<i_{\max}-2}^{n} \frac{\Delta_{i_1 i_2 \ldots i_{\max}-2}}{D_{i_1} D_{i_2} \cdots D_{i_{\max}-2}} + \cdots\cdots + \sum_{1=i_1<i_2}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{1=i_1}^{n} \frac{\Delta_{i_1}}{D_{i_1}} + Q_{\emptyset}$$

Use your favourite generator, (Feynman diagrams, tree-amplitudes, currents,...), and sample *I*(q's) as many time as the number of unknown coefficients

☑ Parametric form of the residues is process independent.

☑ Actual values of the coefficients is process dependent.

- Problem: what is the form of the residues?
- “find the right variables encoding the cut-structure”

Cuts and Residues

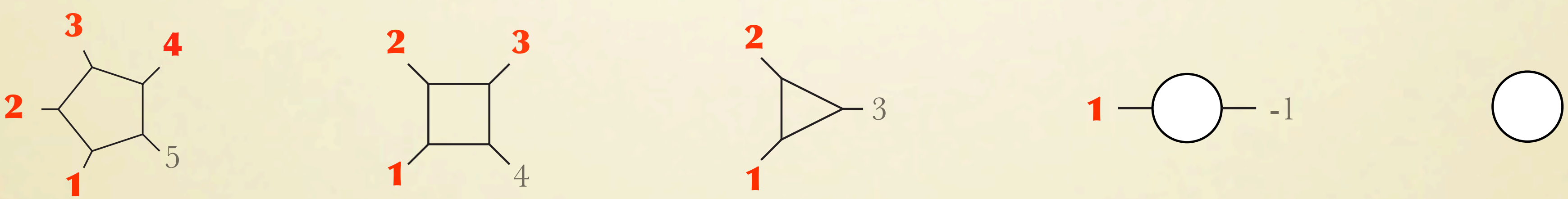
cut-associated basis

For each cut $(ijk\cdots)$, $D_i = D_j = D_k = \cdots = 0$, a basis of four massless vectors

$$\left\{ e_1^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_2^{(ijk\cdots)}, e_4^{(ijk\cdots)} \right\}$$

$$\left(e_i^{(ijk\cdots)}\right)^2 = 0\ , \qquad e_1^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_1^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0\ ,$$
$$e_2^{(ijk\cdots)} \cdot e_3^{(ijk\cdots)} = e_2^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 0\ , \qquad e_1^{(ijk\cdots)} \cdot e_2^{(ijk\cdots)} = -e_3^{(ijk\cdots)} \cdot e_4^{(ijk\cdots)} = 1$$

use independent external momenta + auxiliary orthogonal complement:



4-vectors vs components

- Loop momentum decomposition

$$q + p_i = \sum_{\alpha=1}^{4} x_\alpha \, e_\alpha^{(ijk\cdots)}$$

Problem: what is the form of the residues?

Δ-variables

- ISP's = Irreducible Scalar Products:
  - components of the loop momenta which can *variate* under cut-conditions
  - spurious: vanishing upon integration
  - non-spurious: non-vanishing upon integration $\Rightarrow$ MI's

Integrand-Reduction beyond One-Loop

Ossola & P.M. (2011)

Badger, Frellesvig, Zhang (2011,2012)

Zhang (2012)

Mirabella, Ossola, Peraro, & P.M (2012)

Kleiss, Malamos, Papadopoulos, Verheynen (2012)

Multi-Loop Scattering Amp's from Multivariate Polynomial Division

Multivariate Polynomial Division

Zhang (2012);
Mirabella, Ossola, Peraro, & P.M. (2012)

- **Ideal** $$\mathcal{J}_{i_1\cdots i_n} = \langle D_{i_1}, \cdots, D_{i_n} \rangle \equiv \left\{ \sum_{\kappa=1}^{n} h_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) : h_\kappa(\mathbf{z}) \in P[\mathbf{z}] \right\}$$

- **Groebner Basis** $$\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$$

  n-ple cut-conditions $$D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$$

Multivariate Polynomial Division

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- **Groebner Basis** $\mathcal{G}_{i_1\cdots i_n} = \{g_1(\mathbf{z}), \ldots, g_m(\mathbf{z})\}$

  n-ple cut-conditions $D_{i_1} = \ldots = D_{i_n} = 0 \quad \Leftrightarrow \quad g_1 = \ldots = g_m = 0$

- **Polynomial Division** $\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \Gamma_{i_1\cdots i_n} + \Delta_{i_1\cdots i_n}(\mathbf{z})$,

- **Remainder = Residue** $\Delta_{i_1\cdots i_n}(\mathbf{z})$

- **Quotients** $\Gamma_{i_1\cdots i_n} = \sum_{i=1}^{m} \mathcal{Q}_i(\mathbf{z}) g_i(\mathbf{z})$ belongs to the ideal $\mathcal{J}_{i_1\cdots i_n}$,

$$= \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \ .$$

Multi-Loop Recursive Integrand Reduction

Mirabella, Ossola, Peraro, & P.M. (2012)

remainder = residue

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n} + \frac{\Delta_{i_1\cdots i_n}}{D_{i_1}\cdots D_{i_n}}$$

n-denominator integrand

(n-1)-denominator integrand

Reducibility Criterion

Mirabella, Ossola, Peraro, & P.M. (2012)

Proposition 2.1. *The integrand* $\mathcal{I}_{i_1\cdots i_n}$ *is reducible iff the remainder of the division modulo a Gröbner basis vanishes, i.e. iff* $\mathcal{N}_{i_1\cdots i_n} \in \mathcal{J}_{i_1\cdots i_n}$.

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Proposition 2.2 *An integrand $\mathcal{I}_{i_1\cdots i_n}$ is reducible if the cut $(i_1 \cdots i_n)$ leads to a system of equations with no solution.*

Indeed if the system of equations $D_{i_1}(\mathbf{z}) = \cdots = D_{i_n}(\mathbf{z}) = 0$ has no solution, the *weak Nullstellensatz* theorem ensures that $1 \in \mathcal{J}_{i_1\cdots i_n}$, i.e. $\mathcal{J}_{i_1\cdots i_n} = P[\mathbf{z}]$. Therefore any polynomial in $\mathbf{z}$ is in the ideal. Any numerator function $\mathcal{N}_{i_1\cdots i_n}$ is polynomial in the integration momenta, thus $\mathcal{N}_{i_1\cdots i_n} \in \mathcal{J}_{i_1\cdots i_n}$ and it can be expressed as a combination of the denominators $D_{i_1}(\mathbf{z}), \ldots, D_{i_n}(\mathbf{z})$ [44, 49]. In this case Eq. (2.8) becomes

$$1 = \sum_{\kappa=1}^{n} w_\kappa(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \in \mathcal{J}_{i_1\cdots i_n}, \qquad \text{for some } \omega_\kappa \in P[\mathbf{z}].$$

$$\mathcal{I}_{i_1\cdots i_n} = \sum_{\kappa=1}^{n} \mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1} i_n} .$$

One-Loop Integrand Reduction

In d-dimensions, the generic n-point one-loop integrand reads $\mathcal{I}_{0\cdots(n-1)} \equiv \dfrac{\mathcal{N}_{0\cdots(n-1)}(q,\mu^2)}{D_0(q,\mu^2)\cdots D_{n-1}(q,\mu^2)}$.

for each $\mathcal{I}_{i_1\cdots i_k}$ we define a basis $\mathcal{E}^{(i_1\cdots i_k)} = \{e_i\}_{i=1,\ldots,4}$.

If $k \geq 5$, then $e_i = k_i$, where k_i are four external momenta.
If $k < 5$, then e_i are chosen to fulfill the following relations:

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,,$$
$$e_3^2 = e_4^2 = \delta_{k4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{k4})\,.$$

In terms of $\mathcal{E}^{(i_1\cdots i_k)}$, the loop momentum can be decomposed as, $q^\mu = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu$.

each numerator $\mathcal{N}_{i_1\cdots i_k}$ can be treated as a rank- k polynomial in $\mathbf{z} \equiv (x_1, x_2, x_3, x_4, \mu^2)$,

$$\mathcal{N}_{i_1\cdots i_k} = \sum_{\vec{j}\in J(k)} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,,$$

$J(k) \equiv \{\vec{j} = (j_1,\ldots,j_5) : j_1 + j_2 + j_3 + j_4 + 2\,j_5 \leq k\}$.

☑ *Step 1.* Since $n > 5$, the Proposition 2.2 guarantees that $\mathcal{N}_{0\cdots n-1}$ is reducible, and, by iteration, it can be written as a linear combination of 5-point integrands $\mathcal{I}_{i_1\cdots i_5}$.

☑ *Step 1.* Since $n > 5$, the Proposition 2.2 guarantees that $\mathcal{N}_{0\cdots n-1}$ is reducible, and, by iteration, it can be written as a linear combination of 5-point integrands $\mathcal{I}_{i_1\cdots i_5}$.

☑ *Step 2.* The numerator of each $\mathcal{I}_{i_1\cdots i_5}$ is a rank-5 polynomial in $\mathbf{z}$. We define the ideal $\mathcal{J}_{i_1\cdots i_5}$, and compute the Gröbner basis $\mathcal{G}_{i_1\cdots i_5} = (g_1, \ldots, g_5)$, which is found to have a remarkably simple form:

$$g_i(\mathbf{z}) = c_i + z_i \ , (i = 1, \ldots, 5) \ .$$

[keep it in mind!]

The division of $\mathcal{N}_{i_1\cdots i_5}$ modulo $\mathcal{G}_{i_1\cdots i_5}$ gives a *constant* remainder,

$$\Delta_{i_1\cdots i_5} = c_0 \ .$$

$$\Gamma_{i_1\cdots i_5} = \sum_{\kappa=1}^{5} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \ ,$$

where $\mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}$ are the numerators of the 4-point integrands, $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_5}$, obtained by removing the i_κ-th denominator.

☑ *Step 3.* For each $\mathcal{I}_{i_1\cdots i_4}$, the numerator $\mathcal{N}_{i_1\cdots i_4}$ is a rank-4 polynomial in $\mathbf{z}$. The Gröbner basis $\mathcal{G}_{i_1\cdots i_4}$ of the ideal $\mathcal{J}_{i_1\cdots i_4}$ contains four elements. Dividing $\mathcal{N}_{i_1\cdots i_4}$ modulo $\mathcal{G}_{i_1\cdots i_4}$, we obtain

$$\Delta_{i_1\cdots i_4} = c_0 + c_1 x_4 + \mu^2(c_2 + c_3 x_4 + \mu^2 c_4)\,.$$

$$\Gamma_{i_1\cdots i_4} = \sum_{\kappa=1}^{4} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}(\mathbf{z}) D_{i_\kappa}(\mathbf{z})\ ,$$

contains the numerators of 3-point integrands $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}$.

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$$\Gamma_{i_1\cdots i_4} = \sum_{\kappa=1}^{4} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) \, ,$$

contains the numerators of 3-point integrands $\mathcal{I}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_4}$.

☑ *Step 4.* The Gröbner basis $\mathcal{G}_{i_1 i_2 i_3}$ is formed by three elements, and is used to divide $\mathcal{N}_{i_1 i_2 i_3}$. The remainder $\Delta_{i_1 i_2 i_3}$ is polynomial in μ^2 and in the third and fourth components of q in the basis $\mathcal{E}^{(i_1 i_2 i_3)}$,

$$\Delta_{i_1 i_2 i_3} = c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \, .$$

The term $\Gamma_{i_1 i_2 i_3}$ generates the rank-2 numerators of the 2-point integrands $\mathcal{I}_{i_1 i_2}$, $\mathcal{I}_{i_1 i_3}$, and $\mathcal{I}_{i_2 i_3}$.

☑ *Step 5.* The remainder of the division of $\mathcal{N}_{i_1 i_2}$ by the two elements of $\mathcal{G}_{i_1 i_2}$ is:

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \ .$$

It is polynomial in μ^2 and in the last three components of q in the basis $\mathcal{E}^{(i_1 i_2)}$. The reducible term of the division, $\Gamma_{i_1 i_2}$, generates the rank-1 integrands, $\mathcal{I}_{i_1}$, and $\mathcal{I}_{i_2}$.

Step 5. The remainder of the division of $\mathcal{N}_{i_1 i_2}$ by the two elements of $\mathcal{G}_{i_1 i_2}$ is:

$$\Delta_{i_1 i_2} = c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \ .$$

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Step 6. The numerator of the 1-point integrands is linear in the components of the loop momentum in the basis $\mathcal{E}^{(i_1)}$,

$$\mathcal{N}_{i_1} = \beta_0 + \sum_{j=1}^{4} \beta_j \, x_j \ .$$

The only element of the Gröbner basis $\mathcal{G}_{i_1}$ is D_{i_1}, which is quadratic in $\mathbf{z}$. Therefore the division modulo $\mathcal{G}_{i_1}$, leads to a vanishing quotient, hence

$$\mathcal{N}_{i_1} = \Delta_{i_1} \ .$$

☑ *Step 7.* Collecting all the remainders computed in the previous steps, we obtain the complete decomposition of $\mathcal{I}_{0\cdots n-1}$ in terms of its multi-pole structure

$$\mathcal{I}_{0\cdots n-1} = \sum_{k=1}^{5} \left(\sum_{1=i_1<\ldots<i_k}^{n-1} \frac{\Delta_{i_1\cdots i_k}}{D_{i_1}\cdots D_{i_k}} \right) .$$

which reproduces the well-known one-loop d-dimensional integrand decomposition formula

Ossola, Papadopoulos, Pittau
Ellis, Giele, Kunszt, Melnikov

GroebnerBasis[{$poly_1$, $poly_2$, …}, {x_1, x_2, …}] gives a list of polynomials that form a Gröbner basis for the set of polynomials $poly_i$.

PolynomialReduce[$poly$, {$poly_1$, $poly_2$, …}, {x_1, x_2, …}] yields a list representing a reduction of $poly$ in terms of the $poly_i$. The list has the form {{a_1, a_2, …}, b}, where b is minimal and $a_1\, poly_1 + a_2\, poly_2 + \ldots + b$ is exactly $poly$. ≫

- What can we do within this new framework?

The Maximum-Cut Theorem

Mirabella, Ossola, Peraro, & P.M. (2012)

At ℓ loops, in four dimensions, we define a *maximum-cut* as a (4ℓ)-ple cut

$$D_{i_1} = D_{i_2} = \cdots = D_{i_{4\ell}} = 0 \, ,$$

which constrains completely the components of the loop momenta. In four dimensions this implies the presence of four constraints for each loop momenta.
We assume that:
in non-exceptional phase-space points, a maximum-cut has a finite number n_s of solutions, each with multiplicity one.
Under this assumption we have the following

Theorem 4.1 (Maximum cut). *The residue at the maximum-cut is a polynomial paramatrised by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.*

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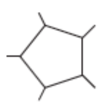
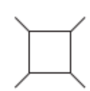
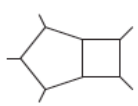
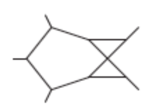
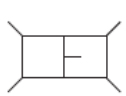
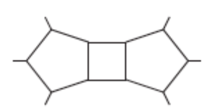
diagram	Δ	n_s	diagram	Δ	n_s
	c_0	1		$c_0 + c_1 z$	2
	$\sum_{i=0}^{3} c_i z^i$	4		$\sum_{i=0}^{3} c_i z^{\,i}$	4
	$\sum_{i=0}^{7} c_i z^{\,i}$	8		$\sum_{i=0}^{7} c_i z^{\,i}$	8

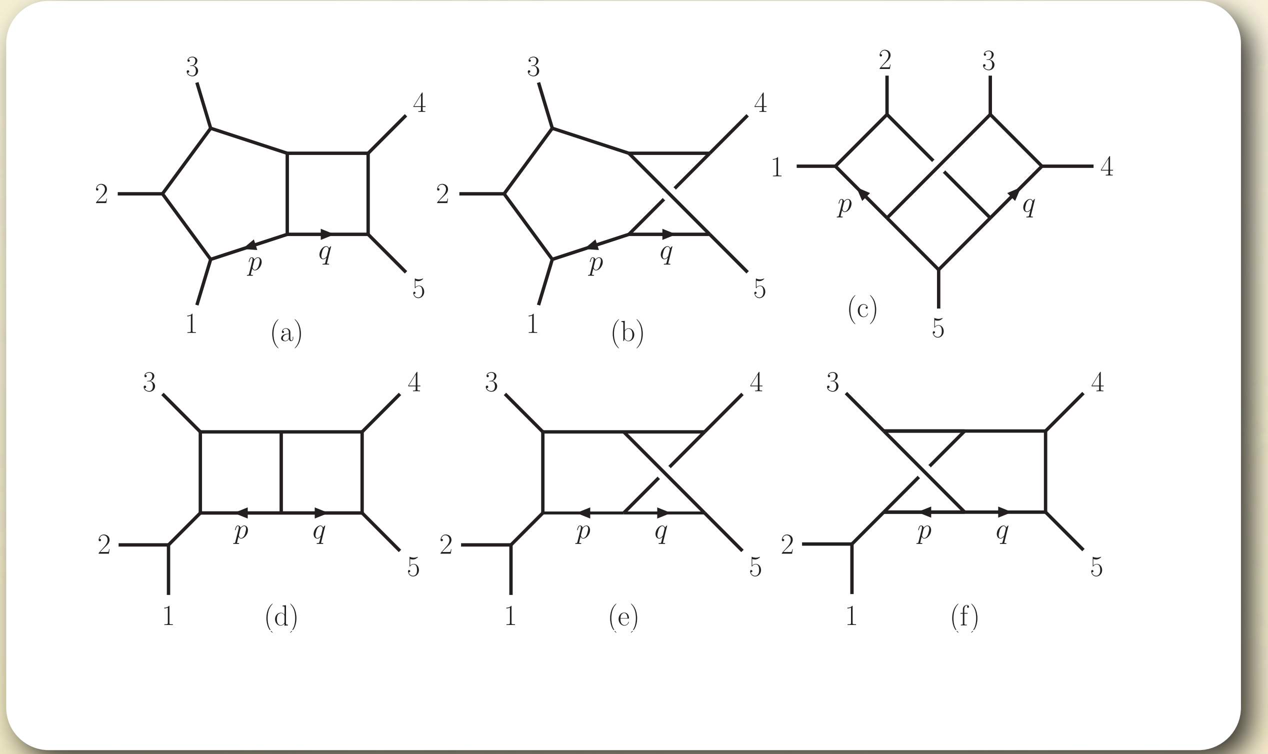
2-loop 5-point amplitudes in N=4 SYM

Bern, Czakon, Kosower, Roiban, Smirnov

Arkani-Hamed, Bourjaily, Cachazo, Caron-Houot, Trnka

Drummond, Henn, Trnka

Carrasco, Johansson



Integrand

$$\mathcal{I}_{1\cdots 8} \equiv \frac{\mathcal{N}_{1\cdots 8}(q,k)}{D_1(q,k)\cdots D_8(q,k)} \ ,$$

Momentum basis

$$q^\mu = \sum_{i=1}^{4} y_i \ \tau_i^\mu \ , \qquad k^\mu = \sum_{i=1}^{4} x_i \ e_i^\mu \ .$$

Generic Numerator

$$\mathcal{N}(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4) = \sum_{\vec{j}\in J(k)} \alpha_{\vec{j}} \ x_1^{j_1} \ x_2^{j_2} \ x_3^{j_3} \ x_4^{j_4} \ y_1^{j_5} \ y_2^{j_6} \ y_3^{j_7} \ y_4^{j_8} \ ,$$

with $J(k)$ being the set of values for the exponents compatible with the renormalizability

Polynomial Division

$$\mathcal{N}_{i_1\cdots i_n}(\mathbf{z}) = \sum_{\kappa=1}^{n} \mathcal{N}_{i_1\cdots i_{\kappa-1} i_{\kappa+1}\cdots i_n}(\mathbf{z}) D_{i_\kappa}(\mathbf{z}) + \Delta_{i_1\cdots i_n}(\mathbf{z}) \ .$$

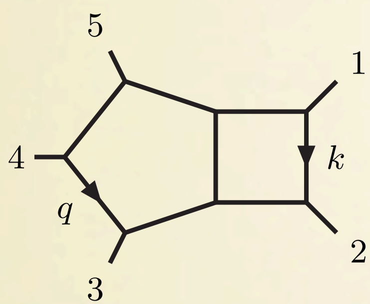
2-Loop Integrand Decomposition Formula (4D)

$$\mathcal{I}_n = \sum_{i_1<<i_8=1}^{n} \frac{\Delta_{i_1\cdots i_8}}{D_{i_1}\cdots D_{i_8}} + \sum_{i_1<<i_7=1}^{n} \frac{\Delta_{i_1\cdots i_7}}{D_{i_1}\cdots D_{i_7}} + \cdots + \sum_{i_1<i_2=1}^{n} \frac{\Delta_{i_1 i_2}}{D_{i_1} D_{i_2}} + \sum_{i=1}^{n} \frac{\Delta_i}{D_i} + \mathcal{Q}_\varnothing$$

The PentaBox diagram in N=4 SYM

Ossola & P.M. (2011)

Mirabella, Ossola, Peraro, & P.M. (2012)

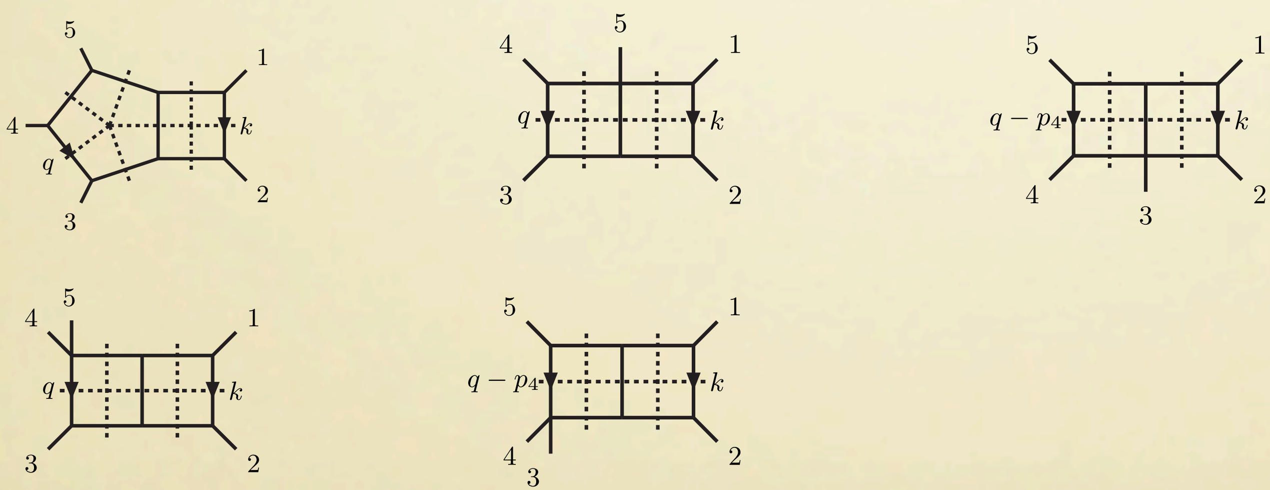


$$
\begin{aligned}
D_1 &= k^2 \\
D_2 &= (k+p_2)^2 \\
D_3 &= (k-p_1)^2 \\
D_4 &= q^2 \\
D_5 &= (q+p_3)^2 \\
D_6 &= (q-p_4)^2 \\
D_7 &= (q-p_4-p_5)^2 \\
D_8 &= (q+k+p_2+p_3)^2 \ .
\end{aligned}
$$

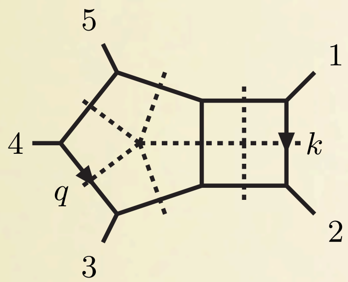
$$N(q,k) = 2\,q\cdot v + \alpha$$ Carrasco & Johansson (2011)

$$v^\mu = \frac{1}{4}\Big(\gamma_{12}(p_1^\mu - p_2^\mu) + \gamma_{23}(p_2^\mu - p_3^\mu) + 2\,\gamma_{45}(p_4^\mu - p_5^\mu) + \gamma_{13}(p_1^\mu - p_3^\mu)\Big)$$

$$\alpha = \frac{1}{4}\Big(2\,\gamma_{12}(s_{45} - s_{12}) + \gamma_{23}(s_{45} + 3s_{12} - s_{13}) + 2\,\gamma_{45}(s_{14} - s_{15}) + \gamma_{13}(s_{12} + s_{45} - s_{13})\Big)$$



5-POINT 8FOLD-CUT $D_1 = \ldots = D_8 = 0$



$$\Delta_{12345678}(q,k) = \mathrm{Res}_{12345678}\left\{\mathcal{N}_{1\cdots 8}(q,k)\right\} .$$

$$q^\mu = \sum_{i=1}^{4} y_i \, \tau_i^\mu \, , \qquad k^\mu = \sum_{i=1}^{4} x_i \, e_i^\mu \, . \qquad e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4.$$

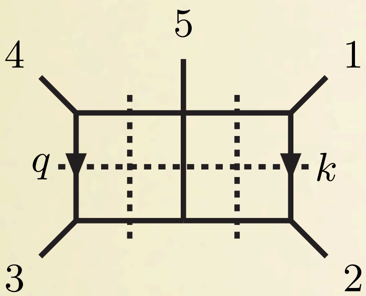
$$\Delta_{12345678}(q,k) = c_{12345678,0} + c_{12345678,1}\, y_4 + c_{12345678,2}\, x_3 + c_{12345678,3}\, x_4 \; .$$

[Maximum Cut Thm]

generic residue

5-POINT 7FOLD-CUT

$$D_1 = \ldots = D_6 = D_8 = 0$$



$$\Delta_{1234568}(q,k) = \text{Res}_{1234568}\left\{\frac{N(q,k) - \Delta_{12345678}(q,k)}{D_7}\right\} .$$

$$q^\mu = \sum_{i=1}^{4} y_i \, \tau_i^\mu \, , \qquad k^\mu = \sum_{i=1}^{4} x_i \, e_i^\mu \, .$$

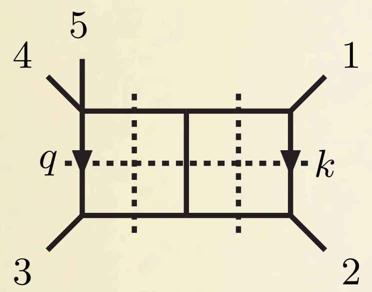
$$e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4.$$

$$\begin{aligned}
\Delta_{1234568} = \; & c_0 + c_1\, x_3 + c_2\, x_3^2 + c_3\, x_3^3 + c_4\, x_3^4 + c_5\, x_4 + c_6\, x_4^2 + c_7\, x_4^3 + c_8\, x_4^4 \\
& + c_9\, y_3 + c_{10}\, x_4\, y_3 + c_{11}\, y_3^2 + c_{12}\, x_4 y_3^2 + c_{13}\, y_3^3 + c_{14}\, x_4 y_3^3 + c_{15}\, y_3^4 \\
& + c_{16}\, x_4 y_3^4 + c_{17}\, y_4 + c_{18}\, x_3 y_4 + c_{19}\, x_3^2 y_4 + c_{20}\, x_3^3 y_4 + c_{21}\, x_3^4 y_4 + c_{22}\, x_4 y_4 \\
& + c_{23}\, x_4^2 y_4 + c_{24}\, x_4^3 y_4 + c_{25}\, x_4^4 y_4 + c_{26}\, y_4^2 + c_{27}\, x_4 y_4^2 + c_{28}\, y_4^3 + c_{29}\, x_4 y_4^3 \\
& + c_{30}\, y_4^4 + c_{31}\, x_4 y_4^4 \, .
\end{aligned} \qquad (3.18)$$

generic residue

4-POINT 7FOLD-CUT

$D_1 = \ldots = D_5 = D_7 = D_8 = 0.$



$$\Delta_{1234578}(q,k) = \text{Res}_{1234578}\left\{\frac{N(q,k) - \Delta_{12345678}(q,k)}{D_6}\right\},$$

$$e_1^\mu = p_1^\mu\,, \qquad e_2^\mu = p_2^\mu\,,$$
$$\tau_1^\mu = p_3^\mu\,, \qquad \tau_2^\mu = P_{45}^\mu - \frac{s_{45}}{2P_{45}\cdot\tau_1}\tau_1^\mu\,.$$

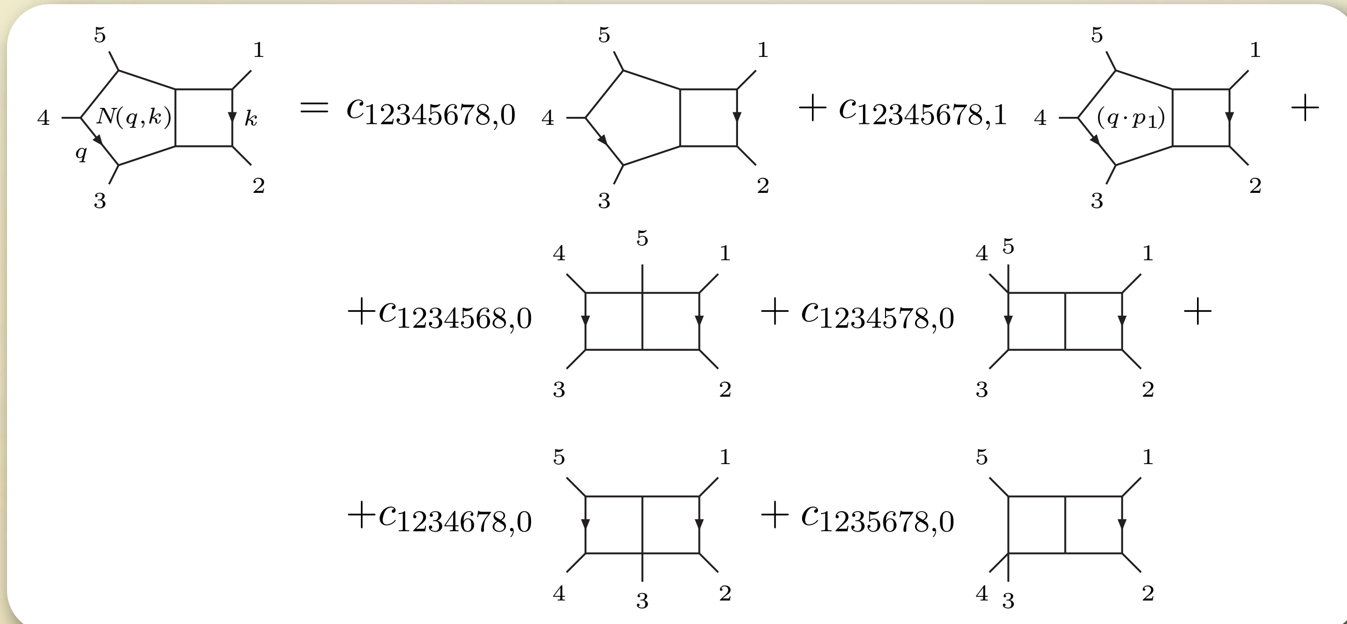
parametrized using thirty-two monomials

$$\{1, x_3, x_3^2, x_3^3, x_3^4, x_4, x_4^2, x_4^3, x_4^4, y_3, x_4y_3, y_3^2, x_4y_3^2, y_3^3, x_4y_3^3, y_3^4, x_4y_3^4, y_4, x_3y_4,$$
$$x_3^2y_4, x_3^3y_4, x_3^4y_4, x_4y_4, x_4^2y_4, x_4^3y_4, x_4^4y_4, y_4^2, x_4y_4^2, y_4^3, x_4y_4^3, y_4^4, x_4y_4^4\}\,.$$

generic residue

PentaBox Integrand Decomposition

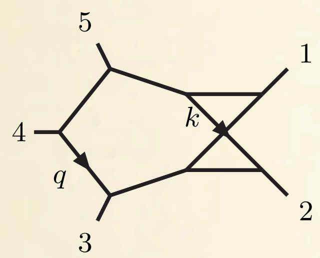
$$
\begin{aligned}
N(q,k) &= \Delta_{12345678}(q,k) + \\
&\quad + \Delta_{1234568}(q,k) D_7 + \Delta_{1234578}(q,k) D_6 + \\
&\quad + \Delta_{1234678}(q,k) D_5 + \Delta_{1235678}(q,k) D_4 = \\
&= c_{12345678,0} + c_{12345678,1}\ (q \cdot p_1) + \\
&\quad + c_{1234568,0} D_7 + c_{1234578,0} D_6 + \\
&\quad + c_{1234678,0} D_5 + c_{1235678,0} D_4\ ,
\end{aligned}
$$



Global (N=N)-test: OK

PentaCross Integrand Decomposition

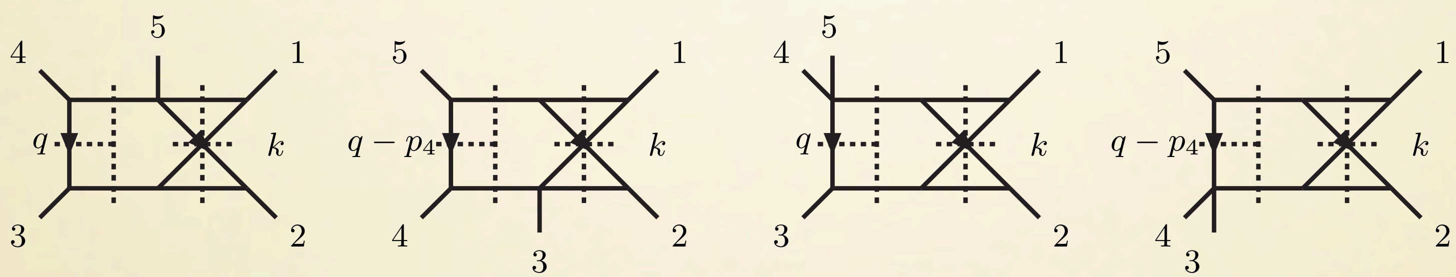
$$
\begin{aligned}
D_1 &= k^2\\
D_2 &= (k+p_2)^2\\
D_3 &= (k+q-p_4-p_5)^2\\
D_4 &= q^2\\
D_5 &= (q+p_3)^2\\
D_6 &= (q-p_4)^2\\
D_7 &= (q-p_4-p_5)^2\\
D_8 &= (q+k+p_2+p_3)^2\ .
\end{aligned}
$$



$$N(q,k) = 2\,q\cdot v + \alpha$$ Carrasco & Johansson (2011)

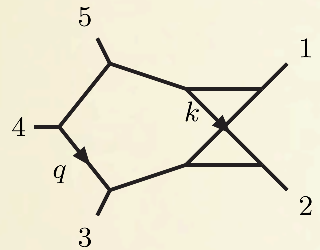
$$v^\mu = \frac{1}{4}\Big(\gamma_{12}(p_1^\mu - p_2^\mu) + \gamma_{23}(p_2^\mu - p_3^\mu) + 2\,\gamma_{45}(p_4^\mu - p_5^\mu) + \gamma_{13}(p_1^\mu - p_3^\mu)\Big)$$

$$\alpha = \frac{1}{4}\Big(2\,\gamma_{12}(s_{45}-s_{12}) + \gamma_{23}(s_{45}+3s_{12}-s_{13}) + 2\,\gamma_{45}(s_{14}-s_{15}) + \gamma_{13}(s_{12}+s_{45}-s_{13})\Big)$$

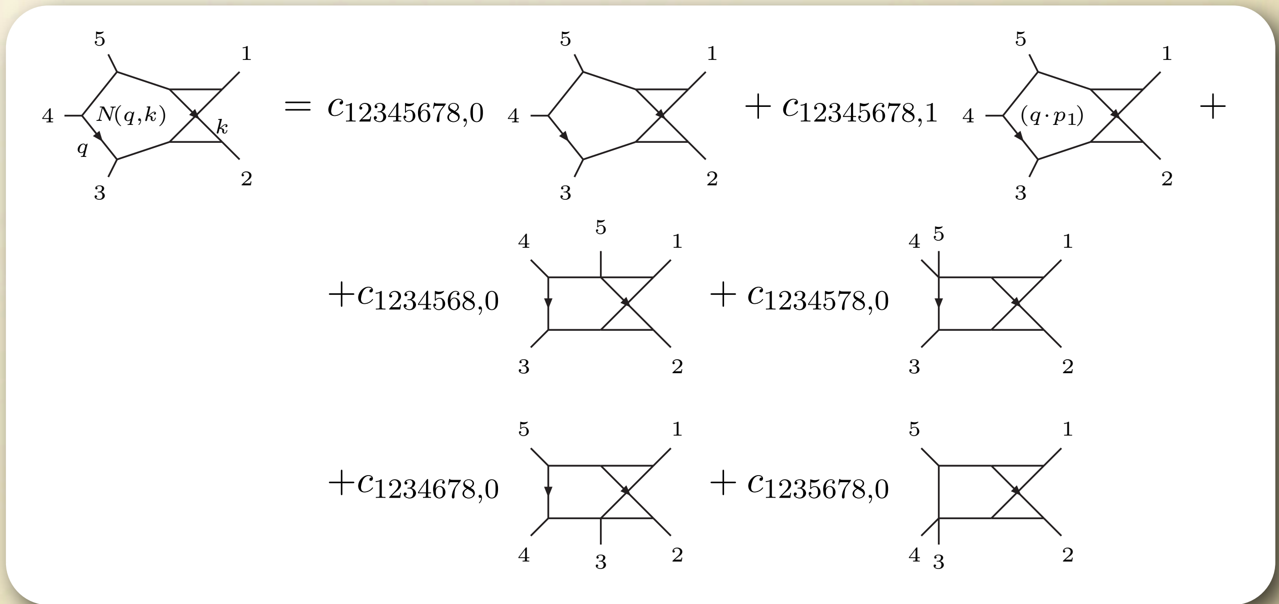


PentaCross Integrand Decomposition

$$
\begin{aligned}
D_1 &= k^2\\
D_2 &= (k+p_2)^2\\
D_3 &= (k+q-p_4-p_5)^2\\
D_4 &= q^2\\
D_5 &= (q+p_3)^2\\
D_6 &= (q-p_4)^2\\
D_7 &= (q-p_4-p_5)^2\\
D_8 &= (q+k+p_2+p_3)^2\ .
\end{aligned}
$$

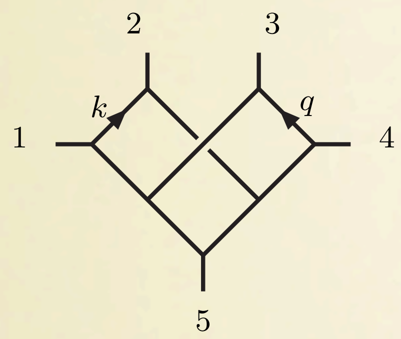


$$
\begin{aligned}
N(q,k) &= \Delta_{12345678}(q,k) + \\
&\quad +\Delta_{1234568}(q,k)D_7 + \Delta_{1234578}(q,k)D_6 + \\
&\quad +\Delta_{1234678}(q,k)D_5 + \Delta_{1235678}(q,k)D_4 = \\
&= c_{12345678,0} + c_{12345678,1}\ (q\cdot p_1) + \\
&\quad +c_{1234568,0}D_7 + c_{1234578,0}D_6 + \\
&\quad +c_{1234678,0}D_5 + c_{1235678,0}D_4\ ,
\end{aligned}
$$



The coefficients are the same of the planar case.

The Last Contribution to the 5-point N=4 SYM



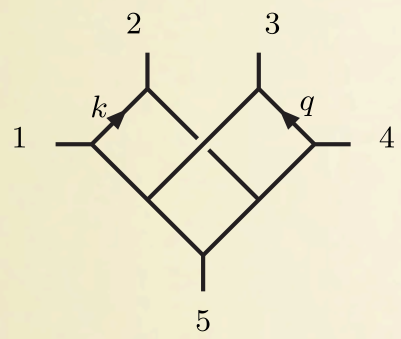
$$
\begin{aligned}
D_1 &= k^2 \\
D_2 &= (k - p_1)^2 \\
D_3 &= (k + p_2)^2 \\
D_4 &= q^2 \\
D_5 &= (q + p_3)^2 \\
D_6 &= (q - p_4)^2 \\
D_7 &= (q - k + p_1 + p_3)^2 \\
D_8 &= (q - k - p_2 - p_4)^2
\end{aligned}
$$

N(q,k) is ***linear*** in the loop momenta

Carrasco & Johansson (2011)

5-POINT 8FOLD-CUT

$D_1 = \ldots = D_8 = 0$ 8 solutions



$$\Delta_{12345678}(q,k) = \text{Res}_{12345678}\left\{\mathcal{N}_{1\cdots8}(q,k)\right\}\,.$$

$$q^\mu = \sum_{i=1}^{4} y_i\,\tau_i^\mu\,, \qquad k^\mu = \sum_{i=1}^{4} x_i\,e_i^\mu\,.$$

$$e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4$$

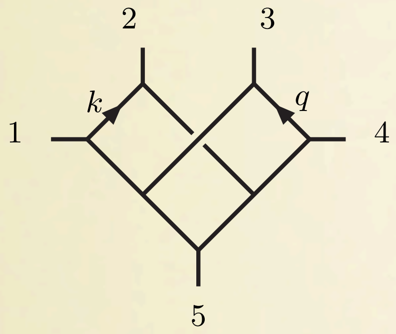
The residue contains 8 monomials $\{1, x_4, y_3, y_3^2, y_4, x_4 y_4, y_4^2, y_4^3\}$

[Maximum Cut Thm]

generic residue

5-POINT 8FOLD-CUT

$D_1 = \ldots = D_8 = 0$ 8 solutions



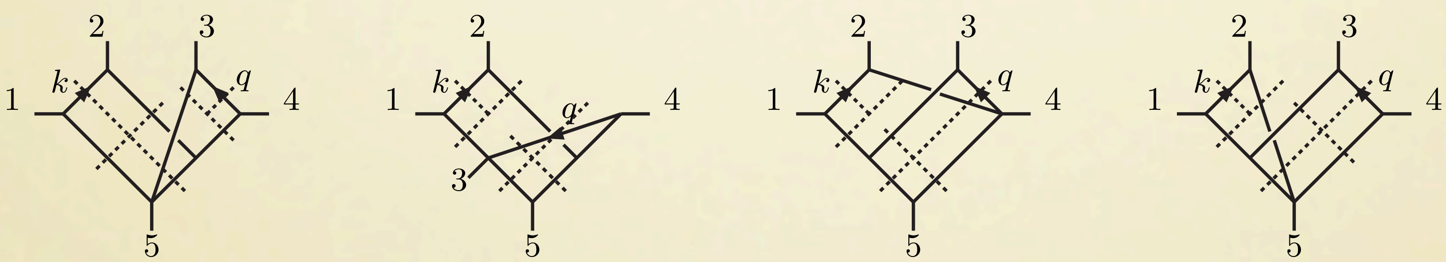
$$\Delta_{12345678}(q,k) = \mathrm{Res}_{12345678}\left\{\mathcal{N}_{1\cdots 8}(q,k)\right\} .$$

$$q^\mu = \sum_{i=1}^{4} y_i\, \tau_i^\mu , \qquad k^\mu = \sum_{i=1}^{4} x_i\, e_i^\mu .$$

$$e_1 = p_1, \qquad e_2 = p_2, \qquad \tau_1 = p_3, \qquad \tau_2 = p_4$$

The residue contains 8 monomials $\{1, x_4, y_3, y_3^2, y_4, x_4 y_4, y_4^2, y_4^3\}$

... FURTHER REDUCTION ...



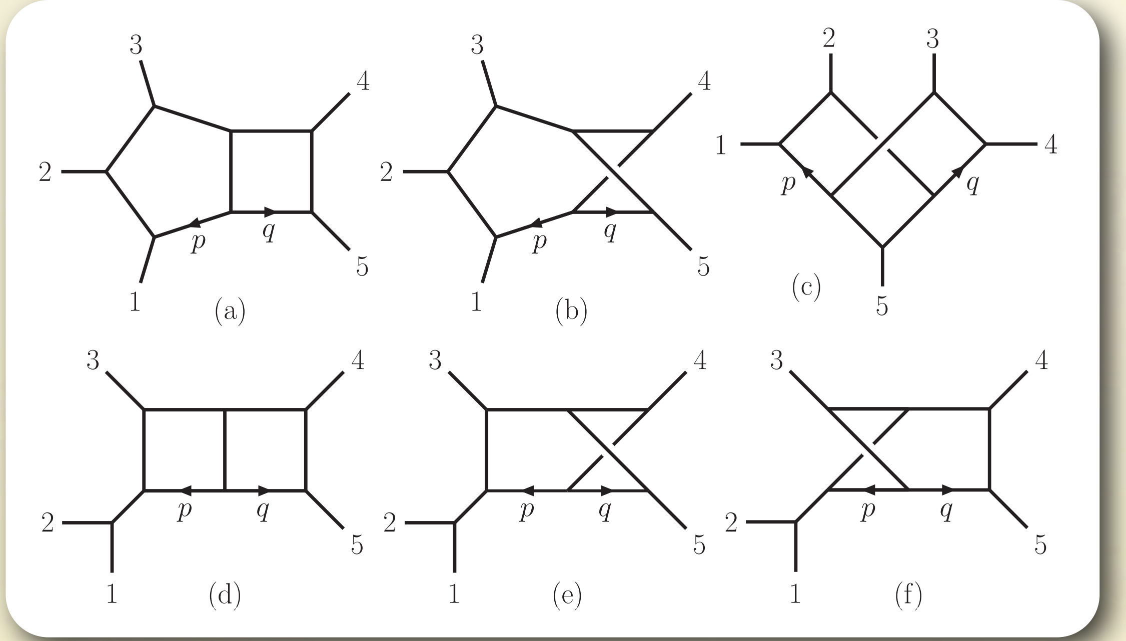
COMPLETE DECOMPOSITION

Global (N=N)-test fulfilled!

2-loop 5-point amplitudes in N=8 SuGra

Same topologies as in the N=4 SYM, but N(q,k) is ***quadratic*** in the loop momenta

Carrasco & Johansson (2011)



The integrand reduction is analogous to the N=4 SYM case, involving the same cuts and residues.

Due to one extra power of loop momenta, the reduction involves also ***6-denominator diagrams***: in the corresponding residues, the constant term is the only non-vanishing coefficient.

Conclusions

- A unique mathematical framework for Amplitudes at any order in Perturbation Theory
  - one ingredient: Feynman denominator
  - one operation: *partial fractioning*
- Multivariate Polynomial Division/Groebner-basis generates the ***residue*** at an arbitrary cut
  - the general expression for the factorized amplitude
- Residues' ***classification*** complementary to Landau's singularity classification
  - byproduct: the Maximum-cut Theorem
- *Recursive* generation of the *Integrand-decomposition Formula* @ any loop
- Amplitude decomposition from the shape of ***residues***
  - ISP's determine a (non-minimal) MI-set

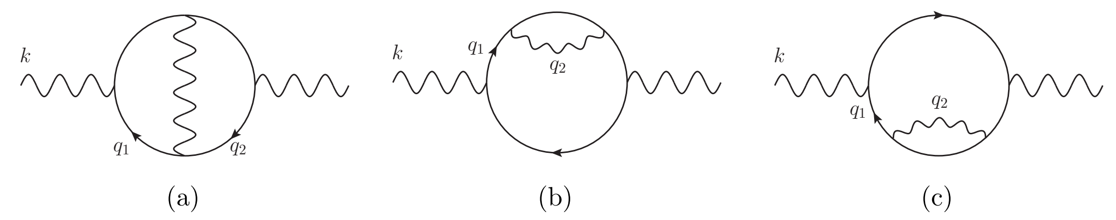
Outlook

- ***Automation***
- ***additional identities*** at the integrand level to reduce the number of MI's

Extra Slides

2-loop Decomposition in DimReg (t'HV)

just sent by Tiziano Peraro



$$\Pi_{\mu\nu} = \left(g^{(d)}_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k^2) \tag{1}$$

where $g^{(d)}_{\mu\nu}$ is the metric tensor in $d = 4 - 2\epsilon$ dimensions and $\Pi(k^2)$ can be obtained by tracing the previous equation[1]

$$\Pi(k^2) = \frac{1}{d-1} \, \Pi^\mu{}_\mu . \tag{2}$$

The two-loop 1PI contributions are given by the diagrams depicted in Fig. 1, hence we may write

$$(d-1)\, \Pi^{(2l)}_{\mathrm{1PI}}(k^2) = \Pi_a(k^2) + \Pi_b(k^2) + \Pi_c(k^2) \tag{3}$$

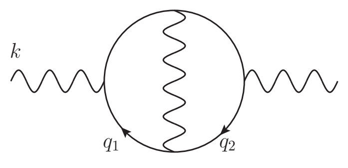
where each contribution is given by the trace of the corresponding diagram. Since the last two diagrams are related by symmetry, I will only give the reduction of the first two.

The d-dimensional loop momenta $\bar{q}_1$ and $\bar{q}_2$ are decomposed as usual

$$\bar{q}_i = q_i + \vec{\mu}_i, \qquad \bar{q}_i \cdot \bar{q}_j = q_i \cdot q_j - \mu_{ij} \tag{4}$$

with $\mu_{ij} \equiv \vec{\mu}_i \cdot \vec{\mu}_j$.

2 Diagram (a)



(a)

The denominators are

$$\begin{aligned} D_1 &= q_1^2 - \mu_{11} \\ D_3 &= q_2^2 - \mu_{22} \\ D_2 &= q_1^2 + k^2 + 2(k \cdot q_1) - \mu_{11} \\ D_4 &= q_2^2 + k^2 + 2(k \cdot q_2) - \mu_{22} \\ D_5 &= q_1^2 + q_2^2 - 2(q_1 \cdot q_2) - \mu_{11} - \mu_{22} + 2\mu_{12}. \end{aligned} \tag{5}$$

The integrand is

$$\mathcal{I}_a = \frac{\mathcal{N}_a}{D_1 \dots D_5} \tag{6}$$

and the numerator

$$\mathcal{N}_a = \mathcal{N}_a^{(0)} + \mathcal{N}_a^{(1)}\epsilon + \mathcal{N}_a^{(2)}\epsilon^2 \tag{7}$$

where

$$\begin{aligned} \mathcal{N}_a^{(0)} =& -32\,\mu_{12}^2 - 32\,(k \cdot q_1)\,(k \cdot q_2) - 32\,(k \cdot q_1)\,(q_1 \cdot q_2) + 32\,(k \cdot q_1)\,\mu_{12} \\ & -32\,(k \cdot q_2)\,(q_1 \cdot q_2) + 32\,(k \cdot q_2)\,\mu_{12} + 64\,(q_1 \cdot q_2)\,\mu_{12} - 32\,(q_1 \cdot q_2)^2 \\ \mathcal{N}_a^{(1)} =& \, 32\,\mu_{12}^2 + 16\,\mu_{11}\,\mu_{22} + 16\,k^2\,(q_1 \cdot q_2) - 16\,k^2\,\mu_{12} + 32\,(k \cdot q_1)\,(k \cdot q_2) \\ & + 32\,(k \cdot q_1)\,(q_1 \cdot q_2) + 16\,(k \cdot q_1)\,q_2^2 - 32\,(k \cdot q_1)\,\mu_{12} - 16\,(k \cdot q_1)\,\mu_{22} \\ & + 16\,(k \cdot q_2)\,q_1^2 + 32\,(k \cdot q_2)\,(q_1 \cdot q_2) - 32\,(k \cdot q_2)\,\mu_{12} - 16\,(k \cdot q_2)\,\mu_{11} \\ & + 16\,q_1^2\,q_2^2 - 16\,q_1^2\,\mu_{22} - 64\,(q_1 \cdot q_2)\,\mu_{12} + 32\,(q_1 \cdot q_2)^2 - 16\,q_2^2\,\mu_{11} \\ \mathcal{N}_a^{(2)} =& -16\,\mu_{11}\,\mu_{22} - 16\,k^2\,(q_1 \cdot q_2) + 16\,k^2\,\mu_{12} - 16\,(k \cdot q_1)\,q_2^2 + 16\,(k \cdot q_1)\,\mu_{22} \\ & -16\,(k \cdot q_2)\,q_1^2 + 16\,(k \cdot q_2)\,\mu_{11} - 16\,q_1^2\,q_2^2 + 16\,q_1^2\,\mu_{22} + 16\,q_2^2\,\mu_{11}. \end{aligned} \tag{8}$$

The complete decomposition of the numerators reads

$$\begin{aligned}
\mathcal{N}_a^{(0)} &= D_5\,\Delta^{(0)}_{1234} + D_4\left(8\,k^2\right) + D_4\,D_5\left(4\right) + D_3\left(8\,k^2\right) + D_3\,D_5\left(4\right) + D_3\,D_4\left(-8\right) \\
&\quad + D_2\left(8\,k^2\right) + D_2\,D_5\left(4\right) + D_2\,D_4\left(-8\right) + D_2\,D_4\,D_5\left(\frac{4}{k^2}\right) + D_2\,D_3\,D_5\left(-\frac{4}{k^2}\right) \\
&\quad + D_1\left(8\,k^2\right) + D_1\,D_5\left(4\right) + D_1\,D_4\,D_5\left(-\frac{4}{k^2}\right) + D_1\,D_3\left(-8\right) + D_1\,D_3\,D_5\left(\frac{4}{k^2}\right) \\
&\quad + D_1\,D_2\left(-8\right) - 8\,(k^2)^2 \\
\mathcal{N}_a^{(1)} &= D_5\,\Delta^{(1)}_{1234} + D_4\left(-8\,k^2\right) + D_4\,D_5\left(-4\right) + D_3\left(-8\,k^2\right) + D_3\,D_5\left(-4\right) + D_3\,D_4\left(8\right) \\
&\quad + D_2\left(-8\,k^2\right) + D_2\,D_5\left(-4\right) + D_2\,D_4\left(8\right) + D_2\,D_4\,D_5\left(-\frac{4}{k^2}\right) + D_2\,D_3\left(8\right) \\
&\quad + D_2\,D_3\,D_5\left(\frac{4}{k^2}\right) + D_1\left(-8\,k^2\right) + D_1\,D_5\left(-4\right) + D_1\,D_4\left(8\right) + D_1\,D_4\,D_5\left(\frac{4}{k^2}\right) \\
&\quad + D_1\,D_3\left(8\right) + D_1\,D_3\,D_5\left(-\frac{4}{k^2}\right) + D_1\,D_2\left(8\right) + 8\,(k^2)^2 \\
\mathcal{N}_a^{(2)} &= +D_5\left(8\,k^2\right) + D_2\,D_3\left(-8\right) + D_1\,D_4\left(-8\right)
\end{aligned} \tag{9}$$

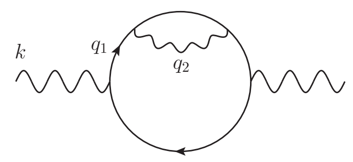
where

$$\begin{aligned}
\Delta^{(0)}_{1234} &= -16\,\mu_{12} - 12\,k^2 + \frac{16\,(q_1\cdot E_2)\,(q_2\cdot E_2)}{E_2^2} + \frac{16\,(q_1\cdot e_3)\,(q_2\cdot e_4)}{(e_3\cdot e_4)} + \frac{16\,(q_1\cdot e_4)\,(q_2\cdot e_3)}{(e_3\cdot e_4)} \\
\Delta^{(1)}_{1234} &= 16\,\mu_{12} + 4\,k^2 - \frac{16\,(q_1\cdot E_2)\,(q_2\cdot E_2)}{E_2^2} - \frac{16\,(q_1\cdot e_3)\,(q_2\cdot e_4)}{(e_3\cdot e_4)} - \frac{16\,(q_1\cdot e_4)\,(q_2\cdot e_3)}{(e_3\cdot e_4)}
\end{aligned} \tag{10}$$

The decomposition in terms of MIs is obtained by plugging these expressions in Eq. (6) and dropping those contributions which vanish upon integration. We obtain

$$
\begin{aligned}
\int d^dq_1 d^dq_2 \mathcal{I}_a = \int d^dq_1 d^dq_2 \Big(& - \frac{8\,(k^2)^2}{D_1D_2D_3D_4D_5} - \frac{12\,k^2}{D_1D_2D_3D_4} + \frac{8\,k^2}{D_1D_2D_3D_5} \\
& + \frac{8\,k^2}{D_1D_2D_4D_5} + \frac{8\,k^2}{D_1D_3D_4D_5} + \frac{8\,k^2}{D_2D_3D_4D_5} \Big) \\
+ \epsilon \int d^dq_1 d^dq_2 \Big(& \frac{8\,(k^2)^2}{D_1D_2D_3D_4D_5} + \frac{4\,k^2}{D_1D_2D_3D_4} - \frac{8\,k^2}{D_1D_2D_3D_5} \\
& - \frac{8\,k^2}{D_1D_2D_4D_5} - \frac{8\,k^2}{D_1D_3D_4D_5} - \frac{8\,k^2}{D_2D_3D_4D_5} \\
& + \frac{8}{D_1D_4D_5} + \frac{8}{D_2D_3D_5} \Big) \\
+ \epsilon^2 \int d^dq_1 d^dq_2 \Big(& \frac{8\,k^2}{D_1D_2D_3D_4} - \frac{8}{D_1D_4D_5} - \frac{8}{D_2D_3D_5} \Big)
\end{aligned}
\tag{11}
$$

3 Diagram (b)



(b)

The denominators are

$$
\begin{aligned}
D_1 &= q_1^2 - \mu_{11} \\
D_3 &= q_2^2 - \mu_{22} \\
D_2 &= q_1^2 + k^2 - 2(k \cdot q_1) - \mu_{11} \\
D_4 &= q_1^2 + q_2^2 + 2(q_1 \cdot q_2) - \mu_{11} - \mu_{22} - 2\mu_{12}.
\end{aligned} \tag{12}
$$

The integrand is

$$\mathcal{I}_b = \frac{\mathcal{N}_b}{D_1^2 D_2 D_3 D_5} \tag{13}$$

and the numerator

$$\mathcal{N}_b = \mathcal{N}_b^{(0)} + \mathcal{N}_b^{(1)}\epsilon + \mathcal{N}_b^{(2)}\epsilon^2 \tag{14}$$

where

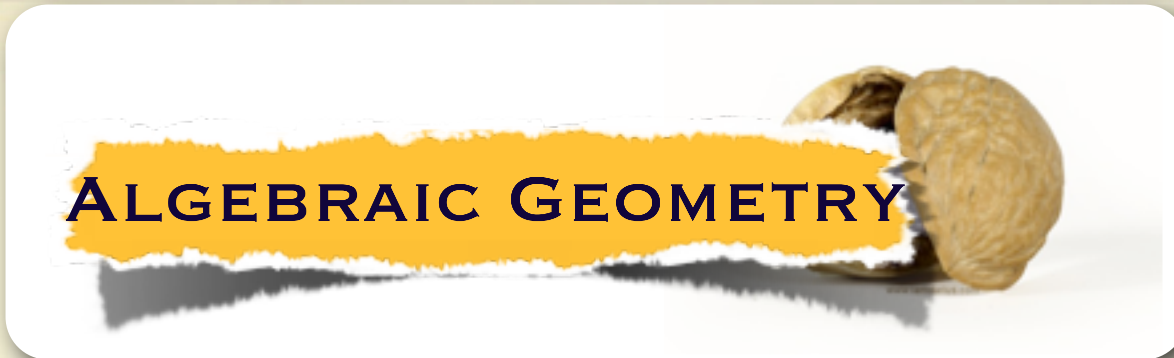
$$
\begin{aligned}
\mathcal{N}_b^{(0)} = {} & 16\,\mu_{11}\mu_{12} + 16\,\mu_{11}2 - 16\,(k \cdot q_1)\,q_1^2 - 32\,(k \cdot q_1)\,(q_1 \cdot q_2) + 32\,(k \cdot q_1)\,\mu_{12} \\
& + 16\,(k \cdot q_1)\,\mu_{11} + 16\,(k \cdot q_2)\,q_1^2 - 16\,(k \cdot q_2)\,\mu_{11} + 16\,q_1^2\,(q_1 \cdot q_2) \\
& - 16\,q_1^2\,\mu_{12} - 32\,q_1^2\,\mu_{11} + 16\,(q_1^2)^2 - 16\,(q_1 \cdot q_2)\,\mu_{11} \\
\mathcal{N}_b^{(1)} = {} & -32\,\mu_{11}\mu_{12} - 32\,\mu_{11}2 + 32\,(k \cdot q_1)\,q_1^2 + 64\,(k \cdot q_1)\,(q_1 \cdot q_2) - 64\,(k \cdot q_1)\,\mu_{12} \\
& - 32\,(k \cdot q_1)\,\mu_{11} - 32\,(k \cdot q_2)\,q_1^2 + 32\,(k \cdot q_2)\,\mu_{11} - 32\,q_1^2\,(q_1 \cdot q_2) \\
& + 32\,q_1^2\,\mu_{12} + 64\,q_1^2\,\mu_{11} - 32\,(q_1^2)^2 + 32\,(q_1 \cdot q_2)\,\mu_{11} \\
\mathcal{N}_b^{(2)} = {} & 16\,\mu_{11}\mu_{12} + 16\,\mu_{11}2 - 16\,(k \cdot q_1)\,q_1^2 - 32\,(k \cdot q_1)\,(q_1 \cdot q_2) + 32\,(k \cdot q_1)\,\mu_{12} \\
& + 16\,(k \cdot q_1)\,\mu_{11} + 16\,(k \cdot q_2)\,q_1^2 - 16\,(k \cdot q_2)\,\mu_{11} + 16\,q_1^2\,(q_1 \cdot q_2) - 16\,q_1^2\,\mu_{12} \\
& - 32\,q_1^2\,\mu_{11} + 16\,(q_1^2)^2 - 16\,(q_1 \cdot q_2)\,\mu_{11}.
\end{aligned} \tag{15}
$$

The complete decomposition of the numerators reads

$$\begin{aligned}\mathcal{N}_b^{(0)} &= D_4\Big(-8\,k^2\Big) + D_3\Big(8\,k^2\Big) + D_2\,D_4\Big(8\Big) + D_2\,D_3\Big(-8\Big) + D_1\Big(16\,(k\cdot q_2)\Big) + D_1^2\Big(8\Big)\\ \mathcal{N}_b^{(1)} &= D_4\Big(16\,k^2\Big) + D_3\Big(-16\,k^2\Big) + D_2\,D_4\Big(-16\Big) + D_2\,D_3\Big(16\Big) + D_1\Big(-32\,(k\cdot q_2)\Big)\\ &\quad + D_1^2\Big(-16\Big)\\ \mathcal{N}_b^{(2)} &= D_4\Big(-8\,k^2\Big) + D_3\Big(8\,k^2\Big) + D_2\,D_4\Big(8\Big) + D_2\,D_3\Big(-8\Big) + D_1\Big(16\,(k\cdot q_2)\Big) + D_1^2\Big(8\Big)\end{aligned} \tag{16}$$

The decomposition in terms of MIs is

$$\int d^dq_1 d^dq_2 \mathcal{I}_b = (1-2\epsilon+\epsilon^2)\int d^dq_1 d^dq_2\Big(\frac{16\,(k\cdot q_2)}{D_1D_2D_3D_4} + \frac{8}{D_2D_3D_4}\Big) \tag{17}$$



- deals with multivariate polynomials in z $= (z_1, z_2, \ldots)$.
- Ideal $\mathcal{J} \equiv \langle \omega_1(\mathsf{z}) \cdots \omega_s(\mathsf{z}) \rangle$ generated by ω_i
  - $\mathcal{J} = \{ \sum_i h_i(\mathsf{z})\ \omega_i(\mathsf{z}) \}$
  - polynomial coefficients $h_i(\mathsf{z})$
- Multivariate polynomial division of $f(\mathsf{z})$ modulo $\omega_1(\mathsf{z}), \ldots, \omega_s(\mathsf{z})$
  - needs an order, i.e. $z_1 z_2 \overset{?}{>} z_1^2$
  - $\rightsquigarrow f(\mathsf{z}) = \sum_i h_i(\mathsf{z}) \omega_i(\mathsf{z}) + \mathcal{R}(\mathsf{z})$
  - $h_i(\mathsf{z})$ & $\mathcal{R}(\mathsf{z})$ not unique
- Gröbner basis $\{g_1(\mathsf{z}), \ldots, g_r(\mathsf{z})\}$
  - exists (Buchberger's algorithm) & generates $\mathcal{J}$
  - $\rightsquigarrow$ unique $\mathcal{R}(\mathsf{z})$
- Hilbert's Nullstellensatz
  - $V(\mathcal{J}) =$ set of common zeros of $\mathcal{J}$
  - ($f = 0$ in $V(\mathcal{J})$) $\Rightarrow$ ($f^r \in \mathcal{J}$ for some r)
  - Weak Nullstellensatz: ($V(\mathcal{J}) = \varnothing$) $\Leftrightarrow$ ($1 \in \mathcal{J}$)

weak Nullstellensatz Theorem

Theorem 1.2.3 (Weak Hilbert Nullstellensatz)**.** *If k is algebraically closed, then $V(S) = \emptyset$ iff there exists $f_1 \dots f_N \in S$ and $g_1 \dots g_N \in k[x_1, \dots x_n]$ such that $\sum f_i g_i = 1$*

The German word nullstellensatz could be translated as "zero set theorem". The Weak Nullstellensatz can be rephrased as $V(S) = \emptyset$ iff $\langle S \rangle = (1)$. Since this result is central to much of what follows, **we will assume that k is alge-**

Radical Ideal

Given an ideal $\mathcal{J}$, the *radical* of $\mathcal{J}$ is $\sqrt{\mathcal{J}} \equiv \{f \in P[\mathbf{z}] : \ \exists\, s \in \mathbb{N},\ f^s \in \mathcal{J}\}$.

$\mathcal{J}$ is radical iff $\mathcal{J} = \sqrt{\mathcal{J}}$.

Finiteness Theorem

The following theorem bounds the number of points in $\mathbf{V}(I)$ whenever I is zero dimensional.

Theorem 3-4. *Let I be a zero-dimensional ideal in $\mathbb{C}[x_1, \dots, x_n]$. Then the number of points in $\mathbf{V}(I)$ is at most $\dim_{\mathbb{C}}(A)$. Equality occurs if and only if I is a radical ideal.*

Shape Lemma

since $p_{red}(x)/x$ is a cubic polynomial in x^2.

If I is a zero-dimensional radical ideal in $S = \mathbb{Q}[x_1, \dots, x_n]$ then, possibly after a linear change of variables, the ring S/I is always isomorphic to the univariate quotient ring $\mathbb{Q}[x_i]/(I \cap \mathbb{Q}[x_i])$. This is the content of the following result.

Proposition 2.3. **(Shape Lemma)** *Let I be a zero-dimensional radical ideal in $\mathbb{Q}[x_1, \dots, x_n]$ such that all d complex roots of I have distinct x_n-coordinates. Then the reduced Gröbner basis of I in the lexicographic term order has the shape*

$$\mathcal{G} \quad = \quad \{\, x_1 - q_1(x_n),\, x_2 - q_2(x_n),\, \dots, x_{n-1} - q_{n-1}(x_n),\, r(x_n)\, \}$$

where r is a polynomial of degree d and the q_i are polynomials of degree $\leq d-1$.

For polynomial systems of moderate size, `Singular` is reall

MCT: PROOF (PART 1)

Proof. Let us parametrize the propagators using 4ℓ variables $\mathbf{z} = (z_1, \ldots z_{4\ell})$. In this parametrization, the solutions of the maximum-cut read,

$$\mathbf{z}^{(i)} = \left(z_1^{(i)}, \ldots, z_{4\ell}^{(i)}\right) \ , \text{ with } i = 1, \ldots, n_s \ .$$

Let $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ be the ideal generated by the on-shell denominators, $\mathcal{J}_{i_1 \cdots i_{4\ell}} = \langle D_{i_1}, \ldots, D_{i_{4\ell}} \rangle$. According to the assumptions, the number n_s of the solutions is finite, and each of them has multiplicity one, therefore $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ is zero-dimensional and radical [1], In this case, the *Finiteness Theorem* ensures that the remainder of the division of any polynomial modulo $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ can be parametrised exactly by n_s coefficients.

MCT: PROOF (PART 2)

Moreover, up to a linear coordinate change, we can assume that all the solutions of the system have distinct first coordinate z_1, i.e. $z_1^{(i)} \neq z_1^{(j)}\ \forall\ i \neq j$. We observe that $\mathcal{J}_{i_1 \cdots i_{4\ell}}$ and z_1 are in the *Shape Lemma* position therefore a Gröbner basis for the lexicographic order $z_1 < z_2 < \cdots < z_n$ is $\mathcal{G}_{i_1 \cdots i_{4\ell}} = \{g_1, \ldots, g_{4\ell}\}$, in the form

$$\begin{cases} g_1(\mathbf{z}) &= \quad f_1(z_1) \\ g_2(\mathbf{z}) &= \quad z_2 - f_2(z_1) \\ & \vdots \\ g_{4\ell}(\mathbf{z}) &= z_{4\ell} - f_{4\ell}(z_1) \,. \end{cases}$$

The functions f_i are univariate polynomials in z_1. In particular f_1 is a rank-n_s square-free polynomial

$$f_1(z_1) = \prod_{i=1}^{n_s} \left(z_1 - z_1^{(i)} \right) \;,$$

i.e. it does not exhibits repeated roots. The multivariate division of $\mathcal{N}_{i_1 \cdots 1_{4\ell}}$ modulo $\mathcal{G}_{i_1 \cdots i_{4\ell}}$ leaves a remainder $\Delta_{i_1 \cdots i_{4\ell}}$ which is a univariate polynomial in z_1 of degree $(n_s - 1)$ in accordance with the *Finiteness Theorem*. $\square$

QCD recursion relations from the largest time equation

Vaman, Yao (2005)

The factorization procedure is to cut these q_i successively by shifting them by $z\eta$. The on-shell conditions will give us a set of solutions, points in the complex plane, namely

$$z_i = \frac{q_i^2 + m_i^2}{2\eta \cdot q_i},$$

The identity which we want to establish is

$$\begin{aligned}
\frac{1}{q_1^2+m_1^2}\frac{1}{q_2^2+m_2^2}\cdots\frac{1}{q_{n-1}^2+m_{n-1}^2} &= \frac{1}{q_1^2+m_1^2}\frac{1}{(q_2-z_1\eta)^2+m_2^2}\cdots\frac{1}{(q_{n-1}-z_1\eta)^2+m_{n-1}^2} \\
&+ \frac{1}{(q_1-z_2\eta)^2+m_1^2}\frac{1}{q_2^2+m_2^2}\cdots\frac{1}{(q_{n-1}-z_2\eta)^2+m_{n-1}^2} \\
&+ \cdots\cdots\cdots \\
&+ \frac{1}{(q_1-z_{n-1}\eta)^2+m_1^2}\cdots\frac{1}{(q_{n-2}-z_{n-1}\eta)^2+m_{n-2}^2}\frac{1}{q_{n-1}^2+m_{n-1}^2}
\end{aligned}$$

making cuts, we have

$$\bar{q}_i^2 + m_i^2 = 0 \;\rightarrow\; q_i^2 + m_i^2 = 2z_i\eta\cdot q_i\,.$$

$$(q_i - z_j\eta)^2 + m_i^2 = q_i^2 + m_i^2 - 2z_j\eta\cdot q_i\,.$$

$$(q_i - z_j\eta)^2 + m_i^2 = 2\eta\cdot q_i(z_i - z_j)\,.$$

Putting these together, we see the identity holds if one can show

$$\begin{aligned}
\frac{(-1)^n}{z_1 z_2\cdots z_{n-1}} &= \frac{1}{z_1(z_1-z_2)(z_1-z_3)\cdots(z_1-z_{n-1})} \\
&+ \frac{1}{(z_2-z_1)z_2(z_2-z_3)\cdots(z_2-z_{n-1})} \\
&\quad\cdots\cdots\cdots \\
&+ \frac{1}{(z_{n-1}-z_1)(z_{n-1}-z_2)\cdots(z_{n-1}-z_{n-2})z_{n-1}}\,. \qquad (7.9)
\end{aligned}$$

This is so, because (7.9) is just a formula of partial fractioning, or it is just a statement that the integral

$$\int \frac{dz}{z(z-z_1)(z-z_2)\cdots(z-z_{n-1})} = 0$$

for a complex variable z over a contour which encloses all the poles.