

LHCphenonet



# Almkvist-Zeilberger-Algorithm for Feynman Integrals.

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joint work with  
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# Two Mathematica Packages

- **MultIntegrate** allows to compute multi-dimensional integrals over hyperexponential integrands in terms of (generalized) harmonic sums. This package uses variations and extensions of the multivariate Almkvist-Zeilberger algorithm.
- **HarmonicSums** allows to deal with nested sums such as harmonic sums, S-sums, cyclotomic sums and cyclotomic S-sums as well as iterated integrals such as harmonic polylogarithms, multiple polylogarithms and cyclotomic polylogarithms in an algorithmic fashion.

Note that for some of the functionalities both packages rely on the **Sigma** package by Carsten Schneider.

# MultIntegrate

We use the Almkvist Zeilberger algorithm to evaluate integrals of the form

$$\int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

where  $F(n; x_1, \dots, x_d)$  is a hyperexponential function i. e.,

$$F(n; x_1, \dots, x_d) = q(n; x_1, \dots, x_d) \cdot e^{\frac{a(x_1, \dots, x_d)}{b(x_1, \dots, x_d)}} \cdot \left( \prod_{p=1}^P S_p(x_1, \dots, x_d)^{\alpha_p} \right) \cdot \left( \frac{s(x_1, \dots, x_d)}{t(x_1, \dots, x_d)} \right)^n,$$

with

$$a(x_1, \dots, x_d), b(x_1, \dots, x_d), s(x_1, \dots, x_d), t(x_1, \dots, x_d), q(n; x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d]$$

and  $S_p(x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d]$ ,  $\alpha_p \in \mathbb{K}$  for  $1 \leq p \leq P$ . With e.g.,  $\mathbb{K} = \mathbb{Q}(\varepsilon)$ .

Our strategy to evaluate integrals of the form

$$\int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

where  $F(n; x_1, \dots, x_d)$  is a hyperexponential function is:

- compute a recurrence for the integrand  $F(n; x_1, \dots, x_d)$

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- compute a recurrence for the integrand  $F(n; x_1, \dots, x_d)$
- use the recurrence for the integrand to derive a recurrence for the integral
- solve the recurrence

# Recurrence for the Integrand

$$F(n; x_1, \dots, x_d) = q(n; x_1, \dots, x_d) \cdot e^{\frac{a(x_1, \dots, x_d)}{b(x_1, \dots, x_d)}} \cdot \left( \prod_{p=1}^P S_p(x_1, \dots, x_d)^{\alpha_p} \right) \cdot \left( \frac{s(x_1, \dots, x_d)}{t(x_1, \dots, x_d)} \right)^n,$$

where

$a(x_1, \dots, x_d), b(x_1, \dots, x_d), s(x_1, \dots, x_d), t(x_1, \dots, x_d), q(n; x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d]$   
and  $S_p(x_1, \dots, x_d) \in \mathbb{K}[x_1, \dots, x_d], \alpha_p \in \mathbb{K}$  for  $1 \leq p \leq P$ . With e.g.,  $\mathbb{K} = \mathbb{Q}(\varepsilon)$ .

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$$\begin{aligned} a(x_1, \dots, x_d), b(x_1, \dots, x_d), s(x_1, \dots, x_d), t(x_1, \dots, x_d), q(n; x_1, \dots, x_d) &\in \mathbb{K}[x_1, \dots, x_d] \\ \text{and } S_p(x_1, \dots, x_d) &\in \mathbb{K}[x_1, \dots, x_d], \alpha_p \in \mathbb{K} \text{ for } 1 \leq p \leq P. \text{ With e.g., } \mathbb{K} = \mathbb{Q}(\varepsilon). \end{aligned}$$

Then there exist

$$L \in \mathbb{N}, e_0(n), e_1(n), \dots, e_L(n) \in \mathbb{K}[n], \text{not all zero, and } R_i(n; x_1, \dots, x_d) \in \mathbb{K}(n, x_1, \dots, x_d)$$

such that

$$G_i(n; x_1, \dots, x_d) := R_i(n; x_1, \dots, x_d) F(n; x_1, \dots, x_d)$$

satisfy

$$\sum_{i=0}^L e_i(n) F(n+i; x_1, \dots, x_d) = \sum_{i=1}^d D_{x_i} G_i(n; x_1, \dots, x_d).$$

# Using the Package MultiIntegrate

In[1]:= << Sigma.m

In[2]:= << HarmonicSums.m

In[3]:= << MultiIntegrate.m

Sigma - A summation package by Carsten Schneider -RISC Linz- V 1.0 (7/7/11)

HarmonicSums by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

MultiIntegrate by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

In[4]:= mAZ[ $\frac{1}{(xy)^{n+1}(1-x-y+xy)^z}$ , n, {x, y}, f]

Out[4]=

$$\left\{ \frac{(-1+x)^{-z}(-1+y)^{-z}(xy)^{-2-n}}{xy}, \left\{ \begin{array}{l} \{nx^2y - nxy + x^2yz + x^2y - xyz - \\ xy, x\}, \{ny^2 - ny + 2y^2 - 2y, y\} \end{array} \right\}, \left( -1 - 2n - n^2 - 2z - 2nz - z^2 \right) f(1+n) + \left( 4 + 4n + n^2 \right) f(2+n) \right\}$$

# Recurrence for the Integral 1

We now consider the integral

$$a(n) := \int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

Where for  $F(n; x_1, \dots, x_d)$  we have

$$\sum_{i=0}^L e_i(n) F(n+i; x_1, \dots, x_d) = \sum_{i=1}^d D_{x_i} G_i(n; x_1, \dots, x_d).$$

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If

$$F(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) = F(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0,$$

we also have

$$G_i(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) = G_i(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0$$

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we also have

$$G_i(n; \dots, x_{i-1}, u_i, x_{i+1}, \dots) = G_i(n; \dots, x_{i-1}, o_i, x_{i+1}, \dots) = 0$$

and hence  $a(n)$  satisfies the homogenous linear recurrence

$$\sum_{i=0}^L e_i(n) a(n+i) = 0.$$

# Using the Package MultiIntegrate

```
In[5]:= mAZIntegrate[(a - x)(-1 + x)(-2 + y)(y - 1), n, {{x, a, 1}, {y, 1, 2}}]
```

```
Out[5]= 
$$\frac{-5 + 15a + 5m - 5am + n - 3an - mn + amn}{(-3 + m)(-2 + m)(-1 + m)(-3 + n)(-2 + n)(-1 + n)} +$$

$$\frac{2^{2-n}(1 - 3a - m + am + n - 3an - mn + amn)}{(-3 + m)(-2 + m)(-1 + m)(-3 + n)(-2 + n)(-1 + n)} +$$

$$a^{-m} \left( \frac{-15a^2 + 5a^3 + 5a^2m - 5a^3m + 3a^2n - a^3n - a^2mn + a^3mn}{(-3 + m)(-2 + m)(-1 + m)(-3 + n)(-2 + n)(-1 + n)} + \right.$$

$$\left. \frac{2^{2-n}(3a^2 - a^3 - a^2m + a^3m + 3a^2n - a^3n - a^2mn + a^3mn)}{(-3 + m)(-2 + m)(-1 + m)(-3 + n)(-2 + n)(-1 + n)} \right)$$

```

## Recurrence for the Integral 2

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

- suppose  $F(n; x_1, \dots, x_d)$  does not vanish at the bounds

## Recurrence for the Integral 2

We again consider the integral

$$a(n) := \int_{u_d}^{o_d} \dots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d,$$

- suppose  $F(n; x_1, \dots, x_d)$  does not vanish at the bounds
- $G_i(n; x_1, \dots, x_d)$  does not have to vanish at the bounds

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- suppose  $F(n; x_1, \dots, x_d)$  does not vanish at the bounds
- $G_i(n; x_1, \dots, x_d)$  does not have to vanish at the bounds
- then force the  $G_i$  to vanish at the integration bounds by modifying the ansatz, and look for  $G_i$  of the form

$$G_i(n; x_1, \dots, x_d) = \overline{G}_i(n; x_1, \dots, x_d)(x_i - u_i)(x_i - o_i),$$

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$$G_i(n; x_1, \dots, x_d) = \overline{G}_i(n; x_1, \dots, x_d)(x_i - u_i)(x_i - o_i),$$

- hence  $a(n)$  satisfies again a homogenous linear recurrence of the form

$$\sum_{i=0}^L e_i(n) a(n+i) = 0.$$

# Example

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n):=} dx_1 dx_2.$$

Note that the integrand does not vanish at the integration bounds.

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Note that the integrand does not vanish at the integration bounds.

Using our ansatz we find

$$\begin{aligned} & 2(n+1)(\varepsilon - n - 2)F(n) - (n+2)(5\varepsilon - 5n - 13)F(n+1) \\ & + (n+3)(4\varepsilon - 4n - 13)F(n+2) - (n+4)(\varepsilon - n - 4)F(n+3) \\ & = D_{x_1}F(n)(x_1 - 1)x_1(x_1 + 1)x_2((n+3)x_1x_2 + 2) \\ & + D_{x_2}F(n)(x_2 - 1)x_2(x_2^2x_1^3(-\varepsilon + n + 4) - (\varepsilon - 3)x_2x_1^2 + (n+2)x_2x_1 + 1). \end{aligned}$$

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Integration of this recurrence yields

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Solving the recurrence leads to

$$I(\varepsilon, n) = \frac{1}{n+1} \left( \sum_{i=1}^n \frac{1}{-i + \varepsilon - 1} - 2^{1-\varepsilon} \sum_{i=1}^n \frac{2^i}{-i + \varepsilon - 1} + \frac{2^{-\varepsilon}(2^\varepsilon - 2)}{\varepsilon - 1} \right)$$

# Using the Package MultiIntegrate

In[6]:= **mAZDirectIntegrate**[ $\frac{(1 + x1 * x2)^n}{(1 + x1)^\varepsilon}$ , n, {{x1, 0, 1}, {x2, 0, 1}}]

$$\text{Out}[6] = \frac{\sum_{\iota_1=1}^n \frac{1}{\iota_1 - \varepsilon + 1}}{-n - 1} - \frac{2 \sum_{\iota_1=1}^n \frac{2^{\iota_1}}{-\iota_1 + \varepsilon - 1}}{(n + 1)2^\varepsilon} + \frac{2^\varepsilon - 2}{(n + 1)(\varepsilon - 1)2^\varepsilon}$$

In[7]:= **mAZDirectIntegrate**[ $\frac{(1 + x1 + x2 + x1 * x2)^n}{(1 + x1)^\varepsilon}$ , n, {{x1, 0, 1}, {x2, 0, 1}}]

$$\text{Out}[7] = \frac{2^{-2n-\varepsilon} (208 - 83 2^{2+n} + 63 2^{1+2n} - 2^{4+\varepsilon} + 7 2^{2+n+\varepsilon} - 13 2^{2n+\varepsilon})}{(1 + n)(1 + n - \varepsilon)}$$

In[8]:= **mAZDirectIntegrate**[ $\frac{(1 + x1 + x2 + x1 * x2)^n}{(1 + x1)^\varepsilon}$ , n, {{x1, 0,  $\frac{1}{2}$ }, {x2, 0, 2}}]

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# Recurrence for the Integral 3

We look again at the integral

$$a(n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d.$$

Suppose that we found

$$\sum_{i=0}^L e_i(n) F(n+i; x_1, \dots, x_d) = \sum_{i=1}^d D_{x_i} G_i(n; x_1, \dots, x_d)$$

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By integration with respect to  $x_1, \dots, x_d$  we get

$$\begin{aligned} \sum_{i=0}^L e_i(n) a(n+i) &= \sum_{i=1}^d \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} O_i(n) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_d \\ &\quad - \sum_{i=1}^d \int_{u_d}^{o_d} \cdots \int_{u_{i-1}}^{o_{i-1}} \int_{u_{i+1}}^{o_{i+1}} \cdots \int_{u_1}^{o_1} U_i(n) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_d \end{aligned}$$

with

$$\begin{aligned} O_i(n) &:= G_i(n; x_1, \dots, x_{i-1}, o_i, x_{i+1}, \dots, x_d) \\ U_i(n) &:= G_i(n; x_1, \dots, x_{i-1}, u_i, x_{i+1}, \dots, x_d). \end{aligned}$$

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Note that there are  $2 \cdot d$  integrals of dimension  $d-1$ , to compute.

# Example

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n; x_1, x_2) :=} dx_1 dx_2,$$

Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

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and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

In the next step apply the algorithm to  $I_1(n)$ ; we find

$$I_1(\varepsilon, n) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon(n+2-\varepsilon)}.$$

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Plugging in yields

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Plugging in yields

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \frac{4 \cdot 2^n - 2^\varepsilon}{2^\varepsilon(n+2-\varepsilon)}.$$

Solving this recurrence yields

$$I(\varepsilon, n) = \frac{1}{n+1} \left( \sum_{i=1}^n \frac{1}{-i + \varepsilon - 1} - 2^{1-\varepsilon} \sum_{i=1}^n \frac{2^i}{-i + \varepsilon - 1} + \frac{2^{-\varepsilon}(2^\varepsilon - 2)}{\varepsilon - 1} \right).$$

# Using the Package MultiIntegrate

In[9]:= **mAZIntegrate**[ $\frac{(1 + x1 * x2)^n}{(1 + x1)^\varepsilon}$ , n, {{x1, 0, 1}, {x2, 0, 1}}]

$$\text{Out}[9] = \frac{\sum_{\iota_1=1}^n \frac{1}{\iota_1 - \varepsilon + 1}}{-n - 1} - \frac{2 \sum_{\iota_1=1}^n \frac{2^{\iota_1}}{-\iota_1 + \varepsilon - 1}}{(n+1)2^\varepsilon} + \frac{2^\varepsilon - 2}{(n+1)(\varepsilon - 1)2^\varepsilon}$$

In[10]:= **mAZIntegrate**[ $\frac{(1 + x1 + x2 + x1 * x2)^n}{(1 + x1)^\varepsilon}$ , n, {{x1, 0, 1}, {x2, 0, 1}}]

$$\text{Out}[10] = \frac{2^{-2n-\varepsilon} (208 - 83 2^{2+n} + 63 2^{1+2n} - 2^{4+\varepsilon} + 7 2^{2+n+\varepsilon} - 13 2^{2n+\varepsilon})}{(1+n)(1+n-\varepsilon)}$$

In[11]:= **mAZIntegrate**[ $\frac{(1 + x1 + x2 + x1 * x2)^n}{(1 + x1)^\varepsilon}$ , n, {{x1, 0, 1/2}, {x2, 0, 2}}]

$$\text{Out}[11] = \frac{2^{-2n-\varepsilon} (208 - 83 2^{2+n} + 63 2^{1+2n} - 2^{4+\varepsilon} + 7 2^{2+n+\varepsilon} - 13 2^{2n+\varepsilon})}{(1+n)(1+n-\varepsilon)}$$

# Laurent Series Expansion of the Integral

- we look again at the integral

$$\mathcal{I}(\varepsilon, n) := \int_{u_d}^{o_d} \cdots \int_{u_1}^{o_1} F(n; x_1, \dots, x_d) dx_1 \dots dx_d.$$

- assume that we can write it in the form

$$\mathcal{I}(\varepsilon, n) = \sum_{l=-L}^{\infty} \varepsilon^l I_l(n).$$

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- compute a recurrence for  $\mathcal{I}(\varepsilon, n)$  in the form

$$a_0(\varepsilon, n) T(\varepsilon, n) + \cdots + a_d(\varepsilon, n) T(\varepsilon, n+d) = h_0(n) + \cdots + h_u(n) \varepsilon^u + O(\varepsilon^{u+1});$$

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- compute a recurrence for  $\mathcal{I}(\varepsilon, n)$  in the form
$$a_0(\varepsilon, n) T(\varepsilon, n) + \cdots + a_d(\varepsilon, n) T(\varepsilon, n+d) = h_0(n) + \cdots + h_u(n) \varepsilon^u + O(\varepsilon^{u+1});$$
use one of the methods presented above
- use algorithm FLSR implemented in Sigma.

# Example

$$I(\varepsilon, n) = \int_0^1 \int_0^1 \underbrace{\frac{(1 + x_1 \cdot x_2)^n}{(1 + x_1)^\varepsilon}}_{F(n; x_1, x_2) :=} dx_1 dx_2,$$

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Applying the algorithm leads to

$$-(n+1)F(n; x_1, x_2) + (n+2)F(n+1; x_1, x_2) = D_{x_1} 0 + D_{x_2} \frac{x_2(x_1 \cdot x_2 + 1)^{n+1}}{(1 + x_1)^\varepsilon}$$

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and hence it follows by integration

$$-(n+1)I(\varepsilon, n) + (n+2)I(\varepsilon, n+1) = \underbrace{\int_0^1 (x_1 + 1)^{n+1-\varepsilon} dx_1}_{I_1(n)} - \int_0^1 0 dx_1.$$

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In the next step apply the method to  $I_1(n)$ ; we find

$$\begin{aligned} I_1(\varepsilon, n) &= \frac{2^{n+2} - 1}{n+2} + \frac{\varepsilon(-2^{n+2}(\mathcal{H}_{-1}(1)(n+2) - 1) - 1)}{(n+2)^2} \\ &\quad + \frac{\varepsilon^2(2^{n+1}(\mathcal{H}_{-1}(1)^2(n+2)^2 - 2\mathcal{H}_{-1}(1)(n+2) + 2) - 1)}{(n+2)^3} + O(\varepsilon^3). \end{aligned}$$

Plugging in and solving the resulting recurrence by means of FLSR and combining the solution with the initial values of the integral yields the result.

$$\begin{aligned}
 I(\varepsilon, n) = & \frac{S_1(2; n)}{n+1} - \frac{S_1(n)}{n+1} + \frac{2(n+1)^3 + 4(-n + 2^n - 1)(n+1)^2 + 2n(n+1)^2}{2(n+1)^4} \\
 & + \varepsilon \left( H_{-1}(1) \left( \frac{-2^{n+2}(n+1) - 2^{n+2}n(n+1)}{2(n+1)^4} - \frac{S_1(2; n)}{n+1} \right) \right. \\
 & + \frac{(2(n+1)n^2 + 4(n+1)n + 2(n+1)) S_2(2; n)}{2(n+1)^4} - \frac{S_2(n)}{n+1} \\
 & \left. + \frac{2^{n+2}(n+1) - 2(n+1)}{2(n+1)^4} \right) + \varepsilon^2 \left( H_{-1}(1)^2 \left( \frac{S_1(2; n)}{2(n+1)} + \frac{2^{n+1}(n^2 + 2n + 1)}{2(n+1)^4} \right) \right. \\
 & + H_{-1}(1) \left( \frac{-2^{n+2}n - 2^{n+2}}{2(n+1)^4} - \frac{S_2(2; n)}{n+1} \right) + \frac{S_3(2; n)}{n+1} - \frac{S_3(n)}{n+1} + \frac{2^{n+2} - 2}{2(n+1)^4} \\
 & \left. + O(\varepsilon^3) \right).
 \end{aligned}$$

# Using the Package MultiIntegrate

In[12]:= **mAZExpandedIntegrate**[ $\frac{(1+x1*x2)^n}{(1+x1)^\varepsilon}$ , n, { $\varepsilon, 0, 2$ }, {{x1, 0, 1}, {x2, 0, 1}}]

Out[12]=

$$\left\{ \left\{ \frac{S[1, \{2\}, n]}{n+1} - \frac{S[1, n]}{n+1} + \frac{2^{n+1}-1}{(n+1)^2}, -\frac{\log(2)S[1, \{2\}, n]}{n+1} + \frac{S[2, \{2\}, n]}{n+1} - \right. \right.$$
$$\frac{S[2, n]}{n+1} + \frac{\log(2)(-2^{n+1})(n+1)+2^{n+1}-1}{(n+1)^3}, \frac{\log(2)^2 S[1, \{2\}, n]}{2n+2} -$$
$$\frac{\log(2)S[2, \{2\}, n]}{n+1} + \frac{S[3, \{2\}, n]}{n+1} - \frac{S[3, n]}{n+1} +$$
$$\left. \left. \frac{\log(2)^2 2^n(n+1)^2 - \log(2)2^{n+1}(n+1)+2^{n+1}-1}{(n+1)^4} \right\}, 0, 2 \right\}$$

In[13]:= **mAZExpandedIntegrate**[ $\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$ , n, { $\varepsilon, 0, 1$ }, {{x1, 0, 1}, {x2, 0, 1}}]

Out[13]=

$$\left\{ \left\{ \frac{(2^{n+1}-1)^2}{(n+1)^2}, -\frac{(2^{n+1}-1)(-2^{n+1}+2^{n+1}n\log(2)+2^{n+1}\log(2)+1)}{(n+1)^3} \right\}, 0, 1 \right\}$$

In[14]:= **mAZExpandedIntegrate**[ $\frac{(1+x1+x2+x1*x2)^n}{(1+x1)^\varepsilon}$ , n, { $\varepsilon, 0, 1$ }, {{x1, 0, 1}, {x2, 0, 2}}]

Out[14]=

$$\left\{ \left\{ -\frac{2^{-n-1}(2^{n+1}-3^{n+1})(3^{n+1}-1)}{(n+1)^2}, -\frac{2^{-n-1}(3^{n+1}-1)(2^{n+1}-3^{n+1}+3^{n+1}n\log(\frac{3}{2})+3^{n+1}\log(\frac{3}{2}))}{(n+1)^3} \right\}, 0, 1 \right\}$$

# Using the Package MultiIntegrate

The following integral occurs in the direct computation of a 3-loop diagram of the ladder-type:

$$\int_0^1 \int_0^1 \left( \frac{(s(x-1) + t(u-1) + 1)^n}{(w-1)(z-1)(sx-s+tu-t-u+1)(sx-s+tu-t-x+1)} \right.$$
$$+ \frac{1}{(z-1)(sx-s+tu-t-x+1)} \frac{(z(-s+tu-t+1) + x((s-1)z+1))^n}{-swx + sw + sxz - sz - tuw + tuz + tw - tz + uw - u - w - xz + x + z}$$
$$+ \left. \frac{1}{(w-1)(sx-s+tu-t-u+1)} \frac{(u((t-1)w+1) - w(s(-x)+s+t-1))^n}{swx - sw - sxz + sz + tuw - tuz - tw + tz - uw + u + w + xz - x - z} \right) du dx$$

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$$\begin{aligned}
& \int_0^1 \int_0^1 \left( \frac{(s(x-1) + t(u-1) + 1)^n}{(w-1)(z-1)(sx-s+tu-t-u+1)(sx-s+tu-t-x+1)} \right. \\
& \quad + \frac{1}{(z-1)(sx-s+tu-t-x+1)} \frac{(z(-s+tu-t+1) + x((s-1)z+1))^n}{-swx + sw + sxz - sz - tuw + tuz + tw - tz + uw - u - w - xz + x + z} \\
& \quad + \left. \frac{1}{(w-1)(sx-s+tu-t-u+1)} \frac{(u((t-1)w+1) - w(s(-x)+s+t-1))^n}{swx - sw - sxz + sz + tuw - tuz - tw + tz - uw + u + w + xz - x - z} \right) du dx \\
& = \frac{1}{(w-1)(z-1)(s+t-1)} \left( S_{1,1} \left( -\frac{(s+t-1)w(z-1)}{sw-sz+z-1}, \frac{sw-sz+z-1}{z-1}; n \right) - S_{1,1} \left( -\frac{(s+t-1)w(z-1)}{sw-sz+z-1}, \frac{z(sw-sz+z-1)}{w(z-1)}; n \right) \right. \\
& \quad - S_{1,1} \left( -\frac{(s+t-1)w(z-1)}{sw-sz+z-1}, \frac{(t-1)(sw-sz+z-1)}{(s+t-1)(z-1)}; n \right) + S_{1,1} \left( \frac{s+t-1}{t-1}, (1-t)w; n \right) + S_{1,1} \left( \frac{s+t-1}{t-1}, -\frac{(s-1)(t-1)}{s+t-1}; n \right) \\
& \quad + S_{1,1} \left( -\frac{(s+t-1)w(z-1)}{sw-sz+z-1}, \frac{(sw-sz+z-1)(tz-1)}{(s+t-1)w(z-1)}; n \right) - S_{1,1} \left( \frac{s+t-1}{t-1}, 1-t; n \right) + S_{1,1} \left( \frac{s+t-1}{s-1}, -\frac{(s-1)(t-1)}{s+t-1}; n \right) \\
& \quad + S_{1,1} \left( \frac{(s+t-1)(w-1)z}{(t-1)w-tz+1}, \frac{-tw+w+tz-1}{w-1}; n \right) - S_{1,1} \left( \frac{s+t-1}{t-1}, -\frac{(t-1)(sw-1)}{s+t-1}; n \right) + S_{1,1} \left( \frac{s+t-1}{s-1}, (1-s)z; n \right) \\
& \quad - S_{1,1} \left( \frac{(s+t-1)(w-1)z}{(t-1)w-tz+1}, \frac{(s-1)(-tw+w+tz-1)}{(s+t-1)(w-1)}; n \right) - S_{1,1} \left( \frac{s+t-1}{s-1}, 1-s; n \right) - S_{1,1} \left( \frac{s+t-1}{s-1}, -\frac{(s-1)(tz-1)}{s+t-1}; n \right) \\
& \quad - S_{1,1} \left( \frac{(s+t-1)(w-1)z}{(t-1)w-tz+1}, \frac{w((t-1)w-tz+1)}{(w-1)z}; n \right) + S_{1,1} \left( \frac{(s+t-1)(w-1)z}{(t-1)w-tz+1}, -\frac{(sw-1)((t-1)w-tz+1)}{(s+t-1)(w-1)z}; n \right) \\
& \quad - S_{1,1}(1, (1-s)z; n) + S_{1,1}(1, z - sz; n) + S_2((1-s)z; n) - S_{1,1}(1, (1-t)w; n) + S_{1,1}(1, w - tw; n) - S_2((-s - t + 1)w; n) \\
& \quad - S_2((-s - t + 1)z; n) + 2S_2(-s - t + 1; n) - S_2(1 - s; n) + S_2((1 - t)w; n) - S_2(1 - t; n) \Big)
\end{aligned}$$

# (Generalized) Harmonic Sums

## Definition (Harmonic Sums (H-Sums))

For  $c_i \in \mathbb{Z}^*$  and  $n \in \mathbb{N}$  we define

$$S_{c_1, \dots, c_k}(n) = \sum_{n \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} \frac{\text{sign}(c_1)^{i_1}}{i_1^{|c_1|}} \dots \frac{\text{sign}(c_k)^{i_k}}{i_k^{|c_k|}}$$

$k$  is called the depth and  $w = \sum_{i=1}^k |c_i|$  is called the weight of the harmonic sum  $S_{c_1, \dots, c_k}(n)$ .

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$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

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$k$  is called the depth and  $w = \sum_{i=1}^k |c_i|$  is called the weight of the harmonic sum  $S_{c_1, \dots, c_k}(n)$ .

$$S_1(n) = \sum_{i=1}^n \frac{1}{i}$$

$$S_{2,-3}(n) = \sum_{i=1}^n \frac{\sum_{j=1}^i \frac{(-1)^j}{j^3}}{i^2}$$

# Harmonic Polylogarithms (H-Logs)

We define  $f : \{0, 1, -1\} \times (0, 1) \mapsto \mathbb{R}$  by

$$\begin{aligned}f(0, x) &= \frac{1}{x}, \\f(1, x) &= \frac{1}{1-x}, \\f(-1, x) &= \frac{1}{1+x}.\end{aligned}$$

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## Definition (Harmonic Polylogarithms (H-Logs))

Let  $m_i \in \{-1, 0, 1\}$  we define for  $x \in (0, 1)$  :

$$\begin{aligned}\mathsf{H}(x) &= 1, \\\mathsf{H}_{m_1, m_2, \dots, m_k}(x) &= \begin{cases} \frac{1}{k!} (\log x)^k, & \text{if } (m_1, \dots, m_k) = \mathbf{0} \\ \int_0^x f_{m_1}(y) \mathsf{H}_{m_2, \dots, m_k}(y) dy, & \text{otherwise.} \end{cases}\end{aligned}$$

## Example

$$H_1(x) = \int_0^x \frac{1}{1-x_1} dx_1 = -\log(1-x)$$

$$H_{-1}(x) = \int_0^x \frac{1}{1+x_1} dx_1 = \log(1+x)$$

$$H_{1,0,-1}(x) = \int_0^x \frac{1}{1-x_1} \int_0^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{1+x_3} dx_3 dx_2 dx_1$$

# Mellin Transform

## Definition (Mellin Transform of Harmonic Polylogarithms)

Let  $f(x)$  be a harmonic polylogarithm and let  $n \in \mathbb{N}$ , the Mellin transform is defined by:

$$M(f(x), n) = \int_0^1 x^n f(x) dx$$

Since harmonic polylogarithms are locally integrable functions on  $(0, 1)$  their Mellin transform is finite.

# Mellin Transform of Harmonic Polylogarithms

## Depth 1 cases

$$M(H_0(x), n) = -\frac{1}{(n+1)^2}$$

$$M(H_1(x), n) = \frac{1}{(n+1)^2} + \frac{S_1(n)}{n+1}$$

$$\begin{aligned} M(H_{-1}(x), n) &= \frac{(-1)^n}{n+1} (S_{-1}(n) + \log(2)) \\ &\quad + \frac{\log(2)}{n+1} - \frac{1}{(n+1)^2} \end{aligned}$$

# Using the Package HarmonicSums

In[1]:= << **HarmonicSums.m**

HarmonicSums by Jakob Ablinger -RISC Linz- Version 1.0 (1/3/12)

In[2]:= **Mellin[H[-1,x],x,n]**

$$\text{Out}[2]= \frac{-((-1)^n + 1)(n+1)S[-1, \infty] + (-1)^n(n+1)S[-1, n] - 1}{(n+1)^2}$$

In[3]:= **Mellin[H[1,0,1,x],x,n]**

$$\text{Out}[3]= \frac{(n+1)^3S[1, n]S[2, \infty] + (n+1)^2S[2, \infty] - (n+1)^3S[2, 1, n] - (n+1)S[1, n] - 1}{(n+1)^4}$$

In[4]:= **Mellin[H[-1,1,0,x],x,n]**

$$\text{Out}[4]= \frac{1}{(n+1)^4} \left( -2(-1)^n(n+1)^3S[3, \infty] - 2(n+1)^3S[3, \infty] + (-1)^n(n+1)^3S[-2, -1, \infty] + (n+1)^3S[-2, -1, \infty] + (-1)^n(n+1)^3S[-1, -2, \infty] + (n+1)^3S[-1, -2, \infty] - (-1)^n(n+1)^3S[-1, 2, n] + (n+1)^2S[2, n] + 1 \right)$$

# Using the Package HarmonicSums

In[5]:= **InvMellin[S[-1, n], n, x]**

$$\text{Out}[5] = \delta_{1-x} S[-1, \infty] + \frac{(-1)^n}{x+1}$$

In[6]:= **InvMellin[S[-1, 2, n], n, x]**

$$\text{Out}[6] = \delta_{1-x} (-2S[3, \infty] + S[-2, -1, \infty] + S[-1, -2, \infty]) - \frac{(-1)^n H[1, 0, x]}{x+1}$$

In[7]:= **InvMellin[S[-1, 1, 3, n], n, x]**

$$\begin{aligned} \text{Out}[7] = & \delta_{1-x} (6S[5, \infty] - S[-3, -2, \infty] - 2S[-2, -3, \infty] - 3S[-1, -4, \infty] - \\ & S[2, 3, \infty] - 2S[3, 2, \infty] - 3S[4, 1, \infty] + S[-3, -1, 1, \infty] + \\ & S[-2, -2, 1, \infty] + S[-2, -1, 2, \infty] + S[-1, -3, 1, \infty] + \\ & S[-1, -2, 2, \infty] + S[-1, -1, 3, \infty]) + \frac{(-1)^n H[1, 1, 0, 0, x]}{x+1} \end{aligned}$$

## Definition (S-Sums)

For  $c_i \in \mathbb{N}$  and  $x_i \in \mathbb{R}^*$  we define

$$S_{c_1, \dots, c_k}(x_1, \dots, x_k; n) = \sum_{n \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{c_1}} \cdots \frac{x_k^{i_k}}{i_k^{c_k}}.$$

$k$  is called the depth and  $w = \sum_{i=1}^k c_i$  is called the weight of the S-sum  $S_{c_1, \dots, c_k}(x_1, \dots, x_k; n)$ .

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$$S_{2,3}(4, 3; n) = \sum_{i=1}^n \frac{4^i \sum_{j=1}^i \frac{3^j}{j^3}}{i^2}$$

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For  $c_i \in \mathbb{N}$  and  $x_i \in \mathbb{R}^*$  we define

$$S_{c_1, \dots, c_k}(x_1, \dots, x_k; n) = \sum_{n \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} \frac{x_1^{i_1}}{i_1^{c_1}} \cdots \frac{x_k^{i_k}}{i_k^{c_k}}.$$

$k$  is called the depth and  $w = \sum_{i=1}^k c_i$  is called the weight of the S-sum  $S_{c_1, \dots, c_k}(x_1, \dots, x_k; n)$ .

$$S_{2,3}(4, 3; n) = \sum_{i=1}^n \frac{4^i \sum_{j=1}^i \frac{3^j}{j^3}}{i^2}$$

$$S_{2,3}(1, -1; n) = \sum_{i=1}^n \frac{1^i \sum_{j=1}^i \frac{(-1)^j}{j^3}}{i^2} = S_{2,-3}(n)$$

## Definition (Cyclotomic Harmonic Sums (C-Sums))

Let  $a_i, k \in \mathbb{N}^*$ ,  $b_i, n \in \mathbb{N}$  and  $c_i \in \mathbb{Z}^*$  we define

$$S_{(a_1, b_1, c_1), (a_2, b_2, c_2), \dots, (a_k, b_k, c_k)}(n) = \\ = \sum_{n \geq i_1 \geq \dots \geq i_k \geq 1} \frac{\text{sign}(c_1)^{i_1}}{(a_1 i_1 + b_1)^{|c_1|}} \frac{\text{sign}(c_2)^{i_2}}{(a_2 i_2 + b_2)^{|c_2|}} \dots \frac{\text{sign}(c_k)^{i_k}}{(a_k i_k + b_k)^{|c_k|}}$$

$k$  is called the depth and  $w = \sum_{i=1}^k |c_i|$  is called the weight of the cyclotomic harmonic sum  $S_{(a_1, b_1, c_1), \dots, (a_k, b_k, c_k)}(n)$ .

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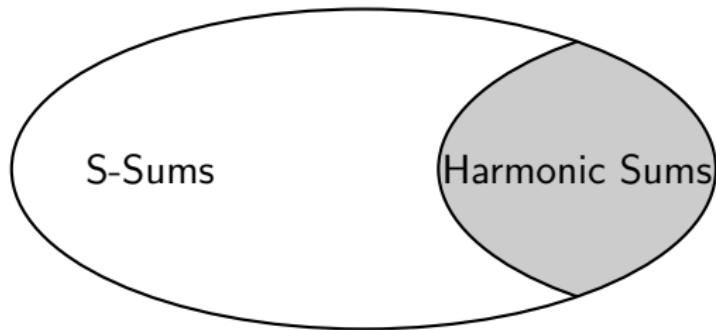
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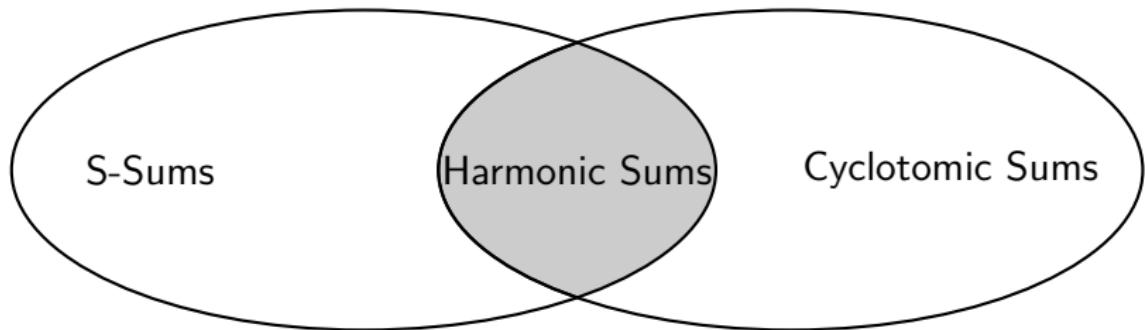
## Connection between these nested sums

Harmonic Sums

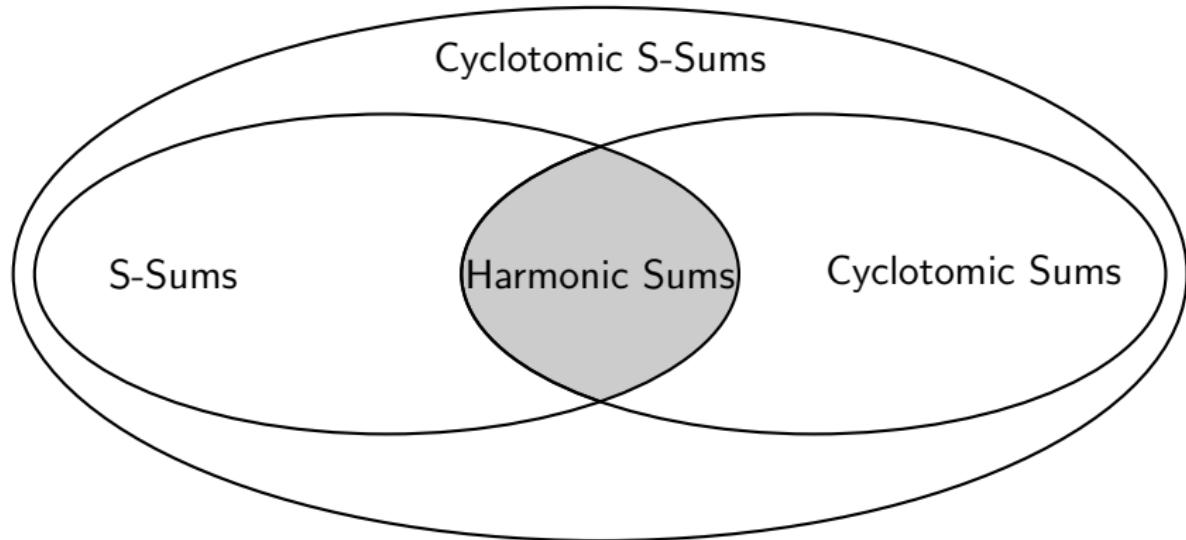
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# Multiple Polylogarithms (M-Logs)

Let  $a \in \mathbb{R}$  and

$$q = \begin{cases} a, & \text{if } a > 0 \\ \infty, & \text{otherwise} \end{cases}$$

We define  $f$  as follows:

$$f_a : (0, q) \mapsto \mathbb{R}$$
$$f_a(x) = \begin{cases} \frac{1}{x}, & \text{if } a = 0 \\ \frac{1}{|a| - \text{sign}(a)x}, & \text{otherwise.} \end{cases}$$

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## Definition (Multiple Polylogarithms (M-Logs))

Let  $m_i \in \mathbb{R}$  and let  $q = \min_{m_i > 0} m_i$  we define for  $x \in (0, q)$  :

$$\mathsf{H}(x) = 1,$$
$$\mathsf{H}_{m_1, m_2, \dots, m_k}(x) = \begin{cases} \frac{1}{k!} (\log x)^k, & \text{if } (m_1, \dots, m_k) = \mathbf{0} \\ \int_0^x f_{m_1}(y) \mathsf{H}_{m_2, \dots, m_k}(y) dy, & \text{otherwise.} \end{cases}$$

## Example

$$H_1(x) = \int_0^x \frac{1}{1-x_1} dx_1 = -\log(1-x)$$

$$H_{-2}(x) = \int_0^x \frac{1}{2+x_1} dx_1 = \log(2+x) - \log(2)$$

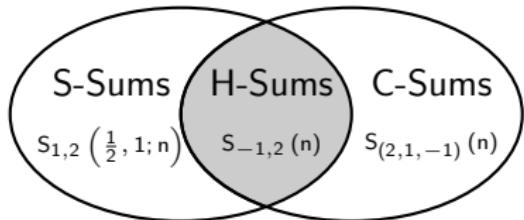
$$H_{2,0,-\frac{1}{2}}(x) = \int_0^x \frac{1}{2-x_1} \int_0^{x_1} \frac{1}{x_2} \int_0^{x_2} \frac{1}{\frac{1}{2}+x_3} dx_3 dx_2 dx_1$$

# Using the Package HarmonicSums

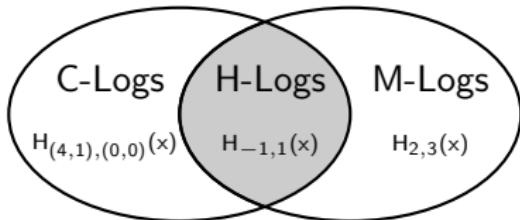
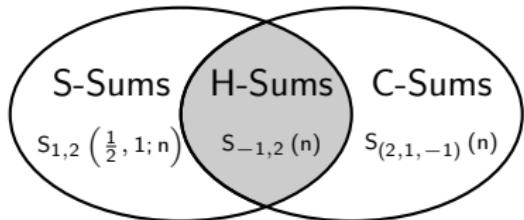
```
In[8]:= InvMellin[S[1,2,1,{1/3,1/2,1}], n, x]
```

$$\begin{aligned} \text{Out}[8] = & -\frac{6^{-n} H[2, 0, 1, x]}{-6 + x} + \frac{6^{-n} H[2, x] S[2, \infty]}{-6 + x} - \frac{6^{-n} S[3, \infty]}{-6 + x} + \frac{3^{-n} S[3, \infty]}{-3 + x} - \\ & \frac{3^{-n} S[2, \infty] S[1, \left\{\frac{1}{2}\right\}, \infty]}{3 - x} + \frac{6^{-n} S[2, \infty] S[1, \left\{\frac{1}{2}\right\}, \infty]}{6 - x} + \\ & \frac{6^{-n} S[1, 2, \left\{\frac{1}{2}, 2\right\}, \infty]}{-6 + x} - \frac{3^{-n} S[1, 2, \left\{\frac{1}{2}, 2\right\}, \infty]}{-3 + x} + \delta_{1-x} \left( -S[4, \infty] - \right. \\ & S[3, \infty] S[1, \left\{\frac{1}{6}\right\}, \infty] + S[3, \infty] S[1, \left\{\frac{1}{3}\right\}, \infty] - \\ & S[2, \infty] S[1, \left\{\frac{1}{6}\right\}, \infty] S[1, \left\{\frac{1}{2}\right\}, \infty] + \\ & S[2, \infty] S[1, \left\{\frac{1}{3}\right\}, \infty] S[1, \left\{\frac{1}{2}\right\}, \infty] - S[2, \infty] S[2, \left\{\frac{1}{2}\right\}, \infty] + \\ & S[2, \infty] S[1, 1, \left\{\frac{1}{6}, 3\right\}, \infty] + S[1, \left\{\frac{1}{6}\right\}, \infty] S[1, 2, \left\{\frac{1}{2}, 2\right\}, \infty] - \\ & S[1, \left\{\frac{1}{3}\right\}, \infty] S[1, 2, \left\{\frac{1}{2}, 2\right\}, \infty] + S[1, 3, \left\{\frac{1}{6}, 6\right\}, \infty] + \\ & \left. S[2, 2, \left\{\frac{1}{2}, 2\right\}, \infty] - S[1, 1, 2, \left\{\frac{1}{6}, 3, 2\right\}, \infty] \right) \end{aligned}$$

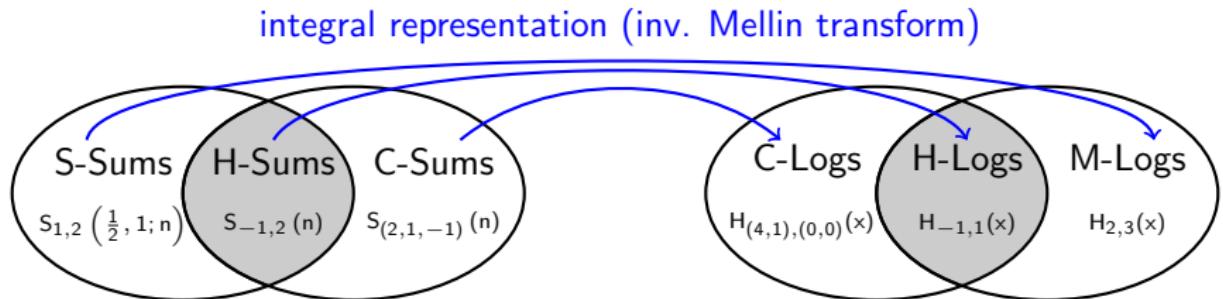
# Connection between these structures



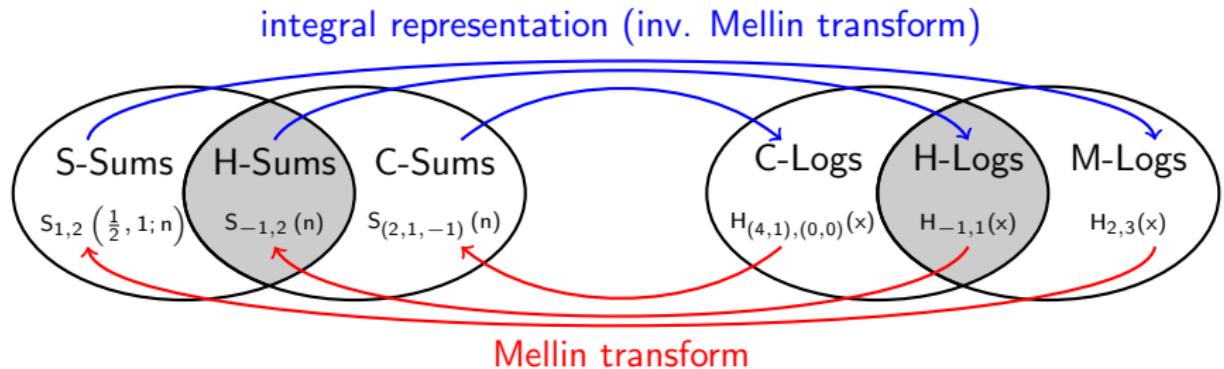
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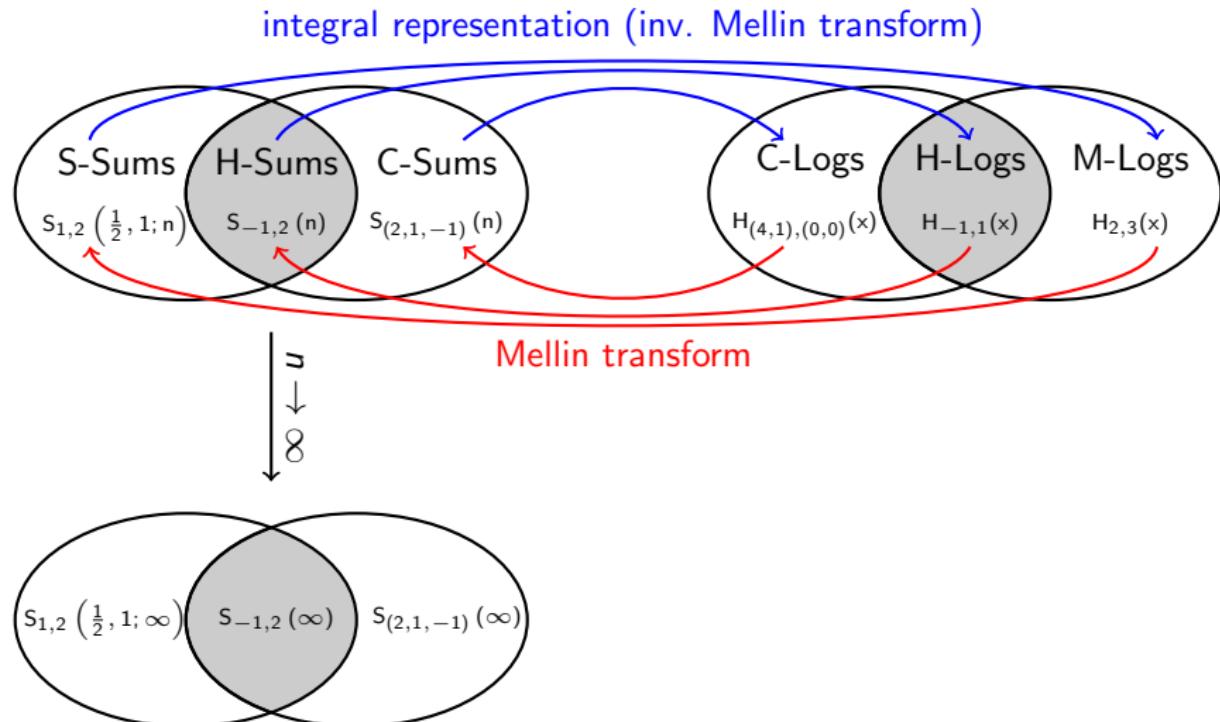
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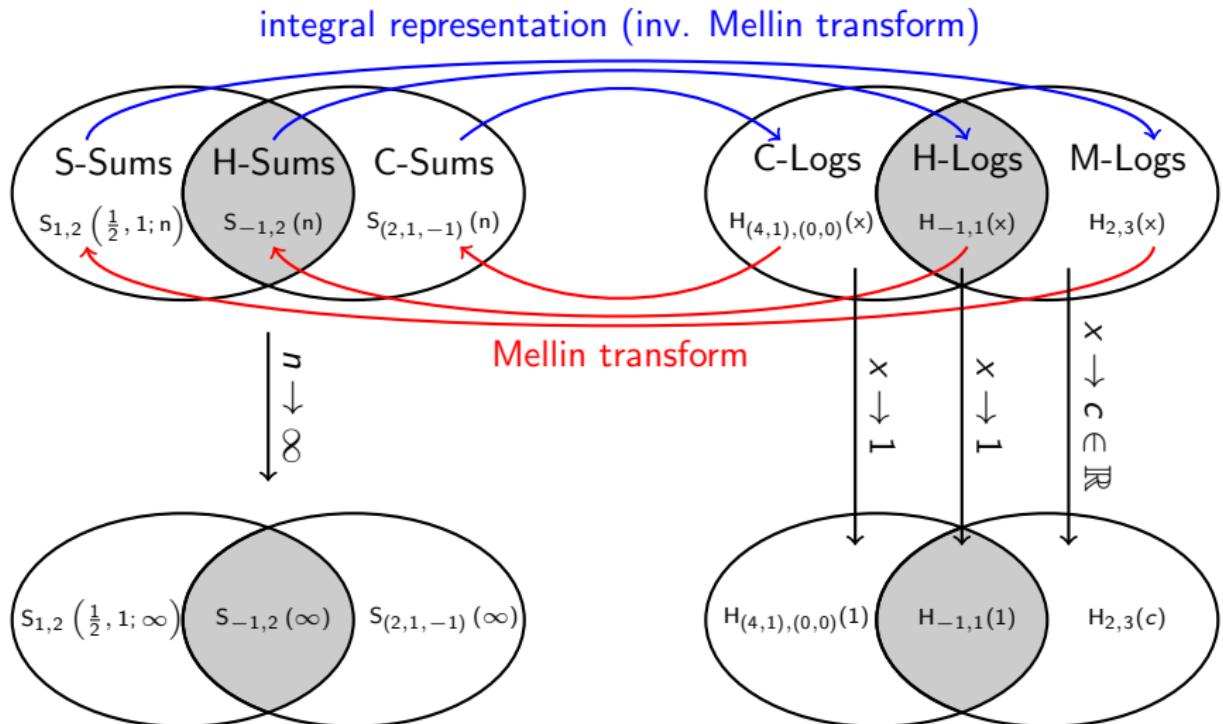
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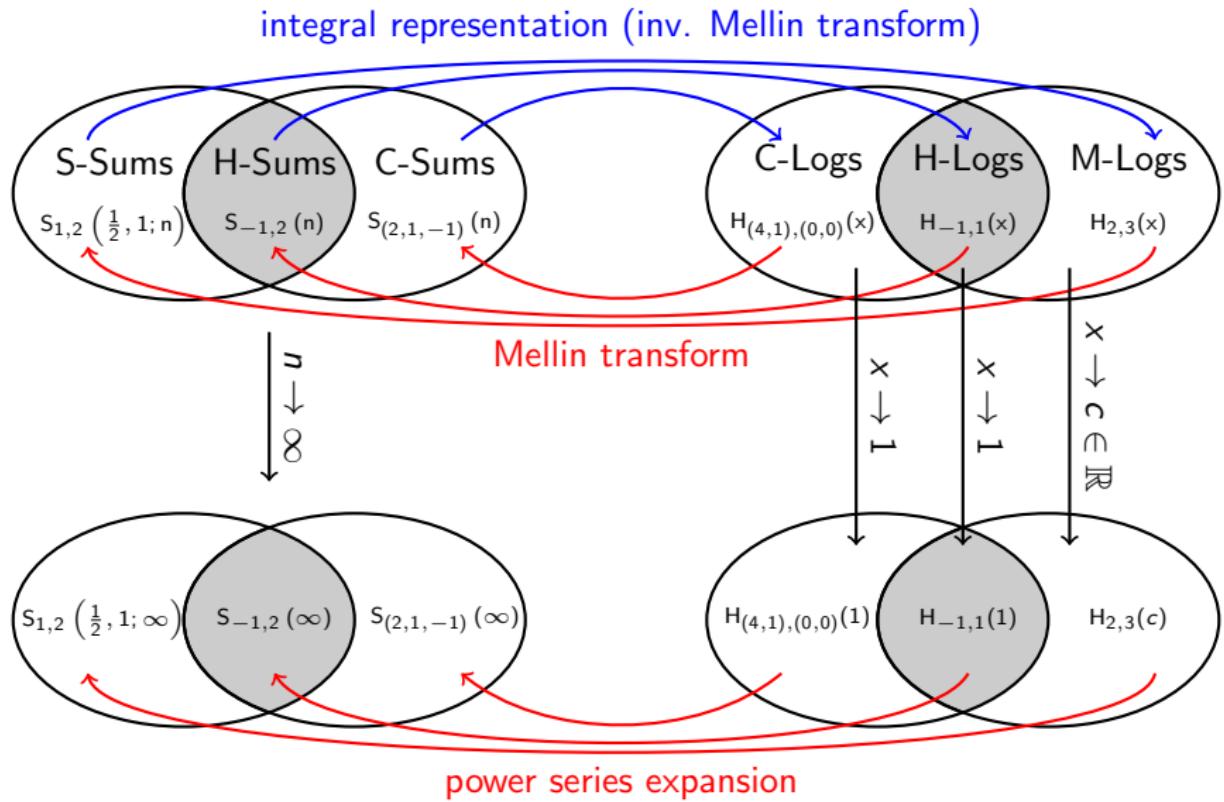
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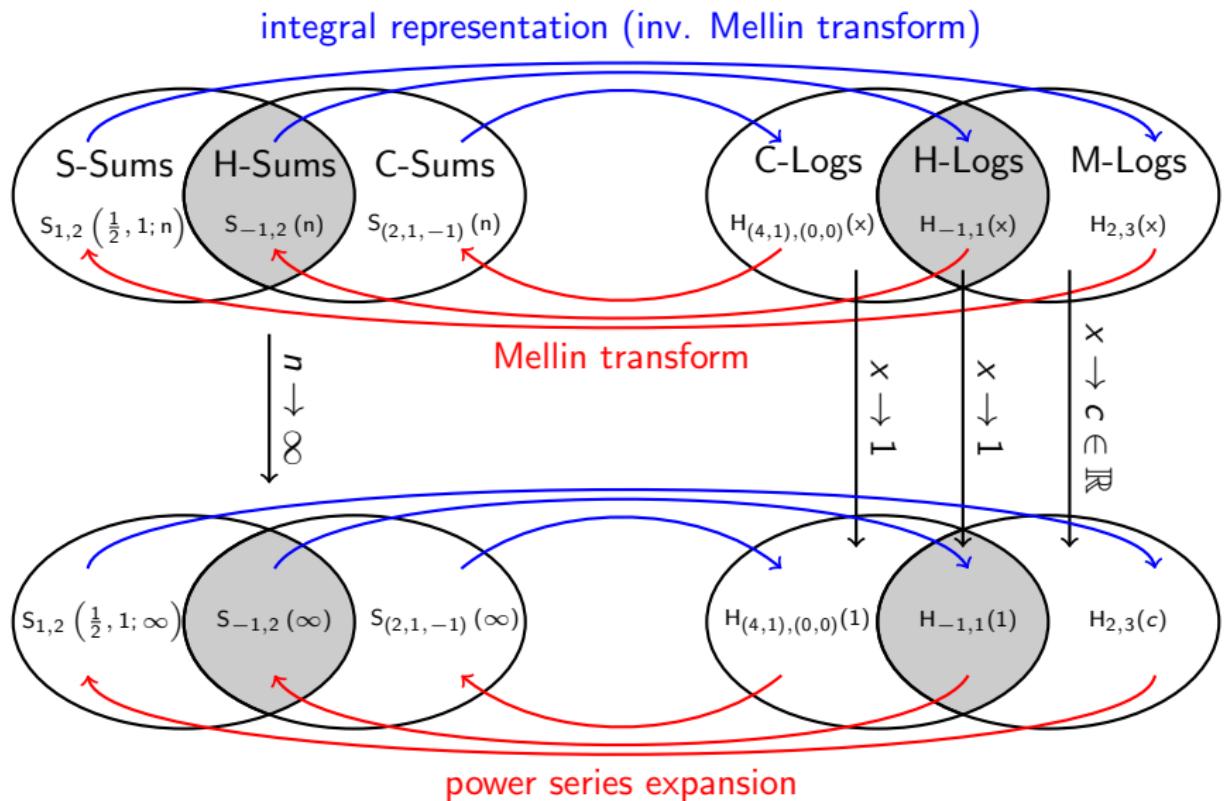
# Connection between these structures



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# Using the Package HarmonicSums

In[9]:= **HToS[H[-1, 3, x] + H[{4, 1}, x]]**

$$\text{Out}[9]= \sum_{\tau_1=1}^{\infty} \frac{3^{-\tau_1} x^{\tau_1}}{\tau_1^2} - \sum_{\tau_1=1}^{\infty} \frac{x^{4\tau_1}}{4\tau_1} - \sum_{\tau_1=1}^{\infty} \frac{(-x)^{\tau_1} S[1, \left\{-\frac{1}{3}\right\}, \tau_1]}{\tau_1} + \sum_{\tau_1=1}^{\infty} \frac{x^{-2+4\tau_1}}{-2+4\tau_1}$$

In[10]:= **SToH[** $\sum_{\tau_1=1}^{\infty} \frac{(-x)^{\tau_1} S[1, \left\{-\frac{1}{3}\right\}, \tau_1]}{\tau_1}$ **]**

$$\text{Out}[10]= -H[-1, 3, x] + H[0, 3, x]$$

In[11]:= **SinfToH[S[2, \infty] - S[-1, -1, \infty] + S[1, \left\{\frac{1}{3}\right\}, \infty]]**

$$\text{Out}[11]= -H[-1, 1, 1] + H[3, 1]$$

In[12]:= **HToSinf[H[-1, 1, 1] + H[3, 1]]**

$$\text{Out}[12]= S[2, \infty] - S[-1, -1, \infty] + S[1, \left\{\frac{1}{3}\right\}, \infty]$$

# Asymptotic Expansion of Harmonic Sums

We say that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is expanded in an asymptotic series

$$f(x) \sim \sum_{k=0}^{\infty} \frac{a_k}{x^k}, \quad x \rightarrow \infty,$$

where  $a_k$  are constants, if for all  $K \geq 0$

$$R_K(x) = f(x) - \sum_{k=0}^K \frac{a_k}{x^k} = o\left(\frac{1}{x^K}\right), \quad x \rightarrow \infty.$$

Why do we need these expansions of harmonic sums?

E.g.,

- for limits of the form

$$\lim_{n \rightarrow \infty} n \left( S_2(n) - \zeta_2 - S_{2,2}(n) + \frac{7 \zeta_2^2}{10} \right) = \zeta_2 - 1$$

- for the approximation of the values of analytic continued harmonic sums at the complex plane

$$S_{2,-3}(-20 + 10i) \approx -0.795096 - 0.105476i$$

Let  $\varphi(x)$  be analytic at  $x = 1$ , with

$$\varphi(1 - x) = \sum_{k=0}^{\infty} a_k x^k.$$

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Consider the Mellin transform

$$\Omega(n, \varphi(x)) = \int_0^1 x^n \varphi(x) dx$$

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For  $\operatorname{Re}(n) > 0$ ,  $\Omega(n, \varphi(x))$  is given by the factorial series

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$$\Omega(n, \varphi(x)) = \sum_{k=0}^{\infty} \frac{a_k k!}{(n+1)(n+2)\cdots(n+k+1)}$$

- $\Omega(n, \varphi(x))$  can be continued analytically to values of  $n \in \mathbb{C}$  as a meromorphic function.
- $\Omega(n, \varphi(x))$  has an analytic asymptotic representation.
- The poles of  $\Omega(n, \varphi(x))$  are situated at the negative integers.

# Asymptotic Expansions

- this basic idea can be turned into an algorithm

$$S_{-1,3}(n) \sim (-1)^n \left( -\frac{1}{4n^3} + \frac{5}{8n^4} - \frac{5}{8n^5} - \frac{5}{16n^6} \right) + \frac{3 \log(2) \zeta_3}{4} + (-1)^n \left( \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{4n^6} \right) \zeta_3 - \frac{19 \zeta_2^2}{40}$$

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- the algorithm can be extended to cyclotomic harmonic sums

$$\begin{aligned} S_{(2,1,2),(1,0,1)}(n) \sim & -4 \log(2) S_{(2,1,-1)}(\infty)^2 - 8 \log(2) S_{(2,1,-1)}(\infty) - 4 \log(2) + \frac{1}{9n^3} - \frac{1}{4n} + \frac{7\zeta_3}{4} \\ & + \left( -\frac{11}{48n^3} + \frac{1}{4n^2} - \frac{1}{4n} \right) (\log(n) + \gamma) \end{aligned}$$

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- extension to a subset of the S-Sums:

$$\begin{aligned} S_{2,1}(1, \frac{1}{3})(n) \sim & -S_{1,2}\left(\frac{1}{3}, 1; \infty\right) + 3^{-n} \left( \frac{3}{2n^3} - \frac{3}{4n^2} + \frac{1}{2n} \right) H_{0,3}(1) + 3^{-n} \left( -\frac{3}{2n^3} + \frac{3}{4n^2} - \frac{1}{2n} \right) S_2\left(\frac{1}{3}; \infty\right) \\ & + \left( 3^{-n} \left( -\frac{3}{2n^3} + \frac{1}{2n^2} + 3^n \left( -\frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \right) + \zeta_2 \right) S_1\left(\frac{1}{3}; \infty\right) \\ & + S_3\left(\frac{1}{3}; \infty\right) + H_3(1) 3^{-n} \left( \frac{3}{2n^3} - \frac{1}{2n^2} \right) + \frac{3^{-n}}{4n^3} \end{aligned}$$

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$$\begin{aligned} S_{(2,1,2),(1,0,1)}(n) \sim & -4 \log(2) S_{(2,1,-1)}(\infty)^2 - 8 \log(2) S_{(2,1,-1)}(\infty) - 4 \log(2) + \frac{1}{9n^3} - \frac{1}{4n} + \frac{7\zeta_3}{4} \\ & + \left( -\frac{11}{48n^3} + \frac{1}{4n^2} - \frac{1}{4n} \right) (\log(n) + \gamma) \end{aligned}$$

- extension to a subset of the S-Sums:

$$\begin{aligned} S_{2,1}\left(1, \frac{1}{3}\right)(n) \sim & -S_{1,2}\left(\frac{1}{3}, 1; \infty\right) + 3^{-n} \left( \frac{3}{2n^3} - \frac{3}{4n^2} + \frac{1}{2n} \right) H_{0,3}(1) + 3^{-n} \left( -\frac{3}{2n^3} + \frac{3}{4n^2} - \frac{1}{2n} \right) S_2\left(\frac{1}{3}; \infty\right) \\ & + \left( 3^{-n} \left( -\frac{3}{2n^3} + \frac{1}{2n^2} + 3^n \left( -\frac{1}{6n^3} + \frac{1}{2n^2} - \frac{1}{n} \right) \right) + \zeta_2 \right) S_1\left(\frac{1}{3}; \infty\right) \\ & + S_3\left(\frac{1}{3}; \infty\right) + H_3(1) 3^{-n} \left( \frac{3}{2n^3} - \frac{1}{2n^2} \right) + \frac{3^{-n}}{4n^3} \end{aligned}$$

- extension to a subset of the cyclotomic S-Sums:

$$S_{(2,1,2)}\left(\frac{1}{2}; n\right) \sim -\frac{\pi^2 - 6 \left( 8H_{0,1}\left(\frac{1}{\sqrt{2}}\right) + \log^2(2) \right)}{24\sqrt{2}} - 2^{-n} \left( -\frac{293}{8n^5} + \frac{99}{16n^4} - \frac{5}{4n^3} + \frac{1}{4n^2} + 2^n \right)$$

# Using the Package HarmonicSums

```
In[13]:= SExpansion[S[-1, 3, n], n, 10]
```

```
Out[13]=
```

$$\begin{aligned} & (-1)^n \left( -\frac{1}{4n^3} + \frac{5}{8n^4} - \frac{5}{8n^5} - \frac{5}{16n^6} + \frac{31}{24n^7} + \frac{133}{96n^8} - \frac{169}{24n^9} - \frac{163}{16n^{10}} \right) + \\ & \frac{3 \ln 2 z3}{4} + (-1)^n \left( \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{8n^4} - \frac{1}{4n^6} + \frac{17}{16n^8} - \frac{31}{4n^{10}} \right) z3 - \frac{19 z2^2}{40} \end{aligned}$$

```
In[14]:= GetApproximation[S[-1,3,n], {-2.5, 2}]
```

```
Out[14]= -0.795096 - 0.105476 i
```

```
In[15]:= HLimit[n*(S[2, n] - z2 - S[2, 2, n] + 7*z2^2/10), n]
```

```
Out[15]= -1+z2
```