Reduction of two loop Amplitudes @ the Integrand level

Ioannis Malamos (IFIC, Valencia)

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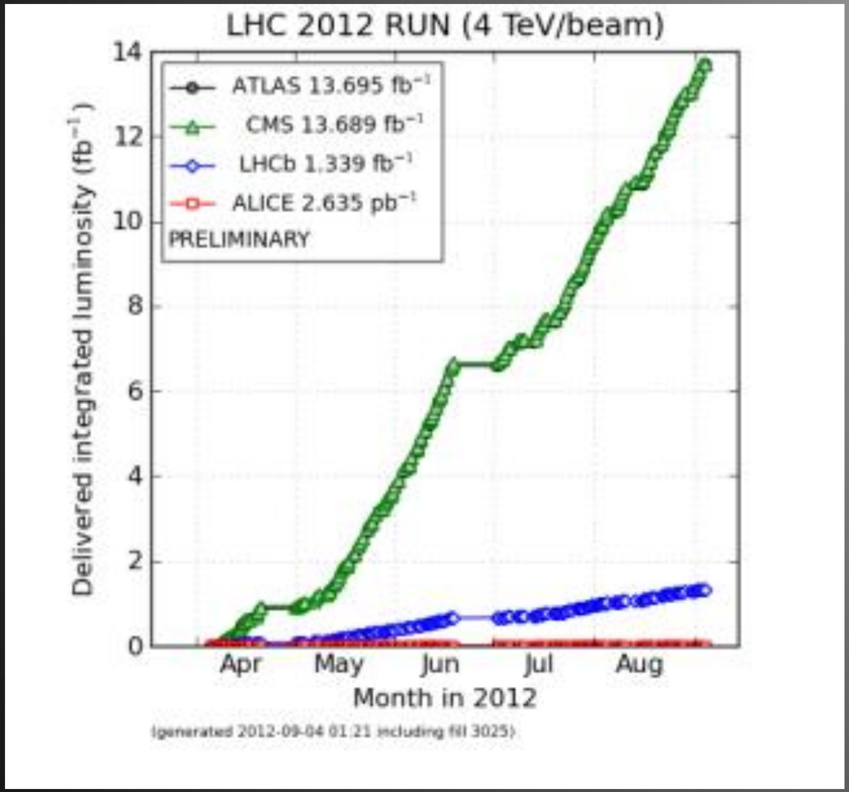
LHC performance calls for serious theoretical work



- Precision becomes important for most of the processes
- Fixed order calculations need to advance
- NLO complete and automated, NNLO in the making

On the virtual part of NLO calculations

- Loop diagrams are considered to be the bottleneck of beyond LO calculations
- Large number of Feynman diagrams– Complicated loop integrals
- Reduction techniques : Minimize the size and the difficulty of such calculations
- Working at the amplitude level, suitable for numerical approach

Historical Background

- D.B.Melrose, G.Källèn–J.Toll (1965)
- Passarino–Veltman
- Unitarity based methods (Bern, Dixon, Dunbar,Kosower– at the amplitude level)
- Generalised Unitarity (Britto, Cachazo, Feng)
- Reduction at the Integrand level (OPP)

OPP Method @ one loop

$$A \to \frac{N(q)}{\prod D_i}$$

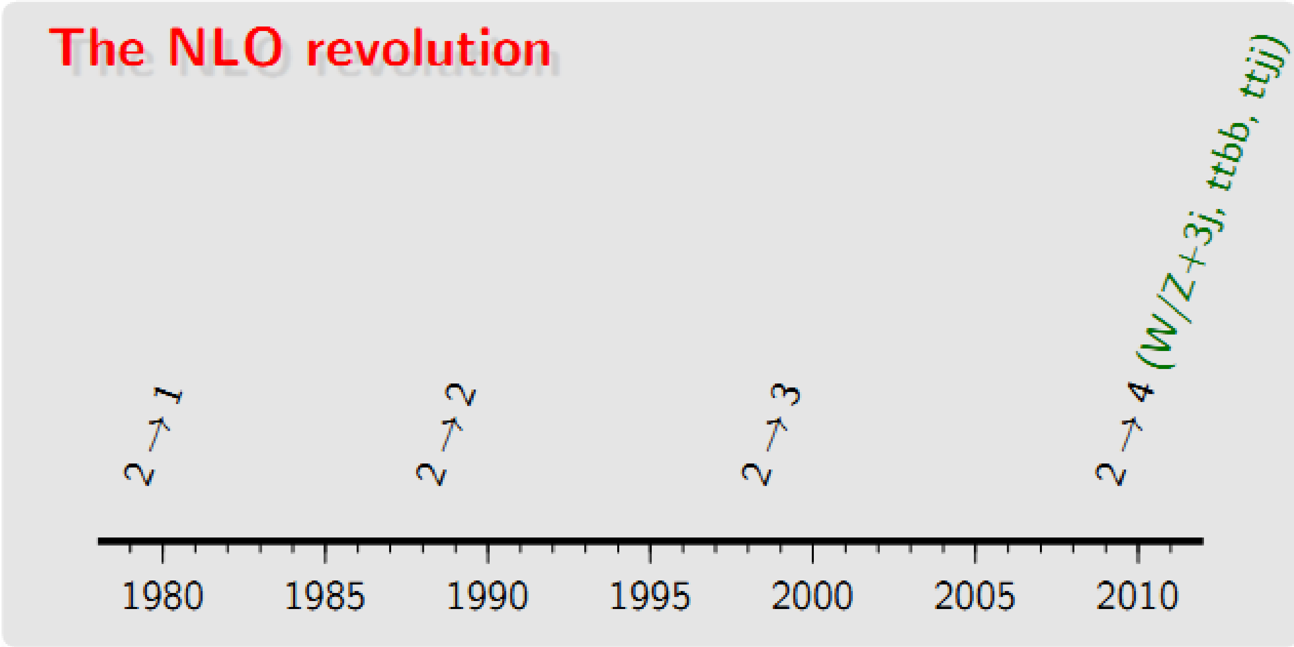
$$\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} [c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} [a(i_0) + \tilde{a}(q; i_0)] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}$$

Solving for known values of the loop momentum q

Completing the NLO

- Quadruple, triple, double and single cuts to obtain the coefficients
- Terms with a tilde vanish upon integration (spurious terms)
- Scalar Integrals
- Rational terms (working in d dimensions)
- Real part

- The NLO revolution



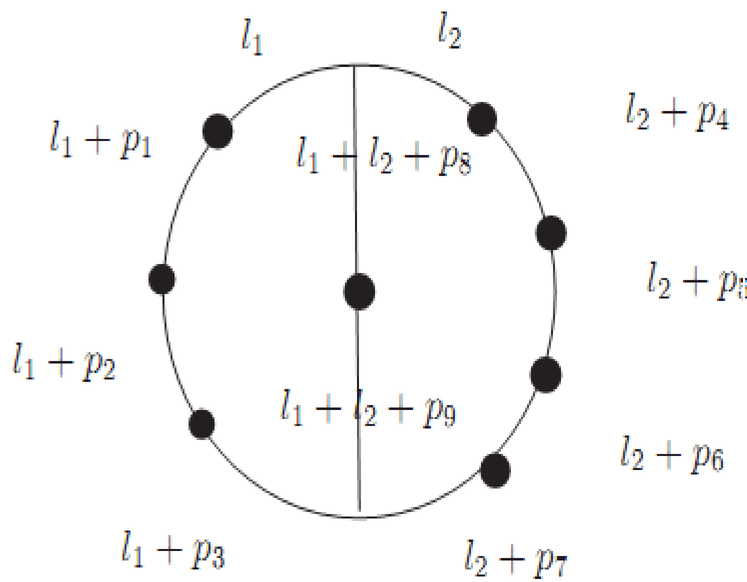
2009: NLO $W+3j$ [Rocket: Ellis, Melnikov & Zanderighi]	[unitarity]
2009: NLO $W+3j$ [BlackHat: Berger et al]	[unitarity]
2009: NLO $t\bar{t}b\bar{b}$ [Bredenstein et al]	[traditional]
2009: NLO $t\bar{t}b\bar{b}$ [HELAC-NLO: Bevilacqua et al]	[unitarity]
2009: NLO $q\bar{q} \to b\bar{b}b\bar{b}$ [Golem: Binoth et al]	[traditional]
2010: NLO $t\bar{t}jj$ [HELAC-NLO: Bevilacqua et al]	[unitarity]
2010: NLO $Z+3j$ [BlackHat: Berger et al]	[unitarity]

Integrand Reduction at two loops

- Ossola, Mastrolia (2011)
- Badger, Frellesvig, Zhang (2011)
- Zhang (2012)
- Mirabella, Ossola, Peraro, Mastrolia (2012)
- Kleiss, I.M. , Papadopoulos, Verheyen (2012)

- Generic two-loop graph: iGraph

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, arXiv:1206.4180 [hep-ph].



$$D(l_1 + p_i)\ ,\ D(l_2 + p_j)\ ,\ D(l_1 + l_2 + p_k)$$

Counting to "one"

- Consider scalar integrals without loss of generality
- Write the numerator (1) of these integrals in terms of Denominators times coefficients (polynomials in the loop momenta)
- Investigate when this systems has solutions
- What is the minimal number of Denominators/ rank of the coefficients ?

In other words solve the equation:

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_2 + p_j) = 1$$

Let us go a step back at one loop

$$1 = T_1(q)D_1 + T_2(q)D_2 + \cdots + T_n(q)D_n$$

Constant terms: $T_j(q) = x_j$

$$q^2 \sum_{j=1}^{n} x_j + 2q_\mu \sum_{j=1}^{n} x_j p_j{}^{\mu} + \sum_{j=1}^{n} x_j \mu_j = 1 \ .$$

$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j p_j{}^{\mu} = 0 \ , \sum_{j=1}^{n} x_j \mu_j = 1$$

- solution exists for $n = 6$ $d = 4$

Linear terms $T(q) = P_1(q)$, count tensor structures:

$$1 \ , \ q^\mu \ , \ q^\mu q^\nu \ , \ q^2 q^\mu \ .$$

There are, for $d = 4$, therefore $1+4+10+4 = 19$ independent tensor structures.
In d dimensions, tensor up to rank k, $N(d,k)$ number of independent tensor structures

$$N(d,k) = \left(\begin{array}{c} d-1+k \\ k \end{array} \right) + \sum_{p=0}^{k+1} \left(\begin{array}{c} d-1+p \\ p \end{array} \right) \ .$$

In the table below we give the results for various ranks and dimensionalities.

k	0	1	2	3	4
$d = 1$	3	4	5	6	7
2	4	8	13	19	26
3	5	13	26	45	71
4	6	19	45	90	161
5	7	26	71	161	322
6	8	34	105	266	588

Values of $N(d,k)$

The OPP-"miracle" is that the OPP equation works with only 10(6) different coefficients

$$1 = \sum_{i=1}^{5} D_i(q)(c_i^{(0)} + c_i^{(1)}\epsilon_i(q))$$

all $c_i^{(1)}$ being equal! rank deficient problems

Return to two loops

- Order of the iGraph= $n_1+n_2+n_3$
- Constraints : $n_{1,2,3} \leq 4$ (one loop constraint)
- $n_1+n_2+n_3 \leq 11$ ($=2d+3$), constant coefficients

Linear terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j)$$

$$T(d) = (4d^2 + 18d + 2)/2$$

n	$d = 6$	$d = 5$	$d = 4$	$d = 3$	$d = 2$	$d = 1$
3	39-0	33-0	27-0	21-0	15-0	9-0
4	52-0	44-0	36-0	28-0	20-0	12-2
5	65-1	55-1	45-1	35-1	25-1	15-5
6	78-3	66-3	54-3	42-3	30-3	
7	91-6	77-6	63-6	49-6	35-8	
8	104-10	88-10	72-10	56-10		
9	111-15	99-15	81-15	63-17		
10	130-21	110-21	90-21			
11	143-28	121-28	99-30			
12	156-36	132-36				
13	169-45	143-47				
14	182-55					
15	195-55					
$T(d)$	127	96	69	46	27	10

Quadratic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \sum_j c_{ij}(l_2 \cdot t_j) + \sum_{j \le k} d_{ijk}(l_1 \cdot t_j)(l_1 \cdot t_k) + \cdots$$

$$T(d) \quad = \quad 4d^3/3 + 10d^2 + 20d/3 - 2$$

n	$d = 4$	$d = 3$	$d = 2$
3	135-4	84-3	45-3
4	180-6	128-6	60-6
5	225-18	140-16	75-15
6	270-38	168-32	90-30
7	315-65	196-53	
8	360-98	224-80	
9	405-136	252-108	
10	450-180		
11	495-225		
$T(d)$	270	144	60

Cubic terms

$$x_i = a_i + \sum_j b_{ij}(l_1 \cdot t_j) + \cdots + \sum_{j \le k} g_{ijkl}(l_1 \cdot t_j)(l_1 \cdot t_k)(l_1 \cdot t_l) + \cdots$$

$$T(d) = 2d^4/3 + 22d^3/3 + 71d^2/6 + d/6 + 1$$

n	$d=6$	$d=5$	$d=4$	$d=3$
5				420/332
6				504/352
7			1155/803	588/360
8			1320/823	672/360
9		2574/1603	1485/831	
10		2860/1623	1650/831	
11	5005/2848	3146/1631		
12	5460/2868	3432/1631		
13	5915/2876			
14	6370/2876			
$T(d)$	2876	1631	831	360

ANSWER:

- Every two loop integral can be written in terms of integrals up to 2d Denominators
- In most cases cubic terms are needed (d=2 special case)
- The 2d basis Integrals are compatible with Unitarity (from the constraint)
- There exist l_1,l_2 such that 2d denominators vanish → no further reduction is possible this way (see also Nullstellensatz theorem – Mirabela, Ossola, Peraro, Mastrolia)

OPP @ two loops

$$1 = \sum D_i R_i + \sum D_i D_j R_{ij} + \sum D_i D_j D_k R_{ijk} + \cdots$$

- Reducible scalar products RSP give rise to terms with higher powers of D_i,

$$D \otimes RSP \to D \otimes D$$

- Parametrizing the "residue" functions with irreducible scalar products ISP

$$R = D \otimes ISP$$

- Solving the master equation

$$1 = 1$$

OPP @ two loops

- Classify all possible residues for every integral of the basis
- Use Unitarity cuts to extract the coefficients (at the maximal cuts the number of solutions matches the number of coefficients)
- Freedom in the choice of the ISP

Finding a minimal basis

- The unitarity basis described above is not a minimal one
- Reduction to true Master Integrals demands the use of IBP identities (Chetyrkin, Tkatchov)
- Removal of double poles
- Combine OPP with IBP's

What about NNLO?

- Integrand reduction for the virtual part (in progress)
- Rational terms
- Computation of Master Integrals (Significant progress)
- Virtual – Real
- Real– Real

Conclusions

- Integrand reductions boosted the NLO computations leading to an NLO revolution the last 5 years
- NNLO results also important for the LHC
- There is a Unitarity based basis for every two loop Integrand
- The Unitarity base is not necessarily the minimal basis–combine with IBP

Conclusions 2

- Significant progress to all pieces of the NNLO
- Extension to more than two loops in the integrand reduction part are obvious
- More results to come, the NNLO revolution has began!

Thank you, enjoy the meeting (and Ravello)