# Signs of analyticity in effective theories

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# Outline

Based on AJ & O'Connell, `hep-th/0609159`

and work to appear soon, with Allan Adams (MIT) and Donal O'Connell (Caltech/IAS)

- I. The S-matrix & the optical theorem
- II. Dispersion relations in the Mandelstam representation
- III. Froissart-Martin bounds
- IV. Analiticity constraints on irrelevant scalar operators
- V. A no-tachyon hypothesis
- VI. The narrow resonance approximation (NRA)
- VII. Fermion-fermion interactions
- VIII. Nuclear bound states
- IX. Model independent constraints on physics beyond the SM

# S-matrix

Consider matrix elements of asymptotic 2-particle states

$$_{\rm out}\langle \boldsymbol{p}_3\boldsymbol{p}_4|\boldsymbol{p}_1\boldsymbol{p}_2\rangle_{\rm in} = \langle \boldsymbol{p}_3\boldsymbol{p}_4|S|\boldsymbol{p}_1\boldsymbol{p}_2\rangle$$

with $S = 1 + iT$. Unitarity: $S^\dagger S = 1 \implies -i(T - T^\dagger) = T^\dagger T$, so

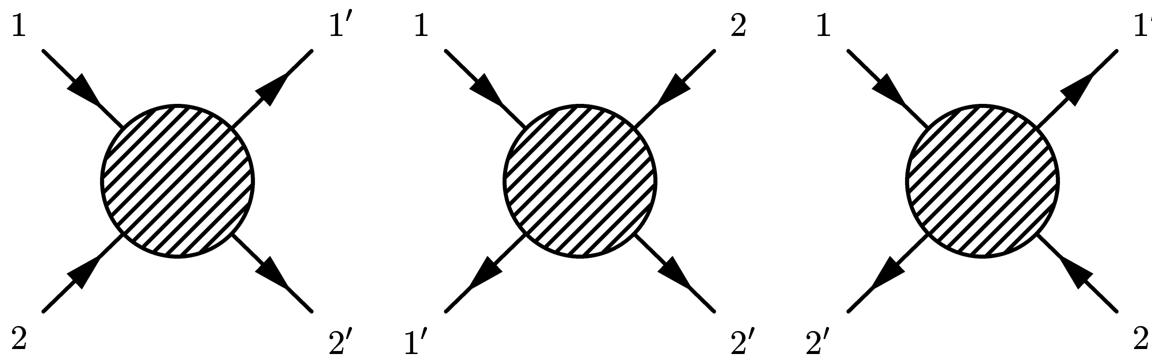
$$\langle j|T|i\rangle - \langle j|T^\dagger|i\rangle = i\sum_f \langle j|T^\dagger|f\rangle\langle f|T|i\rangle$$

$$2\,\mathrm{Im}\langle i|T|i\rangle = \sum_f |\langle f|T|i\rangle|^2 \geq 0$$

Define scattering amplitude $\mathcal{M}$ by

$$\langle \boldsymbol{p}_3\boldsymbol{p}_4\ldots|iT|\boldsymbol{p}_1\boldsymbol{p}_2\rangle = (2\pi)^4\delta^{(4)}(p_1 + p_2 - \sum p_f)\, i\mathcal{M}(p_1 p_2 \to p_f)$$

# Mandelstam representation




$s$-channel: $1\ 2 \to 1'\ 2'$ $t$-channel: $1\ \bar{1}' \to \bar{2}\ 2'$ $u$-channel: $1\ \bar{2}' \to 1'\ \bar{2}$

$$s = (p_1 + p_2)^2;\ t = (p_1 - p_1')^2;\ u = (p_1 - p_2')^2 \ \Rightarrow\ s + t + u = \sum m_i^2$$

- $t = 0$ corresponds to forward scattering in the $s$-channel
- crossing of $1'$ and $2$ exchanges $s$ and $t$
- crossing of $2'$ and $2$ exchanges $s$ and $u$
- optical theorem for $s$-channel:

$$\sigma_{tot} = \frac{1}{2|\boldsymbol{p}_1|\sqrt{s}}\,\mathrm{Im}[\mathcal{M}(s, t = 0)]$$

# Mandelstam representation, contd.

Take $\mathcal{M}(s,t,u)$ to analytic in complex plane, except for poles and cuts dictated by unitarity

Microcausality believed to be thus encoded (*à la* Kramers-Krönig relations in electrodynamics)

- There is a story about the student who wanted to know "Can one prove the Mandelstam representation from field theory?"
- He went to Weisskopf who responded "Field theory? What is field theory?"
- Then he sought out Wigner who said "Mandelstam? Who is Mandelstam?"
- Finally, our persistent student found his way to Chew, repeated the question, and heard "Proof? What is proof?"

Frautschi, '63

Intense interest in applying this to the strong interactions in the 60's

- "One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane."

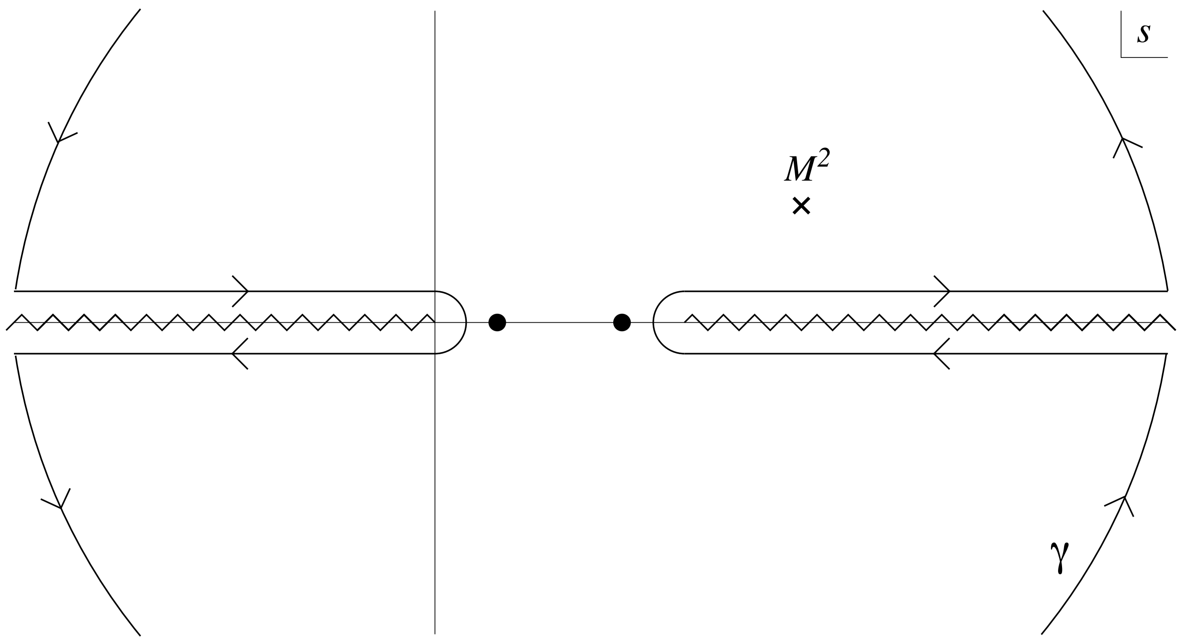
Eden, Landshoff, Olive & Polkinghorne, '66

# Dispersion relations

Fixed $t$, assuming $\lim_{s\to\infty}\mathcal{M}(s,t)=0$:

$$\mathcal{M}(s,t)=\frac{1}{2\pi i}\int_0^\infty ds'\,\frac{D^{(s)}(s',t,u')}{s'-s}+\frac{1}{2\pi i}\int_0^\infty du'\frac{D^{(u)}(s,t',u')}{u'-u}$$

with $u'=4m^2-s'-t$.



# Dispersion relations, contd.

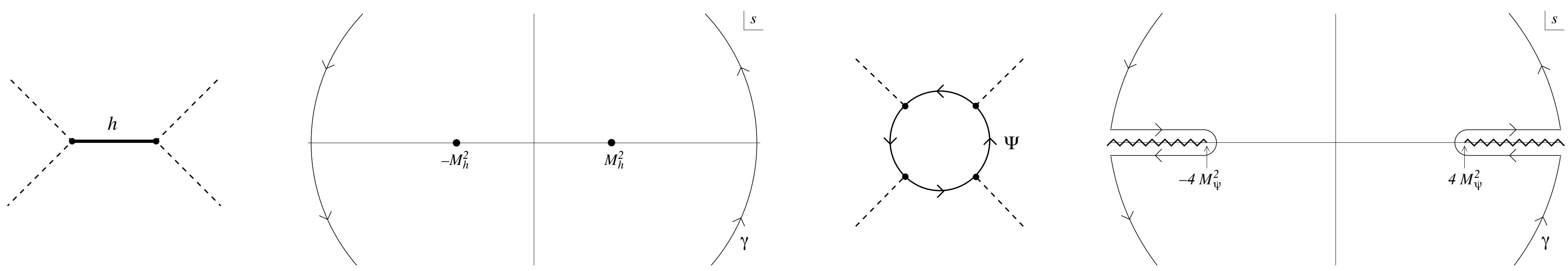
$D^{(s)}(s, t=0)$ includes:

- poles of form $2\pi i g^2 \delta(s - M^2)$ for exactly stable mediators with mass $M$ and coupling $g$
- branch cut discontinuities $2i\,\mathrm{Im}[\mathcal{M}(s, t=0)]$ for $s > 4m^2_{gap}$

By optical theorem $\mathrm{Im}[\mathcal{M}(s, t=0)] > 0$

Cut discontinuity may be approximated by one or more narrow resonances

As width $\Gamma \to 0$, resonance pole moves to physical sheet



## Subtraction

If $|\mathcal{M}(s)|$ doesn't fall off to 0 as $s \to \infty$, contribution from contour at infinity can't be neglected. Consider:

$$\begin{aligned}\mathcal{M}(s) - \mathcal{M}(s_0) &= \frac{1}{2\pi i}\int ds'\, D(s')\left(\frac{1}{s'-s} - \frac{1}{s'-s_0}\right) \\ &= \frac{1}{2\pi i}(s-s_0)\int ds'\, \frac{D(s')}{(s'-s)(s'-s_0)}\end{aligned}$$

Or twice subtracted:

$$\mathcal{M}(s) - \mathcal{M}(s_0) - (s-s_0)\mathcal{M}'(s_0) = \frac{1}{2\pi i}(s-s_0)^2\int ds'\, \frac{D(s')}{(s'-s)(s'-s_0)^2}$$

# Subtraction, contd.

and so forth...

if $|\mathcal{M}(s)| < s^n$ for large $s$, it requires at most $n$ subtractions

Each subtraction introduces an undetermined constant

Assume analyticity around $s_0$:

Each subtraction makes us insensitive to one further coefficient in Taylor series of $\mathcal{M}(s)$ around $s_0$

# Partial wave unitarity

Specialize to center-of-mass frame for particles of equal mass

$$s = 4(\boldsymbol{p}^2 + m^2); \; t = -2\boldsymbol{p}^2(1 - \cos\theta); \; u = -2\boldsymbol{p}^2(1 + \cos\theta)$$

Write

$$\mathcal{M}(s, z \equiv \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1) a_\ell(s) P_\ell(z)$$

Forward scattering: $P_\ell(z = 1) = 1$, for all $\ell$'s

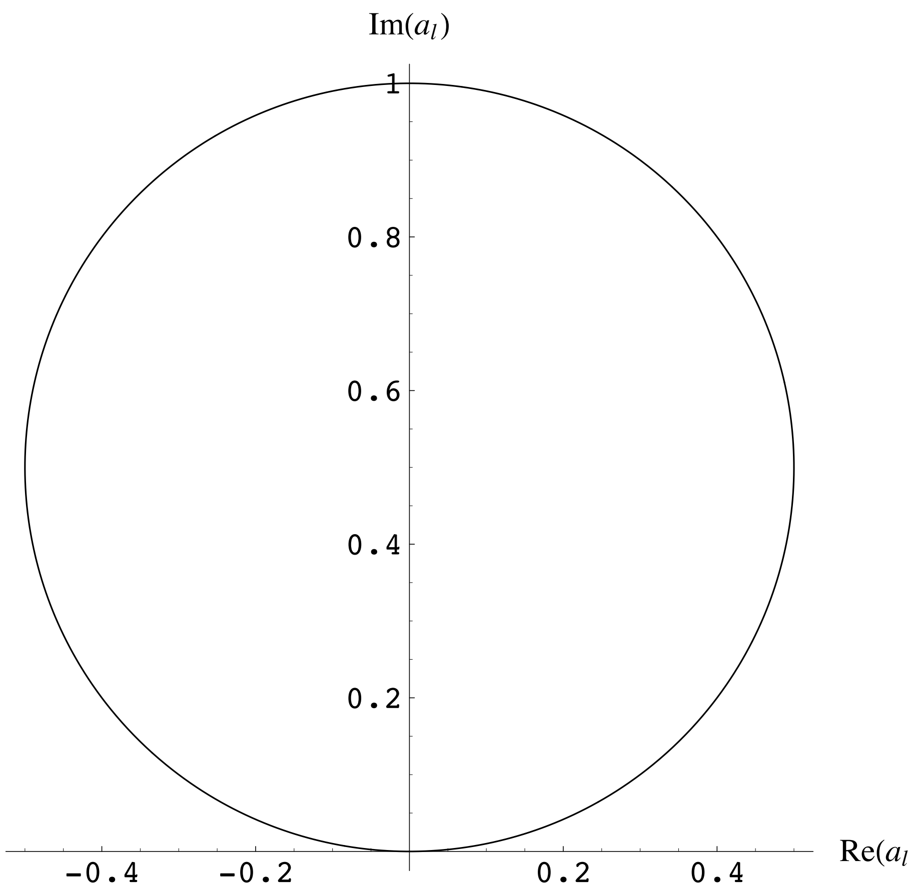
Use orthogonality of $P_\ell$'s and optical theorem to show

$$\mathrm{Im}\, a_\ell(s) \geq \sqrt{\frac{s - 4m^2}{s}} \left| a_\ell(s) \right|^2 \;,$$

since $\sigma_{total} \geq \sigma_{elastic}$

# Partial wave unitarity, contd.

For $(s - 4m^2)/s \to 1$, all $a_\ell$'s must all within the *unitarity circle* of radius $1/2$, centered at $i/2$



# Froissart-Martin bounds

This, plus analyticity in $z-$plane and *temperedness* (i.e. polynomially bounded amplitudes) constrains asymptotics

$$|\mathcal{M}(s,t)| \leq \frac{s}{M^2}\log^2(s/s_0) < s^2$$

which implies, by optical theorem

$$\sigma_{tot} \leq M^{-2}\log^2(s/s_0)$$

Froissart, '61

Dispersion relations generally require 2 subtractions, but at fixed angle

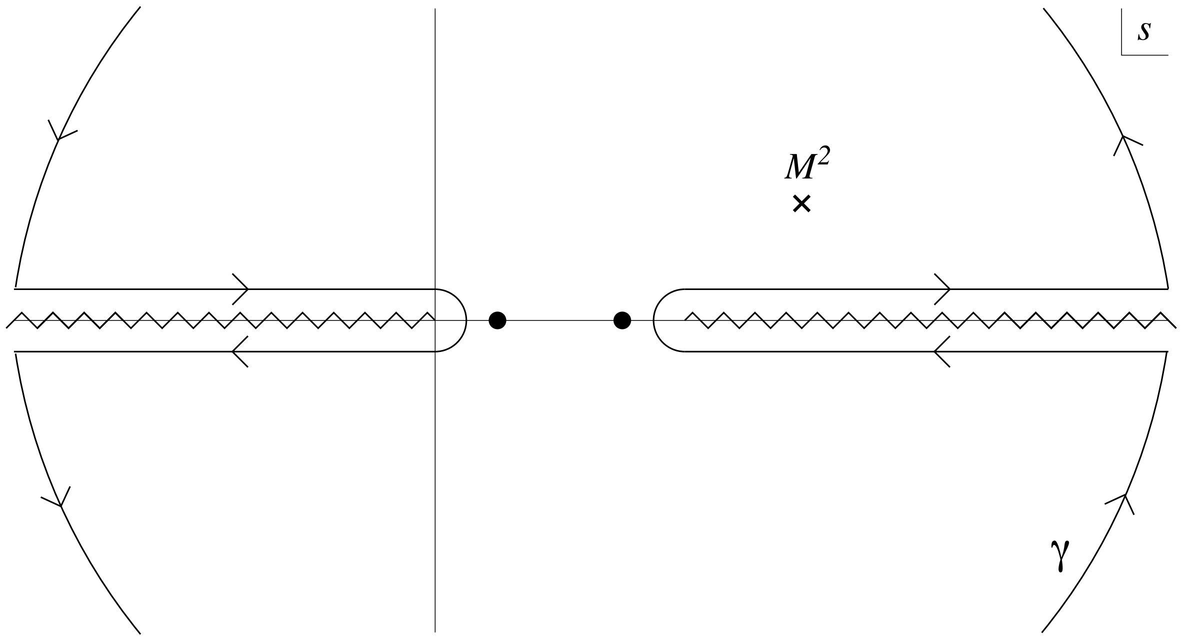
$$|\mathcal{M}(s,\theta \neq 0,\pi)| < (\mathrm{const.})\frac{\log^{3/2}(s/s_0)}{\sin^2\theta} < s$$

Kinoshita, Loeffel & Martin, '63

## Analyticity constraint for bosonic operators

Argument from Adams et al. '06

*Twice-subtracted* dispersion relation for $\mathcal{A}(s) \equiv \mathcal{M}(s, t=0)$:




$$\mathcal{A}(s) - \mathcal{A}(s_0) - (s - s_0)\mathcal{A}'(s_0) = \frac{1}{2\pi i}(s - s_0)^2 \oint_\gamma ds' \frac{\mathcal{A}(s')}{(s' - s_0)^2 (s' - s)}$$

# Analyticity constraint for bosonic operators, contd.

Take $s_0 = 0$ and $s \to 0$

$$\begin{aligned} \mathcal{A}''(0) &= \frac{1}{\pi i} \oint_\gamma ds' \, \frac{\mathcal{A}(s')}{s'^3} \\ A''(0) + \text{neg. pole res.} &= \frac{1}{\pi i} \int_{4m^2}^{\infty} ds' \, \frac{\text{Disc}\mathcal{A}(s')}{s'^3} = \frac{2}{\pi} \int_{4m^2}^{\infty} ds' \, \frac{\text{Im}\,\mathcal{A}(s')}{s'^3} \\ &= \frac{2}{\pi} \int_{4m^2}^{\infty} ds' \, \frac{\sigma_{total}(s')}{s'^2} > 0 \end{aligned}$$

So $A''(0) > 0$

# Analyticity constraint for bosonic operators, contd.

For the theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \pi\right)^2 - \frac{1}{2} m^2 \pi^2 + \frac{c_3}{2\Lambda^4}\left(\partial_\mu \pi\right)^4 + \dots \ ,$$

$\mathcal{A}''(M^2)$ is equal to $2c_3/\Lambda^4$, plus loop corrections suppressed by $M^4/\Lambda^8$

Thus analyticity requires $c_3 > 0$

For $m \to 0$ classical causality requires $c_3 \geq 0$

Linear sigma model

$$\mathcal{L} = \frac{1}{2}\left|\partial_\mu \Phi\right|^2 - \lambda\left(|\Phi|^2 - v^2\right)^2 \ ,$$

for $\Phi = (v + h)\exp(i\pi/v)$ gives $c_3 > 0$ when $h$ is integrated out

- For NLO ops. in $\chi$PT, see Ananthanarayan et al. '95; Pennington & Portoles '95; Adams et al. '06; Distler et al. '07

# No-tachyon conjecture

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \pi\right)^2 + \frac{c_3}{2\Lambda^4}\left(\partial_\mu \pi\right)^4 - \frac{1}{2}m^2\pi^2$$

is equivalent to

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \pi\right)^2 - \frac{1}{2}m^2\pi^2 - \frac{c_3}{2}\Lambda^2 F^2 \pm \frac{c_3}{\Lambda}F\left(\partial_\mu \pi\right)^2$$

so $c_3 < 0$ makes $F$ tachyonic if we add kinetic term $(\partial_\mu F)^2$ to form partial UV-completion

If, for $c_3 < 0$, we add $-\lambda F^4$ term, low energy theory is

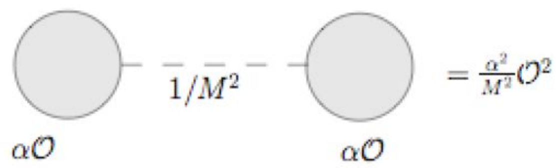
$$\mathcal{L} = X + \frac{c_3}{\Lambda^4}X^2 + \frac{c_4}{\Lambda^8}X^3 + \ldots$$

where the $c$'s are positive if $F$ sits at stable point

# No-tachyon conjecture, contd.

Partial UV completion matches linear sigma model at intermediate scales

Simple inverse OPE




works in all known cases to identify operators that must be positive by causality or stability

Includes constraints on, e.g.,

$$\mathcal{L} = \sqrt{-g}\left(R + \alpha R^2\right)$$

Barrow & Ottewill '83; Strominger '84; Magnano & Sokolowski '93

# No-tachyon conjecture, contd.

Don't expect contraints on ops. of form $\mathcal{O} \propto \mathcal{O}_1 \cdot \mathcal{O}_2$

such as $(\partial_\mu \pi_1)^2 (\partial_\nu \pi_2)^2$, which doesn't contribute to forward scattering

Conjecture: *non-positive EFT's result from integrating out tachyons*

E.g., for the bosonic partial UV completion:

$$\partial^2(\delta F) + c_3 \Lambda^2 (\delta F) = \frac{2c_3}{\Lambda} \left[ (\partial_\mu \pi_0) \partial^\mu (\delta \pi) \right]$$

Causality violation comes from superluminality after throwing away the exponentially growing modes

see Aharanov, Komar & Susskind '69

## How about fermions?

Consider effective description of single Dirac fermion,

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \frac{a}{M^2}\left(\bar{\psi}\psi\right)^2 + \ldots$$

Formally integrate in auxiliary field $F$

$$\mathcal{L}' = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + F\left(\bar{\psi}\psi\right) - \frac{M^2}{4a}F^2.$$

If $a < 0$ the UV path integral diverges, suggesting original theory requires UV and IR modification

# Invariant amplitudes

Amplitude for $\psi\psi \to \psi\psi$ may be expressed as

$$\mathcal{M}(p_1', p_2'; p_1, p_2) = \bar{u}^{s_1'}(p_1')\bar{u}^{s_2'}(p_2') \left( \sum_{i=1}^{5} c_i(s,t)\hat{C}_i \right) u^{s_1}(p_1)u^{s_2}(p_2) \ ,$$

$\hat{C}_i$'s are matrices contracting spinor indices, e.g.

$$\hat{C}_1 = 1^{(1)}1^{(2)}; \qquad \hat{C}_2 = (\gamma_5)^{(1)}(\gamma_5)^{(2)};$$

$$\hat{C}_3 = -\left(\gamma^\mu\right)^{(1)}\left(\gamma_\mu\right)^{(2)}; \quad \hat{C}_4 = -\left(\gamma_5\gamma^\mu\right)^{(1)}\left(\gamma_5\gamma_\mu\right)^{(2)};$$

$$\hat{C}_5 = \left(\sigma^{\mu\nu}\right)^{(1)}\left(\sigma_{\mu\nu}\right)^{(2)}$$

where $\sigma^{\mu\nu} = i\left[\gamma^\mu, \gamma^\nu\right]/2$

superscript indicates whether matrix acts between $\bar{u}^{s_1'}(p_1')$ and $u^{s_1}(p_1)$, or between $\bar{u}^{s_2'}(p_2')$ and $u^{s_2}(p_2)$

# Invariant amplitudes, contd.

Exchange terms from the matrix acting between $\bar{u}^{s'_2}(p'_2)$ and $u^{s_1}(p_1)$, or between $\bar{u}^{s'_1}(p'_1)$ and $u^{s_2}(p_2)$, absorbed into the $c_i$'s through Fierz transformations

Fixed-$s$ dispersion relation

$$c_i(s,t) = \frac{1}{2\pi i}\int_0^\infty dt'\,\frac{D_i^{(t)}(s,t',u')}{t'-t} + \frac{1}{2\pi i}\int_0^\infty du'\,\frac{D_i^{(u)}(s,t',u')}{u'-u}\,,$$

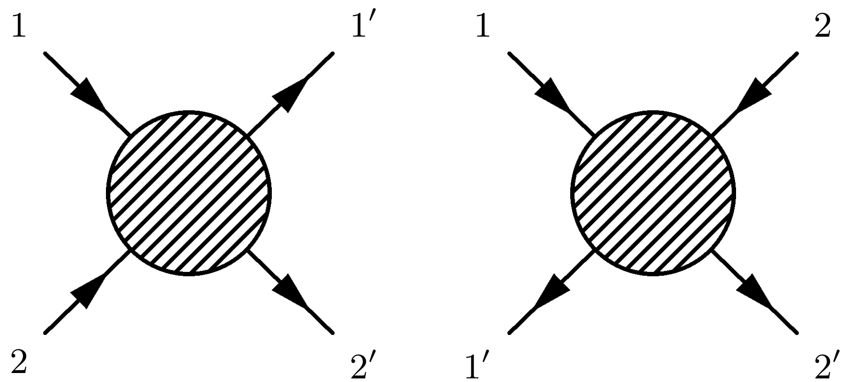
with $u' = 4m^2 - s - t'$

Define $\tilde{c}_i$'s that include only the direct terms. (Can Fierz back to $c_i$'s):

Then

$$\tilde{c}_i(s,t) = \frac{1}{\pi}\int_0^\infty dt'\,\frac{\mathrm{Im}\,\tilde{c}_i(s,t')}{t'-t}$$

## Unitarity




$s$-channel: $\psi_1\psi_2 \to \psi_1'\psi_2'$ $t$-channel: $\psi_1\bar\psi_1' \to \bar\psi_2\psi_2'$

May extract value of $\mathrm{Im}\,\tilde c_i(s,t)$ for unphysical $t > 0$ from the $t$ channel $\psi\bar\psi \to \bar\psi\psi$ process

In partial waves, by unitarity

$$\mathrm{Im}\,\tilde c_i(s,t) \geq \sum_{\ell=0}^{\infty}(2\ell+1)\mathcal{K}_\ell(t)\left|f_\ell^i(t)\right|^2 P_\ell\left(1+\frac{2s}{t-4m^2}\right)$$

Then $\tilde c_i(0,0)$'s must be positive definite if $\tilde c_i(0,t) \to 0$ as $t \to \infty$.

# Leading irrelevant operators

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + a_1\left(\bar{\psi}\psi\right)^2 + a_2\left(\bar{\psi}\gamma_5\psi\right)^2 \\ &\quad -a_3\left(\bar{\psi}\gamma^\mu\psi\right)^2 - a_4\left(\bar{\psi}\gamma_5\gamma^\mu\psi\right)^2 \\ &\quad +a_5\left(\bar{\psi}\sigma^{\mu\nu}\psi\right)^2 + \ldots\end{aligned}$$

In appropriate renormalization scheme, $a_i = \tilde{c}_i(0,0)$ at tree level

Then analyticity requires that $a_i$'s be positive

Equivalent to requiring that narrow resonances have real couplings and be non-tachyons

# Asymptotic behavior

Constraints don't apply if dispersion integrals don't converge

Would correspond to infinite set of resonances in Narrow Resonance Approx. (NRA)

Froissart-Martin allow for $\tilde{c}_i(0,t) \sim \log^2 t$, which requires one subtraction

In the Regge pole model, asymptotic behavior given by

$$\tilde{c}_i(0,t) \sim t^{\alpha(0)-1}$$

where $\alpha(s)$ is leading Regge trajectory associated with narrow fermion-fermion resonances

$\mathrm{Re}\,\alpha(s)$ is decreasing function of $s > 0$, so positivity bounds apply for

$$J = \mathrm{Re}\,\alpha(m_B^2) \leq 0 \ .$$

# Nucleon-nucleon

$$\mathcal{L} = N^\dagger \left[ i\partial_t + \boldsymbol{\nabla}^2/(2m) \right] N - \frac{1}{2} \left[ C_0^S \left( N^\dagger N \right)^2 + C_0^T \left( N^\dagger \boldsymbol{\sigma} N \right)^2 \right] + \ldots$$

for isospin doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Kaplan, Savage & Wise '96, '98

In $\overline{MS}$ scheme, $C_S = -\tilde{c}_1 + \tilde{c}_3;\ C_T = -\tilde{c}_4$

Scattering length $a_{^3S_1} = \frac{m}{4\pi}(C_S + C_T)$, negative as long as $\tilde{c}_1 + \tilde{c}_4 > \tilde{c}_3$

And $a_{^1S_0} = \frac{m}{4\pi}(C_S - 3C_T)$, negative as long as $\tilde{c}_1 > \tilde{c}_3 + 3\tilde{c}_4$

Negative $a$ for attractive potential corresponds to absence of bound states

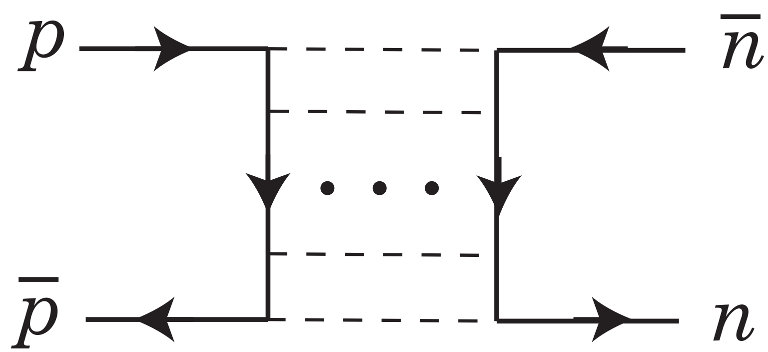
## Nucleon-nucleon, contd.

Use in literature of

$$\tilde{c}_i(s,t) = \frac{1}{\pi}\int_0^\infty dt'\, \frac{\mathrm{Im}\,\tilde{c}_i(s,t')}{t'-t}$$

in nuclear phys. seems to have missed need for subtraction

Nonperturbative when $g^2 m/\mu \sim 1$ (higher loops unsuppressed):

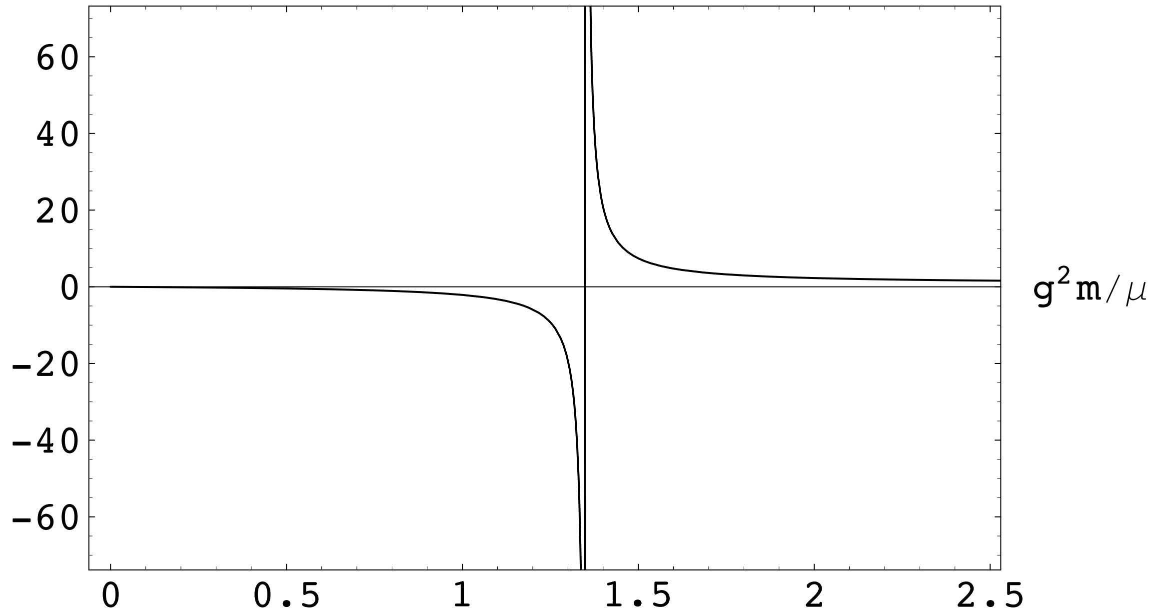



Corresponds to exchanging deuteron

# Nuclear bound states

Pole in scattering length $a$ at critical coupling where bound state forms

Schematically:




As coupling approaches critical value from below, $a \to -\infty$: Dispersion integral is diverging

# Nuclear bound states, contd.

If coupling is greater than critical value, then $a > 0$, which can be accommodated by subtraction

Our positivity bounds appear to fail in nuclear physics due to presence of deuteron bound state requiring subtraction in dispersion integrals:

$$\tilde{c}_i(0,t) \sim t^0$$

for large t

and making 4-fermion operators relevant for energies well above binding scale $B$

$$\mathcal{M} \sim \frac{g^2 m}{s - B^2}$$

(See KSW power counting)

# Relevance to LHC physics?

New physics may be parametrized by higher-dimensional operators

See Buchmüller & Wyler '85

Analyticity constrains the sign of some of those operators

Issue of possible bound state can be disregarded

Completely model independent, if analyticity of UV theory is assumed

NRA argument makes it very easy to identify operators for which such bounds are expected

Thank you.