Threshold resummation in SCET and in perturbative QCD: an analytic comparison

SM@LHC Institute, CERN, October 4$^{\text{th}}$, 2012

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Results obtained in collaboration with Marco Bonvini, Stefano Forte and Margherita Ghezzi, NPB861(2012)337, arXiv:1201.6364 [hep-ph]

Production cross section for a system with invariant mass M^2 in hadron collisions at energy $\sqrt{s}$:

$$\sigma(\tau, M^2) = \int_\tau^1 \frac{dz}{z} \mathcal{L}\left(\frac{\tau}{z}\right) C(z, \alpha_{\rm S}(M^2)); \qquad \tau = \frac{M^2}{s}$$

where $\mathcal{L}$ is a parton luminosity, and the parton cross section $C(z, \alpha_{\rm S})$ has a perturbative expansion in QCD:

$$C(z, \alpha_{\rm S}) = \delta(1-z) + \sum_{n=1}^{\infty} C_n(z)\alpha_{\rm S}^n; \qquad z = \frac{M^2}{\hat{s}}$$

Examples: Higgs, Drell-Yan pairs, heavy quark pairs.

In the threshold region $s \sim M^2$, $\tau \to 1$, and therefore $z \geq \tau$ is also close to 1. Since

$$C_n(z) \sim \left[\frac{\log^{2n-1}(1-z)}{1-z}\right]_+$$

the perturbative expansion is unreliable in this region:

$$\alpha_S^n \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(\frac{\tau}{z}\right)\, C_n(z) \sim \mathcal{L}(\tau)\alpha_S^n \log^{2n}(1-\tau) + \textbf{less singular terms}$$

Resummation is needed.

What happens when τ is *not* that close to 1 is also rather interesting.

Resummation in perturbative QCD

[S. Catani and L. Trentadue, NPB 327 (1989) 323; G. Sterman, NPB 281 (1987) 310; S. Forte and G. Ridolfi, B650 (2003) 229]

Go to Mellin moments:

$$\sigma(N, M^2) = \int_0^1 d\tau\, \tau^{N-1}\, \sigma(\tau, M^2) = \mathcal{L}(N) C(N, \alpha_S(M^2))$$

Resummation techniques in QCD provide a resummed expression for $C(N, \alpha_S(M^2))$:

$$\begin{aligned} C_{QCD}(N, M^2) &= g_0(\alpha_S) \exp \mathcal{S}(\bar{\alpha} L, \bar{\alpha}) \\ \mathcal{S}(\lambda, \bar{\alpha}) &= \frac{1}{\bar{\alpha}} g_1(\lambda) + g_2(\lambda) + \bar{\alpha} g_3(\lambda) + \bar{\alpha}^2 g_4(\lambda) + \ldots \\ \bar{\alpha} &\equiv 2\alpha_S(M^2)\beta_0, \quad L \equiv \ln \frac{1}{N}, \end{aligned}$$

where β_0 is the first coefficient of the QCD β function

$$\mu^2 \frac{d\alpha_S(\mu^2)}{d\mu^2} = -\beta_0 \alpha_S^2(\mu^2) + \mathcal{O}(\alpha_S^2); \qquad \beta_0 = \frac{11 C_A - 2n_f}{12\pi}$$

Logarithmic accuracy in QCD

Powers of $\log N$ correctly predicted by QCD resummation, order by order in the expansion of the coefficient function:

log approx.	g_i **up to**	g_0 **up to order**	**accuracy:** $\alpha_s^n L^k$
LL	$i = 1$	$(\alpha_s)^0$	$k = 2n$
NLL	$i = 2$	$(\alpha_s)^1$	$2n - 2 \leq k \leq 2n$
NNLL	$i = 3$	$(\alpha_s)^2$	$2n - 4 \leq k \leq 2n$
N^{p}LL	$i = p + 1$	$(\alpha_s)^p$	$2n - 2p \leq k \leq 2n$

For the purpose of comparison, a different form proves useful:

$$C_{\mathrm{QCD}}(N, M^2) = \hat{g}_0(\alpha_{\mathrm{S}}(M^2)) \exp \hat{\mathcal{S}}_{\mathrm{QCD}}\left(M^2, \frac{M^2}{\bar{N}^2}\right)$$

with

$$\hat{g}_0(\alpha_{\mathrm{S}}) = 1 + \hat{g}_{01}\alpha_{\mathrm{S}} + \mathcal{O}(\alpha_{\mathrm{S}}^2)$$

$$\hat{\mathcal{S}}_{\mathrm{QCD}}\left(M^2, \frac{M^2}{\bar{N}^2}\right) = \int_{M^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2}\left[A(\alpha_{\mathrm{S}}(\mu^2))\left(\ln\frac{1}{\bar{N}^2} - \ln\frac{\mu^2}{M^2}\right) + \hat{D}_2\alpha_{\mathrm{S}}^2(\mu^2)\right]$$

$$\bar{N} = Ne^{\gamma}$$

$C_{\mathrm{QCD}}(N, M^2)$ **can be expanded in powers of** $\alpha_{\mathrm{S}}(Q^2)$**, and the inverse Mellin transform computed term by term.**

The resulting series is divergent: $C_{\mathrm{QCD}}(N, M^2)$ **has a branch cut on the real positive axis due to the Landau pole of the running coupling, and has therefore no inverse Mellin transform.**

An unavoidable feature of perturbative QCD, which is badly behaved at very low scales.

2. The SCET approach

We will refer to the Drell-Yan resummed cross section as computed in T. Becher, M. Neubert and G. Xu, JHEP 0807 (2008) 030 [arXiv:0710.0680 [hep-ph]]

using SCET.

Resummed coefficient function given directly in momentum (z) space:

$$C_{\text{SCET}}(z, M^2, \mu_s^2) = H(M^2)U(M^2, \mu_s^2)S(z, M^2, \mu_s^2)$$

- $H(M^2)$ **is a hard coefficient, similar to** g_0
- $S(z, M^2, \mu_s^2)$ **soft emission factor**
- $U(M^2, \mu_s^2)$ **evolution factor**

The choice of μ_s determines which logarithms are being resummed.

Some details:

$$S(z, M^2, \mu_s^2) = \tilde{s}_{\mathrm{DY}} \left(\ln \frac{M^2}{\mu_s^2} + \frac{\partial}{\partial \eta}, \mu_s \right) \frac{1}{1-z} \left(\frac{1-z}{\sqrt{z}} \right)^{2\eta} \frac{e^{-2\gamma\eta}}{\Gamma(2\eta)}$$

$$U(M^2, \mu_s^2) = \exp \left\{ - \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} \left[\Gamma_{\mathrm{cusp}} \left(\alpha_{\mathrm{S}}(\mu^2) \right) \ln \frac{\mu^2}{M^2} - \gamma_W \left(\alpha_{\mathrm{S}}(\mu^2) \right) \right] \right\}$$

$$\eta = \int_{M^2}^{\mu_s^2} \frac{d\mu^2}{\mu^2} \Gamma_{\mathrm{cusp}} \left(\alpha_{\mathrm{S}}(\mu^2) \right); \qquad \Gamma_{\mathrm{cusp}}(\alpha_{\mathrm{S}}) = A(\alpha_{\mathrm{S}})$$

The functions $\tilde{s}_{\mathrm{DY}}(L, \mu)$, $\gamma_W(\alpha_{\mathrm{S}})$ have perturbative expansions in powers of $\alpha_{\mathrm{S}}(\mu^2)$.

Logarithmic accuracy in the BNX approach

Powers of $\log \frac{\mu_s}{M}$ correctly predicted by SCET resummation, order by order in the expansion of the coefficient function:

RG-impr. PT PT	Log. approx.	Accuracy $\sim \alpha_s^n \ln^k \frac{\mu_s}{M}$	$\Gamma_{\rm cusp}$	γ_W	H, $\tilde{s}_{\rm DY}$
—	LL	$k = 2n$	1-loop	tree-level	tree-level
LO	NLL	$2n-1 \leq k \leq 2n$	2-loop	1-loop	tree-level
NLO	NNLL	$2n-3 \leq k \leq 2n$	3-loop	2-loop	1-loop
NNLO	NNNLL	$2n-5 \leq k \leq 2n$	4-loop	3-loop	2-loop

Less accurate by one power of log at each perturbative order wrt the QCD counting.

Same accuracy as in QCD achieved by including one more perturbative order in the calculation of $H(M^2)$.

Analytic comparison

The SCET expression can be Mellin-transformed and compared to the QCD result.

If μ_s is kept fixed and independent of z we get, to NNLL accuracy,

$$C_{\rm QCD}(N,M^2) = C_r(N,M^2,\mu_s^2)C_{\rm SCET}(N,M^2,\mu_s^2)$$

$$\begin{aligned} C_r(N,M^2,\mu_s^2) &= \exp\int_{\mu_s^2}^{M^2/\bar{N}^2}\frac{d\mu^2}{\mu^2}\Bigg[\left(A(\alpha_{\rm S}(\mu^2)) - \frac{A_1\alpha_{\rm S}(\mu^2)}{4}\right)\ln\frac{M^2}{\mu^2\bar{N}^2} \\ &\quad + \frac{A_1}{8}\beta(\alpha_{\rm S}(\mu^2))\ln^2\frac{M^2}{\mu^2\bar{N}^2} + \hat{D}_2\alpha_{\rm S}^2(\mu^2)\Bigg] \end{aligned}$$

Observation 1: $C_r(N, M^2, \mu_s^2) = 1$ **for** $\mu_s = \frac{M}{\bar{N}}$**.**

The two expressions coincide exactly (and thus have the same logarithmic accuracy) in N **space if** $\mu_s = \frac{M}{\bar{N}}$**.**

The SCET technique can be viewed as an alternative (more efficient?) way to derive QCD resummed results.

Observation 2: $C_{\text{QCD}}(N, M^2)$ **has no Mellin inverse, while** $C_{\text{SCET}}(N, M^2, \mu_s^2)$ **with** μ_s **fixed and** N**-independent does: the problems with the Landau pole are confined in** $C_r(N, M^2, \mu_s^2)$**. In momentum space**

$$C_{\text{QCD}}(z, M^2) = \int_z^1 \frac{dy}{y} C_r\left(\frac{z}{y}, M^2, \mu_s^2\right) C_{\text{SCET}}(y, M^2, \mu_s^2)$$

which only makes sense at any finite order in α_{S}**.**

BNX suggestion: choose μ_s in a way related to the hadronic, rather than partonic, kinematics, namely

$$\mu_s = M(1-\tau)$$

With this choice, there is no problem with the Landau pole.

However, the standard factorization property is lost:

$$\sigma_{\rm SCET}(\tau, M^2) = \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(\frac{\tau}{z}\right) C_{\rm SCET}(z, M^2, M^2(1-\tau))$$

is no longer a convolution product.

The comparison must be carried on at the level of physical cross sections (because the coefficient function depends on τ) and at fixed order in $\alpha_{\rm S}$ (because of the Landau pole).

Result (to order α_s^2):

$$\sigma_{\mathrm{QCD}}(\tau, M^2) = \sigma_{\mathrm{SCET}}(\tau, M^2) + \alpha_s^2(M^2) \sum_{n=0}^{\infty} \frac{C_n}{n!} (1-\tau)^n \sigma_{\mathrm{SCET}}^{(n)}(\tau, M^2)$$

where the constants C_n are τ-independent:

$$C_0 = -\frac{2}{3}\zeta_3 A_1 \beta_0 - \frac{\pi^2}{48} A_2$$

$$C_n = \frac{A_1 \beta_0}{n}\left(\frac{\pi^2}{6} - \frac{2}{n^2}\right) - \frac{A_2}{4n^2} + \frac{2\hat{D}_2}{n}, \qquad n > 0$$

Log counting: NNLL means that at order α_S^n, the correctly predicted terms are

$$\textbf{BNX counting}: \quad \alpha_S^n \ln^k(1-\tau); \qquad 2n-3 \le k \le 2n$$

$$\textbf{QCD counting}: \quad \alpha_S^n \ln^k(1-\tau); \qquad 2n-4 \le k \le 2n$$

A leading-log term in $\sigma_{\rm SCET}$ generates terms of order

$$\alpha_S^2 \times \alpha_S^n \ln^{2n}(1-\tau) = \alpha_S^m \ln^{2m-4}(1-\tau) \qquad m = n+2$$

in the difference QCD-SCET, which is NNLL according to the QCD counting, but NNNLL according to the SCET/BNX counting.

It can be shown that the same pattern is reproduced at all orders in α_S.

The above conclusions hold provided one does not include in the counting possible powers of $\ln(1-\tau)$ in the parton luminosity. In such case we would have

$$\alpha_{\rm S}^2 \times \alpha_{\rm S}^n \ln^{2n}(1-\tau) \times \ln^p(1-\tau) = \alpha_{\rm S}^m \ln^{2m-4+p}(1-\tau) \qquad m = n+2$$

where the factor $\ln^p(1-\tau)$ arises from the PDFs.

The discrepancy can become arbitrarily large by increasing the value of p.

Small τ

In most cases of interest at the LHC, $\tau \ll 1$. Nevertheless, often resummation provides an improvement over fixed-order calculations (see previous talk).

In these cases, $\mu_s = M(1-\tau)$ is a hard scale, and

$$
\begin{aligned}
C_r(N, M^2, \mu_s^2) &= \exp \int_{\mu_s^2}^{M^2/\bar{N}^2} \frac{d\mu^2}{\mu^2} \left[\left(A(\alpha_{\rm S}(\mu^2)) - \frac{A_1 \alpha_{\rm S}(\mu^2)}{4} \right) \ln \frac{M^2}{\mu^2 \bar{N}^2} \right. \\
&\left. + \frac{A_1}{8} \beta(\alpha_{\rm S}(\mu^2)) \ln^2 \frac{M^2}{\mu^2 \bar{N}^2} + \hat{D}_2 \alpha_{\rm S}^2(\mu^2) \right]
\end{aligned}
$$

is a NLL correction.

This class of NLL terms are resummed by the QCD result but are not resummed at all in $C_{\rm SCET}(z, M^2, M^2(1-\tau))$.

Summary

- **The way QCD resummation and SCET resummation are related depends on the choice of soft scale in the SCET expression.**
- **SCET and QCD resummation coincide in Mellin space (and share the problem of the divergence of the perturbative expansion) with the traditional choice of soft scale.**
- **SCET resummation with the soft scale chosen on the basis of hadron kinematics differs from the QCD result by a non-universal term.**

In the latter case, the SCET approach provides a prescription to tame the Landau singularity. Prices to pay:

1. **accuracy down by one power of log wrt to QCD counting**
2. **the logarithmic accuracy may be spoiled by PDF dependence**
3. **deviations from the perturbative results logarithmically suppressed (as opposed to power or even stronger suppression of other prescriptions)**

back-up slides

An explicit fixed-order calculation yields

$$C_r(N, M^2, \mu_s^2) = 1+\alpha_{\rm S}^2(M^2)\left(-\frac{A_1}{3}\beta_0 \ln^3 \frac{c}{N} + \frac{A_2}{8} \ln^2 \frac{c}{N} + 2\hat{D}_2 \ln \frac{c}{N}\right)+\mathcal{O}(\alpha_{\rm S}^3)$$

The dependence on μ_s is hidden in

$$c = \frac{Me^{-\gamma}}{\mu_s} = \frac{e^{-\gamma}}{1-\tau}.$$

No order-$\alpha_{\rm S}$ term: differences appear at least at order $\alpha_{\rm S}^3$.

Mellin inversion:

$$C_r(z, M^2, \mu_s^2) = \delta(1-z) + \alpha_{\rm S}^2(M^2)\left(-\frac{A_1}{3}\beta_0\frac{\partial^3}{\partial\xi^3} + \frac{A_2}{8}\frac{\partial^2}{\partial\xi^2} + 2\hat{D}_2\frac{\partial}{\partial\xi}\right) c^{\xi} K(z,\xi)\bigg|_{\xi=0} + \mathcal{O}(\alpha_{\rm S}^3),$$

where

$$K(z,\xi) = \frac{1}{\Gamma(\xi)} \ln^{\xi-1}\frac{1}{z}$$

The cross section is obtained by taking the convolution

$$\Sigma(\tau,\xi) = \int_{\tau}^{1} \frac{dz}{z} K(z,\xi)\sigma_{\text{SCET}}\left(\frac{\tau}{z}, M^2\right)$$
$$= (1-\tau)^{\xi}\Delta(\xi)\sum_{n=0}^{\infty}\frac{1}{n+\xi}\frac{1}{n!}(1-\tau)^{n}\sigma_{\text{SCET}}^{(n)}(\tau, M^2)$$

(up to terms suppressed by powers of $1-\tau$)

Technically, the central point is the factor $(1-\tau)^{\xi}$:

$$c^{\xi}(1-\tau)^{\xi} = e^{-\gamma}.$$

Derivatives wrt ξ do not generate extra powers of $\ln(1-\tau)$.

An important point: $\Sigma(\tau,\xi)$ depends (in a non-trivial way) on the parton luminosity.

A different possibility: choose μ_s in the SCET coefficient function to be a parton level soft scale. Typical choice:

$$\mu_s = M(1 - z)$$

and compare $C_{\text{SCET}}(z)$ to the inverse Mellin transform of the QCD result.

- **powers of $\ln(1 - z)$ correctly reproduced order by order**
- **z-space resummed expression useless, because of spurious factorial growth [Catani, Mangano, Nason, Trentadue, NPB 478 (1996) 273]**

  Very roughly speaking, $e^{\alpha_S \ln^2(1-z)}$ diverges at $z = 1$ faster than any power of $1 - z$.