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New techniques in the computation of scattering amplitudes

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Scattering amplitudes in QFT

- LHC physics is dominated by QCD.
- State of the art:
  - ➡ Tree-level: essentially solved (except multi-leg amplitudes).

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  - ➡ One loop:
    - ✓ Integral basis (boxes, triangles, bubbles)
    - ✓ Essentially solved
  - ➡ Two loops:
    - ◉ Two-loop amplitudes in general unknown.
    - ◉ No two-loop integral basis known.

[See talks by Gluza and Malamos]

Multi-loop computations

- Why are multi-loop computations so difficult..?
- Integration-by-parts identities allow to reduce the problem to the computation of a minimal set of master integrals.
- Quantities are divergent:
  - ➡ UV & IR divergences.
- Two-loop integrals are generically polylogarithms of weight 4 in many external physical parameters.
  - ➡ multiple polylogarithms.
  - ➡ need to evaluate these functions numerically in a fast and efficient way, including all the branch cuts, etc.
- In other words, polylogarithms and their generalizations are everywhere!
  - ➡ Need to understand these functions!

The life-cycle of a loop computation

- The final goal is to obtain an expression of the loop integrals in terms of
  - ➡ Transcendental numbers: mutliple zeta values, log 2, *etc.*
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    - ★ Elliptic functions.
- In this talk: will concentrate exclusively on polylogarithms.

The life-cycle of a loop computation

- Recursive definition of multiple polylogarithms:

$$G(a_1,\dots,a_n;z)=\int_0^z \frac{\mathrm{d}t}{t-a_1}\,G(a_2,\dots,a_n;t)\quad\Bigg|\quad \mathrm{Li}_n(z)=\int_0^z\frac{\mathrm{d}t}{t}\,\mathrm{Li}_{n-1}(t)$$

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  - ➡ 2d harmonic polylogarithms: e.g., $a_i \in \{0, 1, a\}$
  - ➡ Cyclotomic harmonic polylogarithms: roots of unity.

The life-cycle of a loop computation

- Even if an amplitude is simple, it might be that our approach to the problem leads to a difficult answer.

- The polylogarithms satisfy various complicated functional equations.

  ➡ The simplicity of the answer might be hidden behind a swath of functional equations.

$$-\mathrm{Li}_2(z) - \ln z \ln(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

- In other words we need to ‘control’ the functional equations among polylogarithms.

Number theory and Loop integrals

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- No! Over the last 20 years polylogarithms were a very active field of research in pure mathematics.
- Mathematicians have discovered very far reaching algebraic structures underlying polylogarithms.
- In particular, mathematicians showed that multiple polylogarithms form a Hopf algebra. [Goncharov]

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- In particular, mathematicians showed that multiple polylogarithms form a Hopf algebra. [Goncharov]
- Consequence: All functional equations are pure combinatorics!
  ➡ You do not even need to know the integral in order to derive the relations among them!

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- Coalgebra: Vector space with an operation that allows one to break two elements apart (comultiplication).
- Hopf algebra: Vector space with both multiplication and comultiplication, i.e., one can 'fuse' and 'break apart' in a consistent manner.
- Goncharov showed that mutliple polylogarithms form a Hopf algebra.

$$\Delta(\ln z) = 1 \otimes \ln z + \ln z \otimes 1$$

$$\Delta(\mathrm{Li}_n(z)) = 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\ln^k z}{k!}$$

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- How can all this be useful to physicists..?

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- How can all this be useful to physicists..?
- Imagine a two-loop multi-scale integral that evaluates to 1000's of Li_4's.
  - ➡ Can the expression be simplified?

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$$\mathrm{Li}_1(z) = -\log(1-z) \qquad\qquad \log(a \cdot b) = \log a + \log b$$

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Symbol

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$$\mathrm{Li}_1(z) = -\log(1-z) \qquad \log(a\cdot b) = \log a + \log b$$

- At the end of this procedure, we have broken everything into little pieces (logarithms = symbol), for which all identities are known.
- We then need to reassemble the pieces to find the simplified expression (This is the most difficult step!)
- At each step information is lost, but in a controlled way:
  - ➡ Can be recovered by going back up one step at the time.

Hopf algebras and Loop integrals

- Understanding the mathematical structure underlying multiple polylogarithms opens new possibilities for the computation of Feynman integrals and scattering amplitudes.

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- Substantial simplifications of complicated expressions:
  - ➡ Six-point MHV amplitude in N=4 Super Yang-Mills.
  - ➡ Two-loop helicity amplitudes for H + 3g.

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- Understanding the mathematical structure underlying multiple polylogarithms opens new possibilities for the computation of Feynman integrals and scattering amplitudes.
- Substantial simplifications of complicated expressions:
  - ➡ Six-point MHV amplitude in N=4 Super Yang-Mills.
  - ➡ Two-loop helicity amplitudes for H + 3g.
- Determine space of functions a priori:
  - ➡ Generalized ladder integrals.
  - ➡ N=4 SYM six-point amplitude in the Regge limit.
  - ➡ Two-loop three mass triangle integrals.

6-point amplitude in N=4 SYM

- Evaluating the individual diagrams one arrives at a very complicated combination of multiple polylogarithms (17 pages),

$$
\begin{aligned}
&R^{(2)}_{6,WL}(u_1,u_2,u_3) = \qquad \text{(H.1)}\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\\
&\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\\
&\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\\
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&\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+
\end{aligned}
$$

[Del Duca, CD, Smirnov]

6-point amplitude in N=4 SYM

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$

[Goncharov, Spradlin, Vergu ,Volovich]

$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

$$x_i^\pm = u_i x^\pm,\quad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},\quad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3.$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(\ell_{4-m}(x^+)+\ell_{4-m}(x^-))$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n(1/x)\right) \qquad J = \sum_{i=1}^{3}(\ell_1(x_i^+) - \ell_1(x_i^-))$$

- This was the first time the new mathematical methods were applied.

Higgs + 3 gluons

- Gehrmann, Jaquier, Glover and Koukoutsakis have recently computed the two-loop helicity amplitudes for a Higgs boson + 3 gluons
  - ➡ in the decay region

$$H \to g^+ g^+ g^+ \qquad\qquad H \to g^+ g^+ g^-$$

  - ➡ and the scattering region

$$g^+ \, g^+ \to g^+ \, H \qquad g^+ \, g^+ \to g^- \, H \qquad g^+ \, g^- \to g^+ \, H$$

- Kinematics (in the decay region):

$$x_1 = \frac{s_{12}}{m_H^2}\,, \qquad x_2 = \frac{s_{23}}{m_H^2}\,, \qquad x_3 = \frac{s_{31}}{m_H^2}$$

$$0 < x_i < 1 \quad \text{and} \quad x_1 + x_2 + x_3 = 1$$

Higgs + 3 gluons

- The result was expressed in terms of complicated combinations of '2d harmonic polylogarithms'.
  - ➡ Symmetries completely lost (e.g. Bose symmetry).
  - ➡ Very long and complicated.
  - ➡ Numerical evaluation of complicated special functions.
  - ➡ Analytic continuation from decay to scattering region very complicated.

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  - ➡ Analytic continuation from decay to scattering region very complicated.
- Brandhuber, Gang and Travaglini observed that the symbol of the leading color weight 4 part (after subtracting the one-loop squared) is equal to the symbol of the form factor of 3 gluons in N=4 Super Yang-Mills.
  - ➡ A simpler representation of the Higgs amplitudes in terms of classical polylogarithms only should exist.

Higgs + 3 gluons

$$
\begin{aligned}
\overline{A}_{\alpha}^{(2)} &= \mathcal{R}_3^{(2)} - \frac{\pi^2}{6} A_{\alpha}^{(1)} - \frac{1}{4}\zeta_3 B_{\alpha}^{(1)} - \frac{\pi^4}{2880} \\
&\frac{11}{6}\Bigg\{ \Lambda_3\left(-\frac{x_1 x_3}{x_2}\right) + \Lambda_3\left(-\frac{x_2 x_3}{x_1}\right) + \Lambda_3\left(-\frac{x_1 x_2}{x_3}\right) - \sum_{i=1}^{3} \mathrm{Li}_3\left(1-\frac{1}{x_i}\right) \\
&- \Lambda_3\left(-\frac{x_1}{x_2}\right) - \Lambda_3\left(-\frac{x_2}{x_1}\right) - \Lambda_3\left(-\frac{x_1}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_1}\right) - \Lambda_3\left(-\frac{x_2}{x_3}\right) - \Lambda_3\left(-\frac{x_3}{x_2}\right) \\
&+ \frac{1}{2}\ln(x_1\, x_2\, x_3)\, A_{\alpha}^{(1)} + \frac{7}{2}\sum_{i=1}^{3} \left[\mathrm{Li}_2\left(1-x_i\right)\, \ln x_i\right] + \frac{3}{4}\ln x_1\, \ln x_2\, \ln x_3 + \frac{1}{6}\ln^3\left(x_1 x_2 x_3\right) \\
&- \frac{5}{16}\pi^2 \ln(x_1 x_2 x_3) - \frac{3}{8}\zeta_3 + i\pi\, A_{\alpha}^{(1)} + \frac{i\pi^3}{16} - \frac{1}{3}\sum_{i=1}^{3}\ln^3 x_i \Bigg\} \\
&+ \frac{1}{36}\sum_{i=1}^{3}\left[\frac{P_1(x_i, x_{i-1}, x_{i+1})}{x_{i-1}^2 x_{i+1}^2}\mathrm{Li}_2(1-x_i) + \frac{P_2(x_i, x_{i-1}, x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1} + \frac{121}{4}\ln^2 x_i\right] \\
&+ \frac{P_3(x_1, x_2, x_3)}{144 x_1^2 x_2^2 x_3^2}\pi^2 - \frac{121}{72} i\pi \ln(x_1 x_2 x_2) + \frac{11}{36} i\pi\,(x_1 x_2 + x_2 x_3 + x_3 x_1) + \frac{185}{24} i\pi \\
&+ \frac{1}{72}\sum_{i=1}^{3}\frac{P_4(x_i, x_{i-1}, x_{i+1})}{x_{i-1}x_{i+1}}\ln x_i - \frac{1}{72}\left(x_1 x_2 + x_3 x_2 + x_1 x_3\right)^2 + \frac{247}{108}\left(x_1 x_2 + x_3 x_2 + x_1 x_3\right) \\
&+ \frac{1321}{216},
\end{aligned}
$$

➡ Kummer' function

$$
\Lambda_n(z) = \int_0^z \mathrm{d}t\, \frac{\ln^{n-1}|t|}{1+t} = (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{k!}\ \ln^k |z|\, \mathrm{Li}_{n-k}(z)
$$

Higgs + 3 gluons

$$\overline{D}_\alpha^{(2)} = -\zeta_3 + \frac{i\pi}{4} - \frac{1}{6}\left(x_1x_2 + x_3x_2 + x_1x_3\right) + \frac{67}{48} + \frac{P_5(x_1,x_2,x_3)}{72x_1^2x_2^2x_3^2}\pi^2$$
$$+ \frac{1}{12}\sum_{i=1}^{3}\left[\frac{P_6(x_i,x_{i-1},x_{i+1})}{x_{i-1}^2x_{i+1}^2}\,\mathrm{Li}_2(1-x_i) + \frac{P_7(x_i,x_{i-1},x_{i+1})}{x_i^2}\ln x_{i-1}\ln x_{i+1}\right. \tag{7.19}$$
$$\left. + \frac{P_8(x_i,x_{i-1},x_{i+1})}{2x_{i-1}x_{i+1}}\ln x_i\right]$$

$$\overline{E}_\alpha^{(2)} = -\frac{i\pi^3}{48} - \frac{i\pi}{3}A_\alpha^{(1)} - \frac{1}{12}\ln\left(x_1x_2x_3\right)\left(\ln x_1\ \ln x_2 + \ln x_1\ \ln x_3 + \ln x_2\ \ln x_3\right)$$
$$+ \frac{P_{13}(x_1,x_2,x_3)}{432} + \frac{7}{12}\ln x_1\ \ln x_2\ \ln x_3 - \frac{5}{48}\pi^2\ln\left(x_1x_2x_3\right) - \frac{29}{24}\zeta_3$$
$$+ \frac{11}{18}i\pi\ \ln(x_1x_2x_3) + \frac{P_{11}(x_1,x_2,x_3)}{288x_1^2x_2^2x_3^2}\pi^2 + \sum_{i=1}^{3}\left[\mathrm{Li}_3(x_i) - \frac{1}{3}\mathrm{Li}_3(1-x_i)\right.$$
$$+ \frac{1}{6}\mathrm{Li}_2(1-x_i)\ \ln x_i + \frac{1}{2}\ln(1-x_i)\ \ln^2 x_i + \frac{1}{6}\ln(x_1x_2x_3)\,\mathrm{Li}_2(1-x_i) \tag{7.20}$$
$$+ \frac{P_9(x_i,x_{i-1},x_{i+1})}{36x_{i-1}^2x_{i+1}^2}\mathrm{Li}_2(1-x_i) + \frac{P_{10}(x_i,x_{i-1},x_{i+1})}{36x_i^2}\ln x_{i-1}\ \ln x_{i+1}$$
$$\left. + \frac{11}{36}\ln^2 x_i + \frac{P_{12}(x_i,x_{i-1},x_{i+1})}{216x_{i-1}x_{i+1}}\ln x_i\right] - \frac{13}{36}i\pi\left(x_1x_2 + x_3x_2 + x_1x_3\right) - \frac{71}{18}i\pi\,,$$

Higgs + 3 gluons

$$
\begin{aligned}
\overline{F}_{\alpha}^{(2)} &= -\frac{i\pi}{18}\ln(x_1x_2x_3) - \frac{11}{144}\pi^2 + \frac{1}{36}\sum_{i=1}^{3}\ln^2 x_i - \frac{5}{54}\ln(x_1x_2x_3) + \frac{5i\pi}{18} \\
&+ \frac{i\pi}{18}(x_1x_2 + x_2x_3 + x_3x_1) + \frac{5}{54}(x_1x_2 + x_3x_2 + x_1x_3) \\
&- \frac{1}{72}(x_1x_2 + x_3x_2 + x_1x_3)^2 - \frac{x_1x_2x_3}{18}\sum_{i=1}^{3}\frac{\ln x_i}{x_i}\,,
\end{aligned}
$$

- Originally, the expressions filled up more than 6 pages!
- Bose symmetry is now completely manifest.
- Only simple functions (classical polylogarithms) with simple arguments.

  ➡ easy numerical evaluation.
- Similar results can be obtained for $H \to g^+g^+g^-$.

Single-valued polylogarithms

- In some cases it is possible to determine the space of functions a priori from general considerations.

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- In particular, in the case of
  - ➡ generalized ladder integrals,
  - ➡ the six-point N=4 SYM amplitude in the Regge limit,
  - ➡ three-mass triangle integrals,

  one can show on general grounds that the space of functions is the (almost) the same, and corresponds to polylogarithms in one complex variable z without any branch cuts in the complex z plane (single-valued).

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  one can show on general grounds that the space of functions is the (almost) the same, and corresponds to polylogarithms in one complex variable z without any branch cuts in the complex z plane (single-valued).
- These functions can be classified [Brown], and it turns out that at a given loop order the amplitude can only be a linear combination of very few functions.

Three-mass triangle integrals

- Consider a a three-point function without internal masses.
- The kinematics is described by the two variables

$$z\,\bar z = u = \frac{p_1^2}{p_3^2} \qquad\qquad (1-z)\,(1-\bar z) = v = \frac{p_2^2}{p_3^2}$$

- It can be shown recursively, by combining the Hopf algebra with Cutkowski's rules, that the such three-point functions must be single-valued functions in the complex variable z.

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- Adding some symemtry considerations, there are only very functions of this type up to weight 4 (= 2 loops)!

weight	+	-
1	$\ln\|z\|^2, \ln\|1-z\|^2$	$-$
2	ζ_2	$\mathcal{P}_2(z)$
3	$\zeta_3, \mathcal{P}_3(z), \mathcal{P}_3(1-z)$	$\mathcal{Q}_3(z)$
4	$\mathcal{Q}_4^+(z), \mathcal{Q}_4^+(1-z)$	$\mathcal{P}_4(z), \mathcal{P}_4(1-z), \mathcal{P}_4(1-1/z), \mathcal{Q}_4^-(z)$

Three-mass triangle integrals

- As an example, the one-loop integral reads in this basis:

$$T_1(p_1^2,p_2^2,p_3^2;\epsilon) = -\,2c_\Gamma\,\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)^2}\,(-p_3^2)^{-1-\epsilon}\,\frac{u^{-\epsilon}\,v^{-\epsilon}}{z-\bar{z}}\,\Bigg\{\mathcal{P}_2(z)+2\epsilon\mathcal{Q}_3(z)$$
$$+\,\epsilon^2\left[\left(\frac{1}{6}\ln u\ln v-\zeta_2\right)\mathcal{P}_2(z)+2\,\mathcal{Q}_4^-(z)\right]+\mathcal{O}(\epsilon^3)\Bigg\}\,.$$

- Similar results at two-loops. [Chavez, CD]

Three-mass triangle integrals

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$$+\,\epsilon^2\left[\left(\frac{1}{6}\ln u\ln v-\zeta_2\right)\mathcal{P}_2(z)+2\,\mathcal{Q}_4^-(z)\right]+\mathcal{O}(\epsilon^3)\Big\}\,.$$

- Similar results at two-loops. [Chavez, CD]
- Note that these integrals were known in principle:
  - ➡ by Davydychev and Ussyukina: in terms of classical polylogarithms, but not expanded high enough in epsilon.
  - ➡ by Birthwright, Glover and Marquard: in terms of complicated iterated integrals involving square roots.
- In our representation: very compact, all symmetries manifest, analytic continuation almost trivial.

More single-valued functions

- Certain generalized multi-loop dual conformal ladder integrals can be expressed in terms of the same type of functions:
  - ➡ write down an ansatz in terms of these functions, and inject it into some differential equation.
  - ➡ generates an infinite number of analytic results for multi-loop 4-point integrals. [Drummond]

More single-valued functions

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  - ➡ write down an ansatz in terms of these functions, and inject it into some differential equation.
  - ➡ generates an infinite number of analytic results for multi-loop 4-point integrals. [Drummond]
- Similar approach can be used for the six-point amplitude in N=4 SYM in the Regge limit:
  - ➡ write down an ansatz in terms of these functions, and match the Taylor expansion of the ansatz to some Mellin-Barnes integral.
  - ➡ generates explicit results for the 6-point amplitude in N=4 SYM up to ten loops in this limit! [Dixon, CD, Pennington]

Conclusion & open questions

- Understanding the mathematics underlying the functions that enter Feynman integrals opens new ways to deal with loop amplitudes:
  - ➡ Simplify complicated expressions.
  - ➡ Determine space of functions a priori and inject into differential equations or match asymptotic expansions.

Conclusion & open questions

- Understanding the mathematics underlying the functions that enter Feynman integrals opens new ways to deal with loop amplitudes:
  - ➡ Simplify complicated expressions.
  - ➡ Determine space of functions a priori and inject into differential equations or match asymptotic expansions.

- Open questions:
  - ➡ So far arguments of polylogarithms need to be rational functions.
  - ➡ The case of elliptic functions is not covered.
  - ➡ Can we define a coproduct directly on Feynman integrals that matches the coproduct on multiple polylogarithms?

Example: inversion relations

- Indeed, the Hopf algebra fixes the inversion relations recursively.

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- Weight 1: trivial

$$\mathrm{Li}_1\left(\frac{1}{x}\right) = -\ln\left(1 - \frac{1}{x}\right) = -\ln(1-x) + \ln(-x) = -\ln(1-x) + \ln x - i\pi$$

with $x = x + i\,\varepsilon\,.$

Example: inversion relations

- Weight 2:

$$\begin{aligned}\Delta_{1,1}\left[\mathrm{Li}_2\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\\ &= \ln(1-x)\otimes\ln x-\ln x\otimes\ln x+i\pi\otimes\ln x\\ &= \Delta_{1,1}\Big[-\mathrm{Li}_2(x)-\frac{1}{2}\ln^2 x+i\pi\ln x\Big]\,.\end{aligned}$$

Example: inversion relations

- Weight 2:

$$\Delta_{1,1}\left[\mathrm{Li}_2\left(\frac{1}{x}\right)\right] = -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)$$
$$= \ln(1-x)\otimes\ln x - \ln x\otimes\ln x + i\pi\otimes\ln x$$
$$= \Delta_{1,1}\left[-\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x\right].$$

- This fixes the inversion relation, up to some zeta value.
  - ➡ At each step we loose a zeta value, they are indecomposable ('primitive').

$$\mathrm{Li}_2\left(\frac{1}{x}\right) = -\mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x + i\pi\ln x + c\,\pi^2$$

and c = 1/3 from x=1.

Example: inversion relations

- Weight 3:

$$\begin{aligned}\Delta_{1,1,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right)\right] &= -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\\ &= -\ln(1-x)\otimes\ln x\otimes\ln x+\ln x\otimes\ln x\otimes\ln x-i\pi\otimes\ln x\otimes\ln x\\ &= \Delta_{1,1,1}\left[\mathrm{Li}_3(x)+\frac{1}{6}\ln^3 x-\frac{i\pi}{2}\ln^2 x\right].\end{aligned}$$

Example: inversion relations

- Weight 3:

$$\Delta_{1,1,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right)\right] = -\ln\left(1-\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)\otimes\ln\left(\frac{1}{x}\right)$$

$$= -\ln(1-x)\otimes\ln x\otimes\ln x + \ln x\otimes\ln x\otimes\ln x - i\pi\otimes\ln x\otimes\ln x$$

$$= \Delta_{1,1,1}\left[\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right].$$

- At this stage however we have lost everything proportional to zeta values.

  ➡ Go one step up!

$$\Delta_{2,1}\left[\mathrm{Li}_3\left(\frac{1}{x}\right) - \left(\mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x\right)\right]$$

$$= \left[-\mathrm{Li}_2\left(\frac{1}{x}\right) - \mathrm{Li}_2(x) - \frac{1}{2}\ln^2 x - i\pi\ln x\right]\otimes\ln x$$

$$= -\frac{1}{3}\pi^2\otimes\ln x \;= \Delta_{2,1}\left(-\frac{\pi^2}{3}\ln x\right)$$

Example: inversion relations

- Finally:

  and $\alpha = \beta = 0$ from x=1.

- We could now go on like this and derive the inversion relations for arbitrary weight.

Example: inversion relations

- Finally:

$$\mathrm{Li}_3\left(\frac{1}{x}\right) = \mathrm{Li}_3(x) + \frac{1}{6}\ln^3 x - \frac{i\pi}{2}\ln^2 x - \frac{\pi^2}{3}\ln x + \alpha\zeta_3 + \beta\, i\pi^3$$

and $\alpha = \beta = 0$ from x=1.

- We could now go on like this and derive the inversion relations for arbitrary weight.

➡ No painful manipulation of the integral representation at any step!

Example: inversion relations

$$G(-z,-z,1-z,1-z;y) = \mathrm{Li}_3(1-x)\log(1-z) + \mathrm{Li}_3(1-z)\log(1-x) + \mathrm{Li}_4\left(1-\frac{1}{x}\right) + \mathrm{Li}_4(1-x)$$
$$-\mathrm{Li}_4(x) - \mathrm{Li}_3(1-x)\log(x) + \mathrm{Li}_4\left(1-\frac{1}{z}\right) + \mathrm{Li}_4(1-z) - \mathrm{Li}_4(z) - \mathrm{Li}_3(1-z)\log(z) + \frac{1}{4}\log^2(1-x)$$
$$\log^2(1-z) + \pi^2\left(-\frac{1}{6}\log(1-x)\log(1-z) + \frac{\log^2(x)}{12} + \frac{\log^2(z)}{12}\right) + \zeta(3)\log(x) - \zeta(3)\log(1-x) +$$
$$\frac{\log^4(x)}{24} - \frac{1}{6}\log(1-x)\log^3(x) - \zeta(3)\log(1-z) + \zeta(3)\log(z) + \frac{\log^4(z)}{24} - \frac{1}{6}\log(1-z)\log^3(z) + \frac{7\pi^4}{360}\,,$$

with $x+y+z=1$, $0 < x,y,z < 1$.