Evaluating multiloop Feynman integrals by dimensional recurrence relations

V.A. Smirnov

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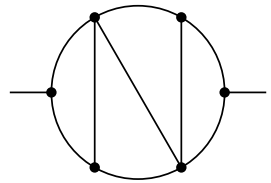
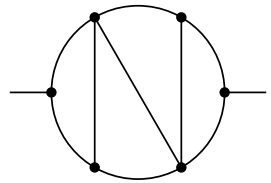
in collaboration with R.N. Lee and A.V. Smirnov

 $= \frac{441}{8}\zeta(7) + O(\varepsilon)$

$$\varepsilon = (4-d)/2$$

 $= \frac{441}{8}\zeta(7) + O(\varepsilon)$

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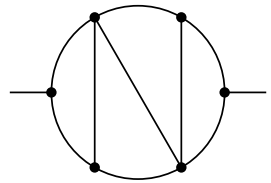 $= \frac{441}{8}\zeta(7) + O(\varepsilon)$

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Five-loop renormalization group calculations in the ϕ^4 theory

[S.G. Gorishny, S.A. Larin, F.V. Tkachov & K.G. Chetyrkin'83, D.I. Kazakov'83]

$$
\begin{aligned}
&= (1-2\epsilon)^3 \left(\frac{\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^3 \left\{ \frac{441\zeta_7}{8} + \epsilon\left(-216\zeta_3\zeta_5 + \frac{5733\zeta_8}{16} - \frac{81\zeta_{2,6}}{2}\right)\right. \\
&+\left(-267\zeta_3^3 - 81\zeta_4\zeta_5 - \frac{675\zeta_3\zeta_6}{2} + \frac{4583\zeta_9}{2}\right)\epsilon^2 + \left(-\frac{2403}{2}\zeta_3^2\zeta_4 - \frac{502287\zeta_5^2}{56} - \frac{7731\zeta_3\zeta_7}{56}\right. \\
&\left.+\frac{1324935\zeta_{10}}{112} + \frac{18441\zeta_{3,7}}{56}\right)\epsilon^3 + \left(-\frac{24315}{2}\zeta_3^2\zeta_5 - \frac{358023\zeta_5\zeta_6}{8} + \frac{139401\zeta_4\zeta_7}{8}\right. \\
&\left.-\frac{59895\zeta_3\zeta_8}{4} + \frac{232767\zeta_2\zeta_9}{4} - \frac{402081\zeta_{11}}{32} - \frac{621}{2}\zeta_3\zeta_{2,6} + \frac{6291}{2}\zeta_{2,1,8}\right)\epsilon^4 \\
&-\left(-6023\zeta_3^4 + 6660\zeta_3\zeta_4\zeta_5 - \frac{650997}{7}\zeta_2\zeta_5^2 + 40507\zeta_3^2\zeta_6 - \frac{1323426}{7}\zeta_2\zeta_3\zeta_7\right. \\
&+\frac{1750957\zeta_5\zeta_7}{2} + \frac{964778\zeta_3\zeta_9}{3} - \frac{104287641323\zeta_{12}}{132672} - 2853\zeta_4\zeta_{2,6} + \frac{48222}{7}\zeta_2\zeta_{3,7} \\
&\left.\left.-\frac{190175\zeta_{3,9}}{6} - 10716\zeta_{2,1,1,8}\right)\epsilon^5 + O\left(\epsilon^6\right)\right\}
\end{aligned}
$$

[R. Lee, A. and V. Smirnovs'11]

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Multiple zeta values $\zeta_m = \zeta(m)$, $\zeta_{m_1,m_2} = \zeta(m_1, m_2), \ldots$

$$
\zeta(m_1, \ldots, m_k) = \sum_{i_1=1}^{\infty} \sum_{1}^{i_1-1} \cdots \sum_{1}^{i_{k-1}-1} \prod_{j=1}^{k} \frac{\mathrm{sgn}(m_j)^{i_j}}{i_j^{|m_j|}} .
$$

Roman Lee's method based on dimensional recurrence relations

[R.N. Lee'09]

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Evaluating Feynman integrals (STMP 211, Springer 2004)
Feynman Integrals Calculus (Springer 2006)

Analytic Tools for Feynman Integrals (STMP ..., Springer 2013)

Master integrals for three-loop form factors of the photon-quark and the effective gluon-Higgs boson vertex originating from integrating out the heavy top-quark loops.

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The missing finite parts of $A_{9,2}$ and $A_{9,4}$

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The evaluation of terms of transcendentality weight 8 characteristic for four loops: $\pi^8, \zeta(5)\zeta(3), \zeta(3)^2\pi^2, \zeta(-6,-2)$, i.e. up to ε^2 for $A_{9,2}$ and $A_{9,4}$ and ε^3 for $A_{9,1}$.

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One-scale Feynman integrals, $p_1^2 = p_2^2 = 0, q^2 = (p_1 + p_2)^2$

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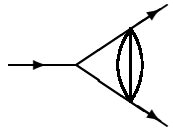
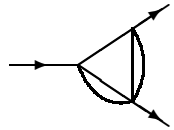
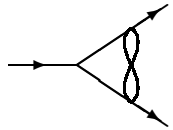


$A_{5,1}$



$A_{5,2}$



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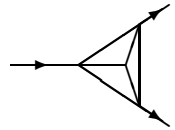

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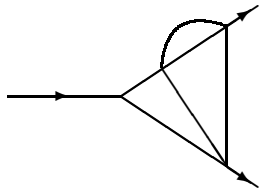
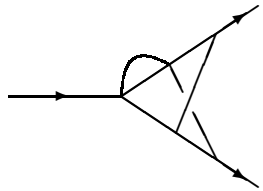
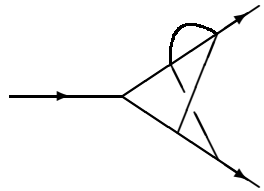


$A_{6,3}$



$A_{7,1}$



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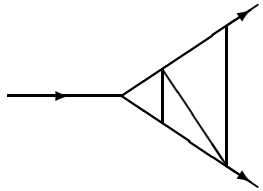
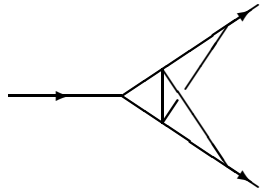
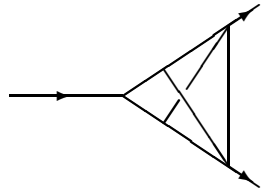


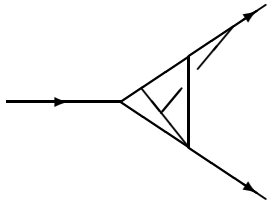
$A_{7,3}$



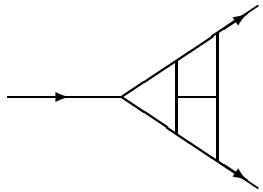
$A_{7,4}$



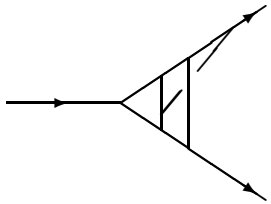
$A_{7,5}$



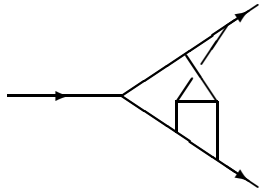
A_8



$A_{9,1}$

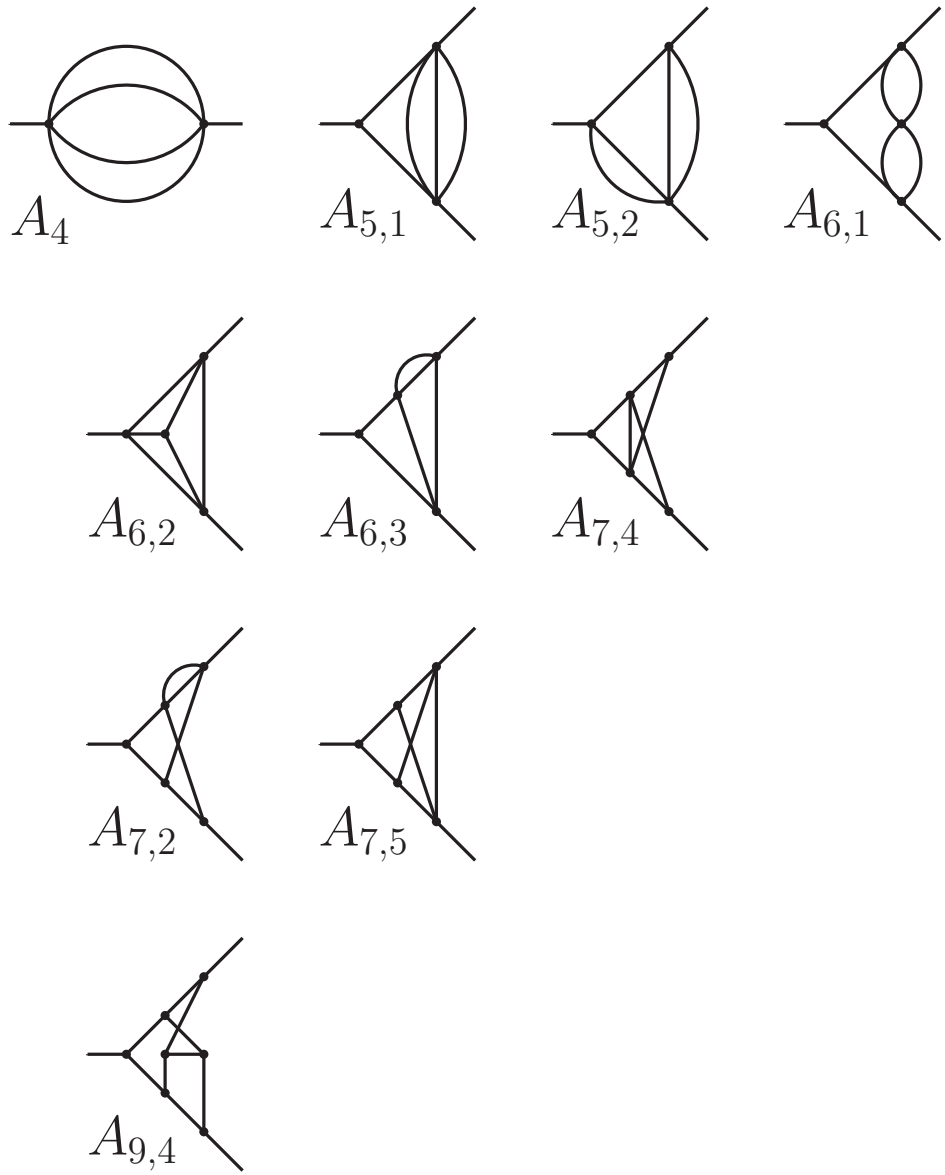


$A_{9,2}$



$A_{9,4}$

$A_{9,4}$ and lower master integrals



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Consider, for example, $A_{6,3}$.

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One lower master integral A_4 which is of complexity level 0.

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- Obtain analytic results using `PSLQ`

Graph $\Gamma \to$ dimensionally regularized Feynman integral

$$F_\Gamma(a_1 \ldots, a_L; d) = \frac{\mathrm{e}^{\mathrm{i}\pi(a+h(1-d/2))/2}\pi^{hd/2}}{\prod_l \Gamma(a_l)}$$
$$\times \int_0^\infty \mathsf{d}\alpha_1 \ldots \int_0^\infty \mathsf{d}\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} \mathrm{e}^{\mathrm{i}\mathcal{V}/\mathcal{U} - \mathrm{i}\sum m_l^2 \alpha_l} ,$$

where $a = \sum a_i$

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For a Feynman integral with $1/(m^2 - k^2 - i0)^{a_l}$ propagators,

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{l \notin T} \alpha_l ,$$
$$\mathcal{V} = \sum_{2-\text{trees } T} \prod_{l \notin T} \alpha_l \left(q^T\right)^2 .$$

Dimensional recurrence relation

[O. Tarasov'96]

$d \to d-2$

$\mathcal{U}^{-d/2} \to \mathcal{U}^{-(d-2)/2} = \mathcal{U}\mathcal{U}^{-d/2}$

Inserting $\alpha_l \to (-ia_l)\mathbf{l}^+$

$$F_\Gamma(a_1 \ldots, a_L; d-2) = \frac{1}{\pi} \sum_T \prod_{l \notin T} a_l \mathbf{l}^+ F_\Gamma(a_1 \ldots, a_L; d)$$

For $F = A_{6,3}$:

$$\begin{aligned} F(1,1,1,1,1,1;d-2) &= F(1,1,1,2,2,2;d) + F(1,1,2,1,2,2;d) \\ &+ F(1,1,2,2,1,2;d) + F(1,2,1,2,1,2;d) + F(1,2,1,2,2,1;d) \\ &+ F(1,2,2,1,1,2;d) + F(1,2,2,1,2,1;d) + F(1,2,2,2,1,1;d) \\ &+ F(2,1,1,2,1,2;d) + F(2,1,1,2,2,1;d) + F(2,1,2,1,1,2;d) \\ &+ F(2,1,2,1,2,1;d) + F(2,1,2,2,1,1;d) \end{aligned}$$

The integrals with raised indices are reduced to master integrals by IBP relations
FIRE [A. Smirnov] and R.Lee's code
$A_{6,3}$ is of complexity level 1.

$$A_{6,3}(d-2) = \frac{8(-3+d)(-9+2d)(-7+2d)(-10+3d)}{-16+3d} A_{6,3}(d)$$
$$+\frac{32(-3+d)(-9+2d)(-7+2d)(-5+2d)(-10+3d)}{(-5+d)(-4+d)^2(-16+3d)}$$
$$\times(-8+3d)(-32+7d)\ A_4(d)$$

where

$$A_4(d) = \frac{\Gamma(4-3d/2)\Gamma(d/2-1)^4}{\Gamma(2d-4)}$$

To solve this difference equation for $f(d) = A_{6,3}(d)$ turn to a new function $f(d) = S(d)g(d)$ with such a function $S(d) \equiv 1/\Sigma(d)$ that the corresponding equation for $g(d)$ will be simpler.

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One variant is to choose $S(d)$ as

$$(-256)^{-d/2} \frac{(3/2 - d/2)(9/4 - d/2)(7/4 - d/2)(5/3 - d/2)}{8/3 - d/2}$$

with the replacements $(a - d/2) \to \Gamma(a - d/2)$ for all the linear factors involved.

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Any factor periodic in d with the period 2 can be included into $S(d)$.

Another variant is to choose $S(d)$ as

$$(-256)^{-d/2}\frac{(3/2-d/2)(9/4-d/2)(7/4-d/2)(5/3-d/2)}{8/3-d/2}$$

with the replacements $(a-d/2)\to 1/\Gamma(1-a+d/2)$ for all the linear factors involved.
Any factor periodic in d with the period 2 can be included into $S(d)$.
Let us choose

$$\frac{1}{S(d)}=\Sigma(d)=\frac{1}{\sqrt{\pi}}32^{4d-\frac{5}{2}}\left(\frac{d}{2}-\frac{5}{3}\right)\sin\left(\frac{1}{2}\pi(d-5)\right)\sin\left(\frac{1}{2}\pi\left(d-\frac{14}{3}\right)\right)$$
$$\times\sin^2\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}-\frac{5}{4}\right)\Gamma\left(\frac{d}{2}-\frac{3}{4}\right)\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)$$

Equation for g:

$$g(d-2) = g(d) + h(d)$$

with

$$h(d) = \frac{1}{3(d-5)} 1024\pi(32-7d)\sin\left(\frac{\pi d}{2}\right)\sin(\pi d)\cos\left(\frac{1}{6}(3\pi d+\pi)\right)$$
$$\times\Gamma\left(7-\frac{3d}{2}\right)\Gamma\left(\frac{d}{2}-2\right)^3$$

$$
\begin{aligned}
g(d) &= g(d-2) - h(d) \\
&= g(d-4) - h(d-2) - h(d) \\
&= \ldots \\
&= g(-\infty) - \sum_{k=0}^{\infty} h(d-2k) \\
&= g(+\infty) + \sum_{k=1}^{\infty} h(d+2k)
\end{aligned}
$$

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\end{aligned}
$$

$r_+(d) = -\sum_{k=0}^{\infty} h(d-2k)$ and $r_-(d) = +\sum_{k=1}^{\infty} h(d+2k)$ are solutions.

However, $r_-(d)$ is a divergent series (for $A_{6,3}$).
$r_+(d)$ converges very well because

$$h(d+2(k+1))/h(d+2k) \to -\frac{1}{27}, \quad k \to \infty$$

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In the general case

$$r\,(d) = r_+\,(d-2) + r_-\,(d)\,, \quad r_\pm\,(d \pm 2k) \overset{k\to\infty}{<} a^k, \quad |a| < 1\,.$$

$$A_{6,3}(d) = (r_+(d) + \omega(d))/\Sigma(d)$$

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For $A_{6,3}$, FIESTA says that in the band $(3,5]$ there can be simple poles at $d = 10/3, 4, 14/3, 5$.

Taking into account this information and analyzing behaviour at infinity (Im$d \to \pm\infty$) with the help of the alpha representation provides the following Ansatz for $\Omega(z)$, with $z = e^{i\pi d}$:

$$a_0 + \frac{a_1}{z - e^{-2i\pi/3}} + \frac{a_2}{z - 1}$$

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$$a_0 + \frac{a_1}{z - e^{-2i\pi/3}} + \frac{a_2}{z - 1}$$

which corresponds to the following Ansatz for $\omega(d)$:

$$b_0 + b_1 \cot\left(\frac{1}{2}\pi(d-4)\right) + b_2 \cot\left(\frac{1}{2}\pi\left(d - \frac{10}{3}\right)\right)$$

The constants b_i are fixed by using an MB representation for $A_{6,3}$

$$\omega(d) = 512\pi^4 3^{-1/2}\left(\cot\left(\frac{1}{2}\pi\left(d-\frac{10}{3}\right)\right) - \cot\left(\frac{1}{2}\pi(d-4)\right)\right)$$

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MB tools at http://projects.hepforge.org/mbtools/

MB.m [M. Czakon'05]
MBresolve.m [A. Smirnov'09]
barnesroutines.m [D. Kosower'08]

Result:

$$A_{6,3}(d) = \frac{1}{\Sigma(d)} \left(\sum_{k=0}^{\infty} r_+(d+2k) + \omega(d) \right)$$

where

$$r_+(d+2k) = -\frac{2048\pi^2(-1)^k(7d+14k-32)}{3(d+2k-5)}$$
$$\times \frac{\sin^2\left(\frac{\pi d}{2}\right)\cos\left(\pi\left(\frac{d}{2}+\frac{1}{6}\right)\right)\cos\left(\frac{\pi d}{2}\right)\csc\left(\frac{3\pi d}{2}\right)\Gamma\left(\frac{d}{2}+k-2\right)^3}{\Gamma\left(\frac{3d}{2}+3k-6\right)}$$

PSLQ

[H.R.P. Ferguson & D.H. Bailey'91]

is an algorithm that provides the best approximation of a given numerical number as a linear combination, with rational coefficients, of a given set of transcendental numbers.

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For example, $54 \times 7 = 378$ digits.

Available in `Matematica 8`

FindIntegerNullVector[]

PSLQ gives

$$
\begin{aligned}
A_{6,3}(4-2\epsilon) \quad = \quad & \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(1-3\epsilon)(1-2\epsilon)}\left\{\frac{1}{6\epsilon^3}+\frac{\pi^2}{8\epsilon}-\frac{35\zeta_3}{6}-\frac{77\pi^4\epsilon}{2880}\right. \\
& -\epsilon^2\left(\frac{49\pi^2\zeta_3}{24}+\frac{651\zeta_5}{10}\right)+\epsilon^3\left(\frac{1141\zeta_3^2}{12}-\frac{93451\pi^6}{725760}\right) \\
& -\epsilon^4\left(\frac{713\pi^2\zeta_5}{40}-\frac{511}{320}\pi^4\zeta_3+\frac{9017\zeta_7}{14}\right)+\epsilon^5\left(\frac{623}{48}\pi^2\zeta_3^2\right. \\
& \left.\left.-\frac{544}{9}\zeta_{-6,-2}+\frac{11195\zeta_5\zeta_3}{6}-\frac{2022493\pi^8}{11612160}\right)+O\left(\epsilon^6\right)\right\},
\end{aligned}
$$

$$A_{9,4}(4-2\varepsilon) = e^{-3\gamma_E\varepsilon}\left\{-\frac{1}{9\varepsilon^6} - \frac{8}{9\varepsilon^5} + \left[1+\frac{43\pi^2}{108}\right]\frac{1}{\varepsilon^4}\right.$$

$$+\left[\frac{109\zeta(3)}{9} + \frac{14}{9} + \frac{53\pi^2}{27}\right]\frac{1}{\varepsilon^3}$$

$$+\left[\frac{608\zeta(3)}{9} - 17 - \frac{311\pi^2}{108} - \frac{481\pi^4}{12960}\right]\frac{1}{\varepsilon^2}$$

$$+\left[-\frac{949\zeta(3)}{9} - \frac{2975\pi^2\zeta(3)}{108} + \frac{3463\zeta(5)}{45} + 84 + \frac{11\pi^2}{18} + \frac{85\pi^4}{108}\right]\frac{1}{\varepsilon}$$

[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

[G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$+\left[\frac{434\zeta(3)}{9} - \frac{299\pi^2\zeta(3)}{3} - \frac{3115\zeta_3^2}{6} + \frac{7868\zeta(5)}{15} - 339\right.$$

$$\left.+\frac{77\pi^2}{4} - \frac{2539\pi^4}{2592} - \frac{247613\pi^6}{466560}\right]$$ [R. Lee, A. and V. Smirnovs'10]

$$+\left[-1242+112\pi^2-589\zeta_3+\frac{487\pi^4}{432}-\frac{19499\pi^2\zeta_3}{108}+\frac{30067\zeta_5}{45}+\frac{25567\pi^6}{30240}\right.$$

$$\left.+\frac{18512\zeta_3^2}{9}+\frac{38903\pi^4\zeta_3}{2592}+\frac{113629\pi^2\zeta_5}{540}-\frac{8564\zeta_7}{63}\right]\varepsilon$$

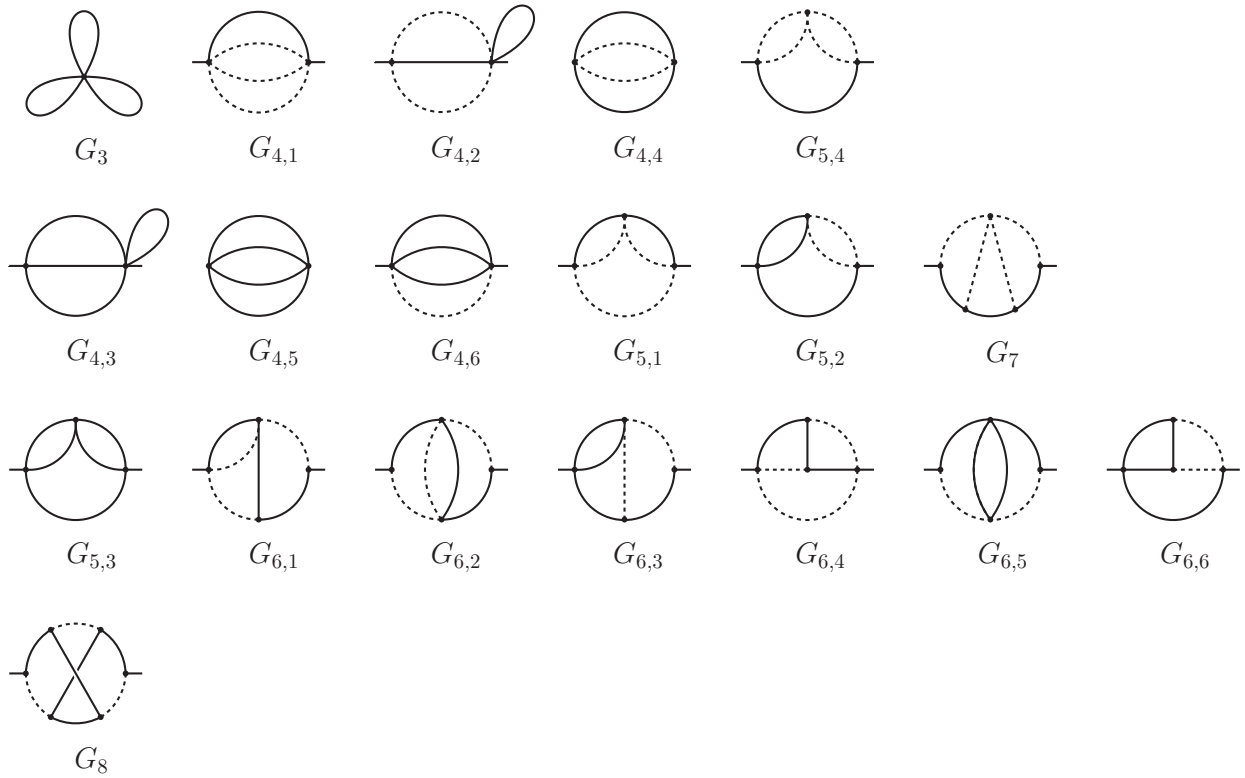
$$+\left[4293-\frac{1887\pi^2}{4}+3756\zeta_3-\frac{491\pi^4}{32}+\frac{4019\pi^2\zeta_3}{18}+\frac{7874\zeta_5}{15}\right.$$

$$-\frac{9901847\pi^6}{3265920}-\frac{26291\zeta_3^2}{6}+\frac{9037\pi^4\zeta_3}{135}+\frac{35728\pi^2\zeta_5}{45}-\frac{72537\zeta_7}{14}$$

$$\left.\left.+\frac{30535087\pi^8}{31352832}-\frac{152299}{216}\pi^2\zeta_3^2+\frac{730841\zeta_3\zeta_5}{135}-\frac{76288}{81}\zeta_{-6,-2}\right]\varepsilon^2+O(\varepsilon^3)\right\}$$

[R. Lee and V. Smirnov'10]

master integrals for 3-loop $g-2$



[S. Laporta and E. Remiddi'96-97]

[K. Melnikov and T. van Ritbergen'00]

A pure numerical approach to evaluate 4-loop $g-2$:

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[S. Laporta'01]

Analytic evaluation of 3-loop master integrals up to transcendentality weight 7:

[R. Lee and V. Smirnov'10]

$$G_8(4-2\epsilon) = \Gamma(1+\epsilon)^3 \Bigg\{ \left(-\frac{\pi^4}{6} + 4\pi^2 \ln^2 2 \right)$$

$$+\epsilon \left(\frac{2\pi^4}{3} - 16\pi^2 \ln^2 2 + \frac{166}{45}\pi^4 \ln 2 - \frac{208}{9}\pi^2 \ln^3 2 - \frac{32 \ln^5 2}{15} \right.$$

$$\left. +256 a_5 + \frac{17\pi^2 \zeta_3}{6} - 291 \zeta_5 \right)$$

$$+\epsilon^2 \left(-\frac{14\pi^4}{3} + 112\pi^2 \ln^2 2 - \frac{664}{45}\pi^4 \ln 2 + \frac{832}{9}\pi^2 \ln^3 2 + \frac{128 \ln^5 2}{15} \right.$$

$$-1024 a_5 - \frac{34\pi^2 \zeta_3}{3} + 1164 \zeta_5 - \frac{21743\pi^6}{11340} - \frac{197}{9}\pi^4 \ln^2 2 + \frac{713}{9}\pi^2 \ln^4 2$$

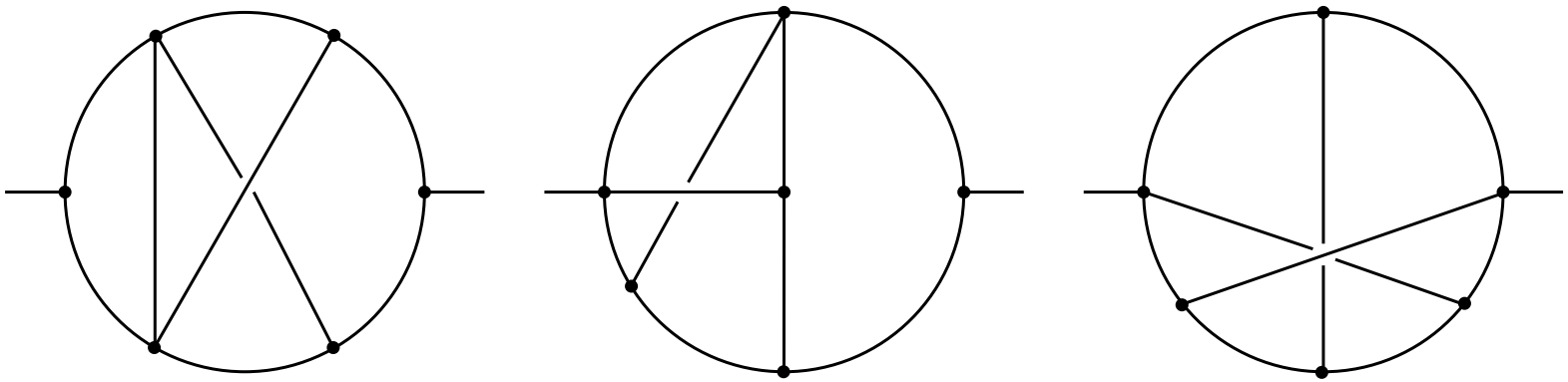
$$\left. +\frac{64 \ln^6 2}{9} - 104\pi^2 a_4 + 5120 a_6 + 2688 s_6 - 51\pi^2 \ln 2\, \zeta_3 - 953 \zeta_3^2 \right)$$

$$
\begin{aligned}
&+\epsilon^3\Bigg(\frac{80\pi^4}{3}-640\pi^2\ln^2 2+\frac{4648}{45}\pi^4\ln 2-\frac{5824}{9}\pi^2\ln^3 2-\frac{896\ln^5 2}{15}+7168 \\
&+\frac{238\pi^2\zeta_3}{3}-8148\zeta_5+\frac{21743\pi^6}{2835}+\frac{788}{9}\pi^4\ln^2 2-\frac{2852}{9}\pi^2\ln^4 2 \\
&-\frac{256\ln^6 2}{9}+416\pi^2 a_4-20480a_6-10752s_6+204\pi^2\ln 2\,\zeta_3+3812\zeta_3^2 \\
&+\frac{4868}{189}\pi^6\ln 2+\frac{8492}{135}\pi^4\ln^3 2-\frac{7288}{45}\pi^2\ln^5 2-\frac{4864\ln^7 2}{315} \\
&+1776\pi^2\ln 2\,a_4+1520\pi^2 a_5+77824a_7-\frac{106880}{7}\ln 2\,s_6+\frac{4003\pi^4\zeta_3}{21} \\
&+\frac{2167}{3}\pi^2\ln^2 2\,\zeta_3-\frac{316}{3}\ln^4 2\,\zeta_3-2528a_4\zeta_3+\frac{133600}{7}\ln 2\,\zeta_3^2 \\
&+\frac{875561\pi^2\zeta_5}{84}+37200\ln^2 2\,\zeta_5-\frac{1325727\zeta_7}{7}+\frac{106880s_{7,a}}{7} \\
&-\frac{161920s_{7,b}}{7}\Bigg)+O\left(\epsilon^4\right)\Bigg\}
\end{aligned}
$$

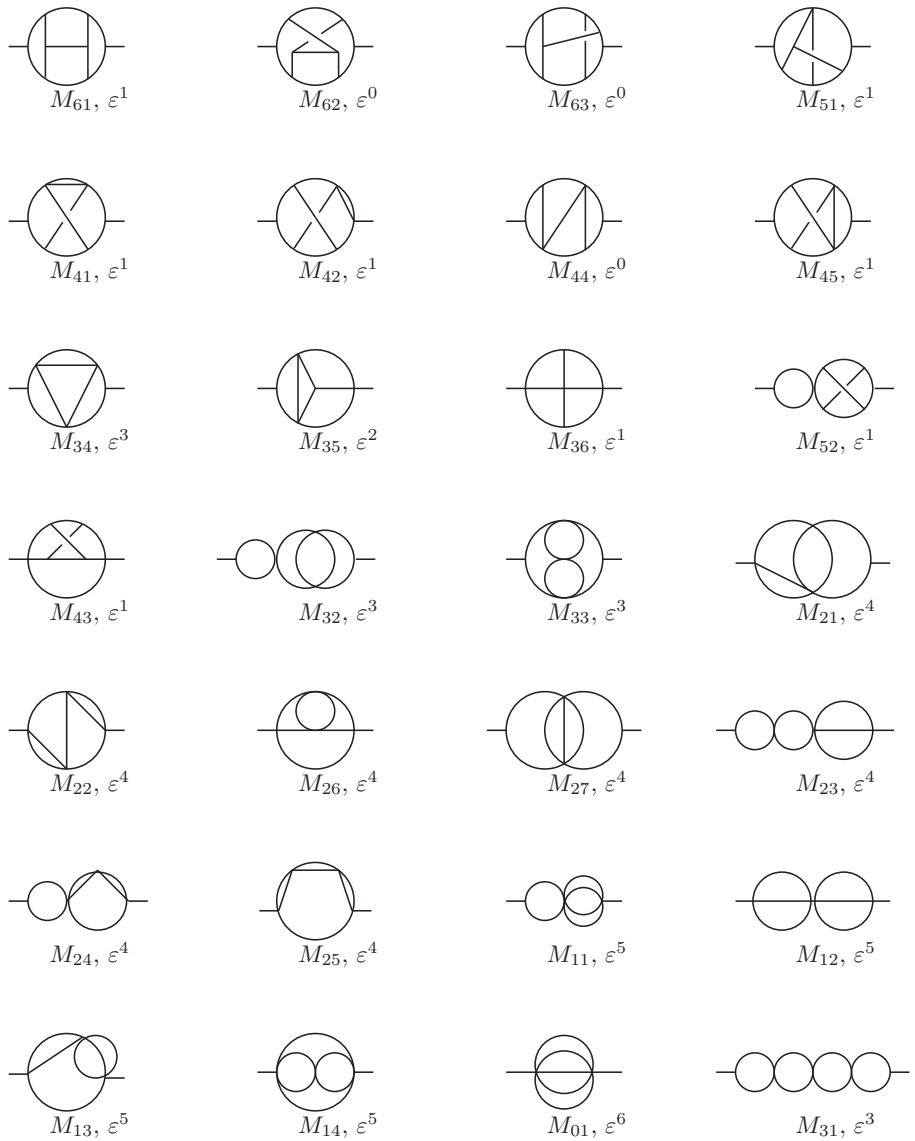
$$
\begin{aligned}
a_i &= \mathsf{Li}_i\left(\frac{1}{2}\right) \ , \\
s_6 &= \zeta(-5,-1) + \zeta(6) \ , \\
s_{7a} &= \zeta(-5,1,1) + \zeta(-6,1) + \zeta(-5,2) + \zeta(-7) \ , \\
s_{7b} &= \zeta(7) + \zeta(5,2) + \zeta(-6,-1) + \zeta(5,-1,-1)
\end{aligned}
$$

The coefficients in the ε-expansion of planar massless propagator diagrams up to five loops should be expressed in terms of multiple zeta values. [Brown'08]

For the following three diagrams, every coefficient in the Taylor expansion in ε is a rational linear combination of multiple zeta values and multiple polylogarithms with 6th roots of unity as arguments.



$$\mathsf{Li}_{n_1,\ldots,n_r}(x_1,\ldots,x_r) = \sum_{0<k_1<\ldots<k_r} \frac{x_1^{k_1}\ldots x_r^{k_r}}{k_1^{n_1}\ldots k_r^{n_r}}$$


M_{61}, ε^1
M_{62}, ε^0
M_{63}, ε^0
M_{51}, ε^1
M_{41}, ε^1
M_{42}, ε^1
M_{44}, ε^0
M_{45}, ε^1
M_{34}, ε^3
M_{35}, ε^2
M_{36}, ε^1
M_{52}, ε^1
M_{43}, ε^1
M_{32}, ε^3
M_{33}, ε^3
M_{21}, ε^4
M_{22}, ε^4
M_{26}, ε^4
M_{27}, ε^4
M_{23}, ε^4
M_{24}, ε^4
M_{25}, ε^4
M_{11}, ε^5
M_{12}, ε^5
M_{13}, ε^5
M_{14}, ε^5
M_{01}, ε^6
M_{31}, ε^3

$$
\begin{aligned}
M_{4,4} = (1-2\epsilon)^3 \left(\frac{\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^3 \Bigg\{ & \frac{441\zeta_7}{8} + \epsilon\left(-216\zeta_3\zeta_5 + \frac{5733\zeta_8}{16} - \frac{81\zeta_{2,6}}{2}\right) \\
& + \left(-267\zeta_3^3 - 81\zeta_4\zeta_5 - \frac{675\zeta_3\zeta_6}{2} + \frac{4583\zeta_9}{2}\right)\epsilon^2 + \left(-\frac{2403}{2}\zeta_3^2\zeta_4 - \frac{502287\zeta_5^2}{56} - \frac{7731\zeta_3\zeta_7}{56}\right. \\
& \left. + \frac{1324935\zeta_{10}}{112} + \frac{18441\zeta_{3,7}}{56}\right)\epsilon^3 + \left(-\frac{24315}{2}\zeta_3^2\zeta_5 - \frac{358023\zeta_5\zeta_6}{8} + \frac{139401\zeta_4\zeta_7}{8}\right. \\
& \left. - \frac{59895\zeta_3\zeta_8}{4} + \frac{232767\zeta_2\zeta_9}{4} - \frac{402081\zeta_{11}}{32} - \frac{621}{2}\zeta_3\zeta_{2,6} + \frac{6291}{2}\zeta_{2,1,8}\right)\epsilon^4 \\
& - \left(-6023\zeta_3^4 + 6660\zeta_3\zeta_4\zeta_5 - \frac{650997}{7}\zeta_2\zeta_5^2 + 40507\zeta_3^2\zeta_6 - \frac{1323426}{7}\zeta_2\zeta_3\zeta_7\right. \\
& + \frac{1750957\zeta_5\zeta_7}{2} + \frac{964778\zeta_3\zeta_9}{3} - \frac{104287641323\zeta_{12}}{132672} - 2853\zeta_4\zeta_{2,6} + \frac{48222}{7}\zeta_2\zeta_{3,7} \\
& \left. - \frac{190175\zeta_{3,9}}{6} - 10716\zeta_{2,1,1,8}\right)\epsilon^5 + O\left(\epsilon^6\right) \Bigg\}
\end{aligned}
$$

[R. Lee, A. and V. Smirnovs'11]

$$
\begin{aligned}
M_{6,3} = \left(\frac{\Gamma(1-\varepsilon)^2\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^3 \frac{(1-2\epsilon)^3}{(1+3\epsilon)(1+4\epsilon)} \Bigg\{ & -\frac{5\zeta_5}{\epsilon} - \left(20\zeta_5 + 41\zeta_3^2 + \frac{25\zeta_6}{2} - \frac{161\zeta_7}{2}\right) \\
& + \left(-308\zeta_3^2 - 50\zeta_6 - 123\zeta_3\zeta_4 + 514\zeta_7 + 4862\zeta_3\zeta_5 - \frac{24451\zeta_8}{4} + 1566\zeta_{2,6}\right)\epsilon \\
& + \Bigg(-924\zeta_3\zeta_4 - 1500\zeta_7 + 68636\zeta_3\zeta_5 - \frac{744639\zeta_8}{8} + 23220\zeta_{2,6} + \frac{1526\zeta_3^3}{3} \\
& -2103\zeta_4\zeta_5 + 4325\zeta_3\zeta_6 + \frac{111709\zeta_9}{36}\Bigg)\epsilon^2 + \Bigg(235200\zeta_3\zeta_5 - \frac{710311\zeta_8}{2} \\
& +85536\zeta_{2,6} + \frac{22048\zeta_3^3}{3} - 36366\zeta_4\zeta_5 + 55695\zeta_3\zeta_6 + \frac{237103\zeta_9}{12} + 2289\zeta_3^2\zeta_4 \\
& + \frac{1341143\zeta_5^2}{56} + \frac{3816969\zeta_3\zeta_7}{56} - \frac{7815019\zeta_{10}}{112} - \frac{500565\zeta_{3,7}}{56}\Bigg)\epsilon^3 + \Bigg(\frac{61040\zeta_3^3}{3} \\
& -160416\zeta_4\zeta_5 + 161860\zeta_3\zeta_6 - \frac{460411\zeta_9}{9} + 33072\zeta_3^2\zeta_4 + \frac{60035137\zeta_3\zeta_7}{56} \\
& + \frac{12859479\zeta_5^2}{56} - \frac{134815227\zeta_{10}}{112} - \frac{7724781\zeta_{3,7}}{56} - 453668\zeta_3^2\zeta_5 + \frac{280574047\zeta_{11}}{64} \\
& + \frac{1346777\zeta_5\zeta_6}{6} - \frac{4654793\zeta_4\zeta_7}{8} + 1309878\zeta_3\zeta_8 - 2749211\zeta_2\zeta_9 - 87752\zeta_3\zeta_{2,6} \\
& -148606\zeta_{2,1,8}\Bigg)\epsilon^4
\end{aligned}
$$

$$
+\left(91560\zeta_3^2\zeta_4+\frac{5319415\zeta_5^2}{14}+\frac{55472425\zeta_3\zeta_7}{14}-\frac{705626\zeta_3^4}{9}\right.
$$

$$
-\frac{7239597\zeta_{3,7}}{14}-5183674\zeta_3^2\zeta_5+\frac{3338505\zeta_5\zeta_6}{2}-1597650\zeta_{2,1,8}-\frac{1752859}{18}\zeta_3^2\zeta_6
$$

$$
+\frac{62741559\zeta_3\zeta_8}{4}-29556525\zeta_2\zeta_9+\frac{2990096591\zeta_{11}}{64}-\frac{29828681054659\zeta_{12}}{2388096}
$$

$$
-\frac{53503881\zeta_4\zeta_7}{8}-\frac{142150835\zeta_{10}}{28}-1729457\zeta_3\zeta_4\zeta_5-\frac{25832277}{7}\zeta_2\zeta_5^2
$$

$$
-\frac{157544998}{21}\zeta_2\zeta_3\zeta_7+\frac{311533051\zeta_5\zeta_7}{18}+\frac{1792012205\zeta_3\zeta_9}{108}-1085340\zeta_3\zeta_{2,6}
$$

$$
\left.-227636\zeta_4\zeta_{2,6}+\frac{1913502}{7}\zeta_2\zeta_{3,7}-\frac{105698899\zeta_{3,9}}{108}-\frac{1275668}{3}\zeta_{2,1,1,8}\right)\epsilon^5
$$

$$
\left.+O\left(\epsilon^6\right)\right\}
$$

Evaluating multicomponent master integrals

Let $\mathbf{F}$ be the column-vector composed of master integrals in a given sector, $F_1, \ldots, F_k$, with $k \geq 2$.

Evaluating multicomponent master integrals

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matrix difference equation

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To solve the corresponding homogeneous equation, we have to find k solutions.
Our tool: cut-integrals.

$$1/(E_j + i0)^{a_j} \to 1/(E_j + i0)^{a_j} - 1/(E_j - i0)^{a_j} \sim \delta^{(a_j - 1)}(E_j)$$

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Such replacement for $a_j \leq 0$ gives a zero result. Therefore $\Delta F(d)$ satisfies the homogeneous equation:

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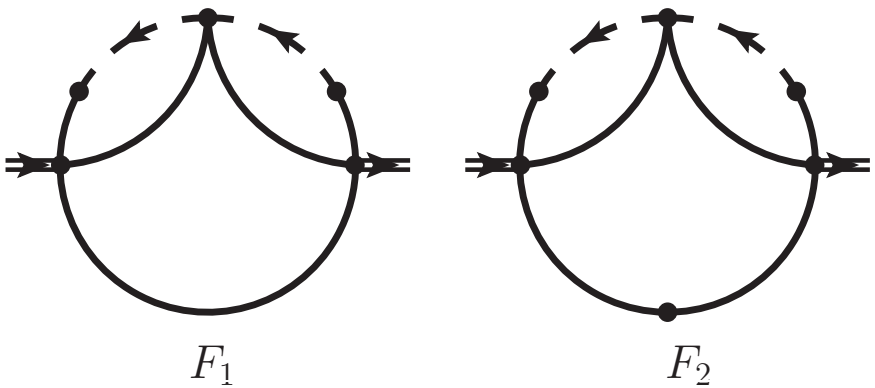
Solving this equation is equivalent to solving the second-order difference equation for

$$F(\nu + 2) + C_1(\nu)\, F(\nu + 1) + C_2(\nu)\, F(\nu) = 0\ ,$$

where C_1 and C_2 can be expressed in terms of $\mathbb{C}(\nu)$.

An example



where dashed lines denote static propagators $1/(v \cdot k + \mathrm{i}0)$ and $v \cdot q = 0$. One chooses $v = (1, \vec{0})$ and $q = (0, \vec{q})$.

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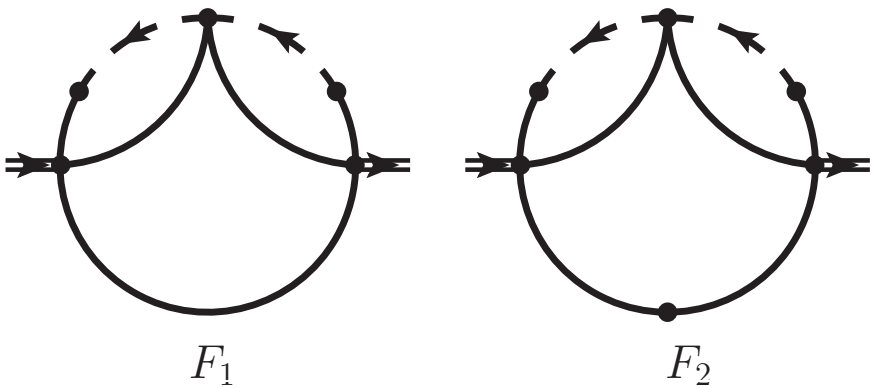

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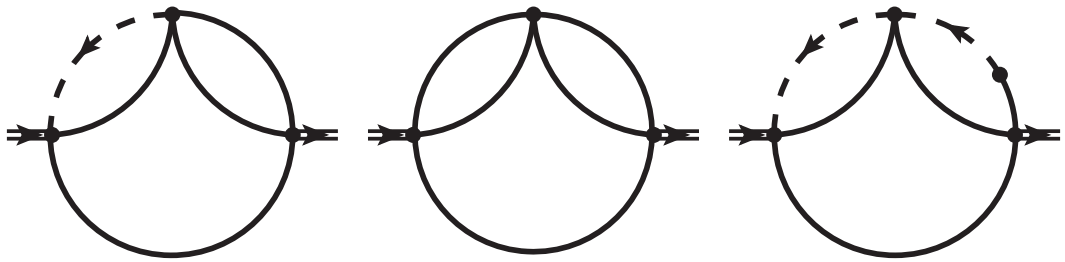
$$F_a(d) = \frac{(-1)^a}{\left(\mathrm{i}\pi^{d/2}\right)^3} \int\int\int \frac{\mathrm{d}^d k \; \mathrm{d}^d l \; \mathrm{d}^d r}{k^2 r^2 ((l+q)^2)^a (k-l)^2 (l-r)^2 (v \cdot k)(v \cdot r)}$$

where $a = 1$ and 2 and $+\mathrm{i}0$ is implied

The lower master integrals



P_{51} P_{53} P_{62}

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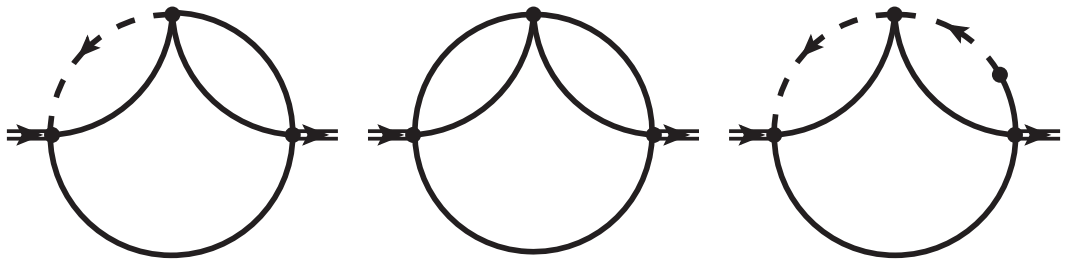

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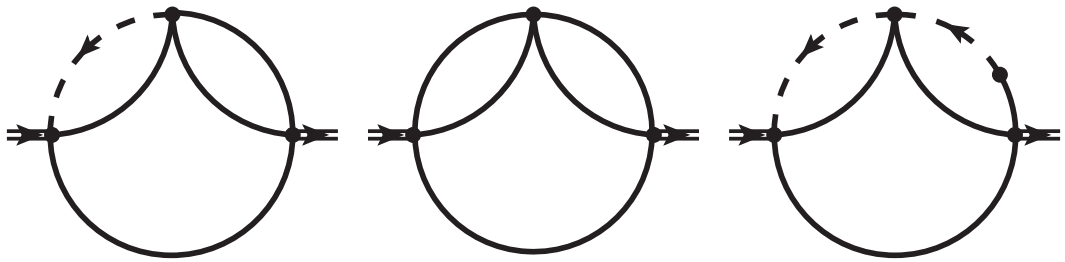

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Turn to the cut integral $\Delta\mathbf{F}$ where cutting the lower line is not necessary,
i.e. $1/(k^2+i0)\to\delta(k^2)$ and $1/(v\cdot k+i0)\to\delta(v\cdot k)=\delta(k_0)$
for all the propagators apart from $1/(l+q)^2$

Two identical one-loop subdiagrams consisting of one static and two usual propagators

$$J(l) = \int \frac{\mathrm{d}^d k}{\pi^{d/2}} \delta(k_0) \delta(k^2) \delta(l^2 - 2l \cdot k)$$

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A trick: instead of Minkowskian metrics let us use the metric signature $(1, 1, -1, -1, \ldots)$, so that
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Result

$$J(l) = 2^{2-d} \frac{\Omega(d-2)}{\pi^{d/2}} \frac{(-l^2)^{d-4}}{(\mathbf{l}^2)^{(d-3)/2}}$$

where $\mathbf{l}^2 = -l_1^2 + \vec{l}^2$, and $\Omega(d) = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit hypersphere in Euclidean d-dimensional space.

The final integral

$$\Delta F_1(\nu) = \int \frac{\mathrm{d}^d l}{\pi^{d/2}} \frac{J(l)^2}{(-(l+q)^2)}$$

Turn to Euclidean space and separate the two terms in the denominator of $1/(l_0^2 + (\mathbf{l}+\mathbf{q})^2)$ introducing a onefold MB representation.

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Result:

$$\Delta F_1(\nu) = \frac{2^{4-4\nu}\Gamma(6-3\nu)}{\Gamma(\nu-1)^2\,\Gamma\left(8-4\nu, 4\nu-\frac{13}{2}\right)} \frac{1}{2\pi\mathrm{i}} \int \mathrm{d}z \frac{\Gamma(-z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(z+5-2\nu)}$$
$$\times\Gamma\left(3\nu - \frac{11}{2} - z\right)\Gamma(z-4\nu+8)\,\Gamma(z+\nu-1)$$

There are two series of poles from the right of the integration contour and three series of poles from the left:

$$z_1 = n, \quad z_2 = 3\nu - \frac{11}{2} + n,$$

$$z_3 = -\frac{1}{2} - n, \quad z_4 = 4\nu - 8 - n, \quad z_5 = 1 - \nu - n,$$

with $n = 0, 1, \ldots$.

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Let us choose the contribution of the series of residues at z_1 and z_4

$$
\begin{aligned}
F^1_{1,h}(\nu) &= \frac{\sqrt{\pi}2^{4-4\nu}\Gamma(6-3\nu)\Gamma\left(3\nu-\frac{11}{2}\right)}{\Gamma(5-2\nu)\Gamma(\nu-1)\Gamma\left(4\nu-\frac{13}{2}\right)} \\
&\quad {}_3F_2\left(\left.\begin{array}{c} 8-4\nu, \frac{1}{2}, \nu-1 \\ 5-2\nu, \frac{13}{2}-3\nu \end{array}\right| 1\right), \\
F^2_{1,h}(\nu) &= \frac{32\Gamma(6-3\nu)\Gamma(5\nu-9)\Gamma\left(\frac{5}{2}-\nu\right)}{2^{4\nu}(8\nu-15)\Gamma(\nu-1)^2\Gamma(2\nu-3)} \\
&\quad \times {}_3F_2\left(\left.\begin{array}{c} 8-4\nu, \frac{5}{2}-\nu, 4-2\nu \\ 10-5\nu, \frac{17}{2}-4\nu \end{array}\right| 1\right)
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\end{aligned}
$$

The series converge at Re $\nu < 5/2$. One can represent them as ${}_4F_3$ convergent uniformly in ν.

Therefore, the matrix of fundamental solutions has the form $\mathbb{F}_h(\nu) = \begin{pmatrix} F^1_{1,h}(\nu) & F^2_{1,h}(\nu) \\ F^1_{2,h}(\nu) & F^2_{2,h}(\nu) \end{pmatrix}$, where $F^1_{2,h}(\nu)$ and $F^2_{2,h}(\nu)$ are obtained form the first equation of the homogeneous system of equations:

$$\begin{aligned} F^1_{2,h}(\nu) &= \frac{F^1_{1,h}(\nu+1) - C_{11}(\nu) F^1_{1,h}(\nu)}{C_{12}(\nu)}, \\ F^2_{2,h}(\nu) &= \frac{F^2_{1,h}(\nu+1) - C_{11}(\nu) F^2_{1,h}(\nu)}{C_{12}(\nu)} \end{aligned}$$

Construct the summing factor using the fundamental solution

[R. Lee & I.S. Terekhov'11]

$$\mathbb{S}(\nu) = \mathbb{W}(\nu)\, S(\nu) \begin{pmatrix} F^2_{2,h}(\nu) & -F^2_{1,h}(\nu) \\ -F^1_{2,h}(\nu) & F^1_{1,h}(\nu) \end{pmatrix} ,$$

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where

$$S\left(\nu\right) = \frac{2^{2\nu}(\nu-2)\Gamma(2\nu-3)^{2}\Gamma\left(4\nu-\frac{13}{2}\right)}{\Gamma\left(2\nu-\frac{7}{2}\right)^{2}\Gamma(2-\nu)^{2}\sin(\pi\nu)}$$

is a solution of the equation $S\left(\nu\right) = S\left(\nu+1\right)\det\mathbb{C}\left(\nu\right)$ and $\mathbb{W}\left(\nu\right)$ is an arbitrary periodic matrix.

We obtain

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where $\mathbf{W}(\nu)$ is an arbitrary periodic column-vector and

$$\begin{aligned} \sum_{+\infty} f(\nu) &= -\sum_{n=0}^{\infty} f(\nu+n) \ , \\ \sum_{-\infty} f(\nu) &= \sum_{n=1}^{\infty} f(\nu-n) \end{aligned}$$

We choose

$$\mathbb{W}(\nu) = \frac{(1+c)(1+2c)}{c^2}$$
$$\times \begin{pmatrix} 2^5(1-c)\left(1-2c-4c^2\right) & 2^5\frac{1+c}{2c^2-1}(1-2c)^2 \\ -\frac{c}{\sqrt{2}}\left(1-2c-4c^2\right) & \frac{c(1-2c)}{\sqrt{2}(2c^2-1)}\left(1-2c-4c^2\right) \end{pmatrix},$$

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where $c = \cos(2\pi\nu)$.
With this choice of the summing factor, $(\mathbb{S}\mathbf{F})$ is holomorphic in the stripe Re $\nu \in [2,3)$ and grows at $\nu \to \pm i\infty$ slower than $\exp(2\pi|\nu|)$.

Taking into account the singularities of $\Sigma_{+\infty}\,\mathbb{S}\,(\nu-1)\,\mathbf{R}\,(\nu)$, we obtain

$$\mathbf{W}\,(\nu) = \frac{4\pi^2}{\sin^2(\pi\nu)}\left(\pi - 2\arctan\left(4\sqrt{5}\right)\cos^2(\pi\nu)\right)\begin{pmatrix} -64 \\ \sqrt{2} \end{pmatrix}$$

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Evaluating with a high accuracy (e.g. 1000 digits) and using PSLQ $\rightarrow$

$$
\begin{aligned}
F_1(2-\epsilon) = {} & \frac{28\pi^4}{135\epsilon} + \frac{116\pi^2\zeta(3)}{9} + \pi^4\left(\frac{224}{135} - 4\ln(2)\right) + \frac{226\zeta(5)}{3} \\
& + \left(-192 s_6 + \frac{1808\zeta(5)}{3} - \frac{8\zeta(3)^2}{3} + \frac{928\pi^2\zeta(3)}{9} + 64\pi^2 \mathsf{Li}_4\left(\frac{1}{2}\right) + \frac{8}{3}\pi^2\ln^4(2)\right. \\
& \left. - \frac{20}{3}\pi^4\ln^2(2) - 32\pi^4\ln(2) - \frac{428\pi^6}{2835} + \frac{1792\pi^4}{135}\right)\epsilon \\
& + \left(-768\mathsf{Li}_4\left(\frac{1}{2}\right)\zeta(3) - 128\pi^2\mathsf{Li}_5\left(\frac{1}{2}\right) + 512\pi^2\mathsf{Li}_4\left(\frac{1}{2}\right) - 1536 s_6\right. \\
& + \frac{384}{7}s_6\ln(2) - \frac{384 s_{7a}}{7} - \frac{3072 s_{7b}}{7} + \frac{4960\zeta(7)}{21} + \frac{35519\pi^2\zeta(5)}{42} \\
& + \frac{14464\zeta(5)}{3} - \frac{64\zeta(3)^2}{3} - \frac{31457\pi^4\zeta(3)}{945} + \frac{7424\pi^2\zeta(3)}{9} - 32\zeta(3)\ln^4(2) \\
& + 372\zeta(5)\ln^2(2) + 32\pi^2\zeta(3)\ln^2(2) - \frac{480}{7}\zeta(3)^2\ln(2) - \frac{3424\pi^6}{2835} + \frac{14336\pi^4}{135} \\
& + \frac{16}{15}\pi^2\ln^5(2) + \frac{64}{3}\pi^2\ln^4(2) - \frac{40}{9}\pi^4\ln^3(2) - \frac{160}{3}\pi^4\ln^2(2) - \frac{3079}{315}\pi^6\ln(2) \\
& \left. - 256\pi^4\ln(2)\right)\epsilon^2 + O(\epsilon^3)\,,
\end{aligned}
$$

$$F_2\left(2-\epsilon\right) = -\frac{\pi^4}{\epsilon} - 93\zeta(5) - 14\pi^2\zeta(3) - 2\pi^4\ln(2) + \Big(-96 s_6 + 120\zeta(3)^2$$

$$+32\pi^2 \mathsf{Li}_4\left(\frac{1}{2}\right) + \frac{4}{3}\pi^2\ln^4(2) - \frac{10}{3}\pi^4\ln^2(2) - \frac{989\pi^6}{420}\Bigg)\epsilon$$

$$+\left(-384\mathsf{Li}_4\left(\frac{1}{2}\right)\zeta(3) - 64\pi^2\mathsf{Li}_5\left(\frac{1}{2}\right) + \frac{192}{7}s_6\ln(2) - \frac{192 s_{7a}}{7}\right.$$

$$-\frac{1536 s_{7b}}{7} - \frac{32666\zeta(7)}{7} - \frac{40585\pi^2\zeta(5)}{84} + \frac{35047\pi^4\zeta(3)}{630} - 16\zeta(3)\ln^4(2)$$

$$+186\zeta(5)\ln^2(2) + 16\pi^2\zeta(3)\ln^2(2) - \frac{240}{7}\zeta(3)^2\ln(2) + \frac{8}{15}\pi^2\ln^5(2)$$

$$\left.-\frac{20}{9}\pi^4\ln^3(2) - \frac{3079}{630}\pi^6\ln(2)\right)\epsilon^2 + O(\epsilon^3)\ ,$$

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- The method is under construction.

extra slides

The full color dependence of the 4-loop 4-gluon amplitude in N=4 SUSY YM in terms of 50 4-loop 4-point integrals.

[Bern, Carrasco, Johansson & Roiban'10]

The critical dimension at which the amplitude first diverge. For 4 loops, this is d=11/2.

The subleading-color parts of the divergence require the three-loop propagator non-planar master integral.

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We have

$$-6.198399226749495916820092547981936876347898798967915 2\ldots$$

Most complicated master integrals for the three-loop static quark potential

[A. and V. Smirnovs & M. Steinhauser'09]