Multi-particle interactions within the BFKL approximation

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1 Regge ansatz for amplitudes

Regge kinematics

$$s = 4E^2 >> -t = \vec{q}^2$$

t-channel partial wave expansion

$$A^p(s,t) = s \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left((-s)^\omega - p s^\omega\right) f^p_\omega(t)\,,\ \sigma_t = \frac{\Im A^p(s,0)}{s}$$

Regge pole hypothesis and the Pomeron trajectory

$$f^p_\omega(t) = \frac{\gamma^2(t)}{\omega - \omega_p(t)}\,,\ \omega_P(t) \approx \Delta - \alpha' \vec{q}^2\,,\ \Delta \approx 0.1$$

Pomeron contribution

$$A^+_P(s,t) \approx \xi_P\, s^{1+\omega_P(t)}\, \gamma^2(t)\,,\ \xi_P(t) = e^{-i\pi\omega_p(t)} - p \approx i\pi\Delta$$

2 Mandelstam cuts

Mandelstam cut contribution

$$A^p_{Mand}(s,t) = \xi_p \, s \int \frac{d^2k}{(2\pi)^2} \Phi^2(k, q-k) s^{\omega_{p_1}(-k^2)} s^{\omega_{p_2}(-(q-k)^2)}$$

Gribov signature conservation rule

$$p = p_1 \, p_2$$

Impact factor and the non-planarity

$$\Phi(k, q-k) = \int_L ds_1 f(s_1, k^2, (q-k)^2, q^2)$$

Mandelstum diagram and two parton (non-perturbative) GPD

$$\Phi(\Delta, -\Delta) \sim \int dx_1 dx_2 \, {}_2D(x_1, x_2, Q_1^2, Q_2^2, \vec{\Delta}) \,, \;\; Q_1^2 \sim Q_2^2 \sim \Lambda^2_{QCD}$$

3 Gribov Pomeron calculus

Multi-particle unitarity conditions in t-channel

$$\Im_t f_\omega(t) \sim \sum_n \int d\Omega_n |f_\omega^{(n)}|^2$$

Separation of particle clusters in their rapidities

$$0 < y_1 < y_2 < ... < y_k < \ln s \,,\ 1 << y_k - y_{k-1} << \ln s$$

Non-relativistic reggeon propagators

$$G_0 = \frac{1}{E + \Delta - \frac{k^2}{2m}} \,,\ E = -\omega \,,\ \alpha' = \frac{1}{2m}$$

Gribov effective Pomeron action

$$S = \int dy\, d^2\rho \left(\phi^* (\partial_y - \Delta)\phi + \frac{1}{2m} \, |\partial_\mu \phi|^2) + \lambda \phi^* \phi^2 + ... \right)$$

4 Gluon reggeization in QCD

QCD Born amplitude

$$M_{AB}^{A'B'}(s,t)|_{Born} = g\,T^{c}_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{2s}{t}\,g\,T^{c}_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Leading logarithmic approximation

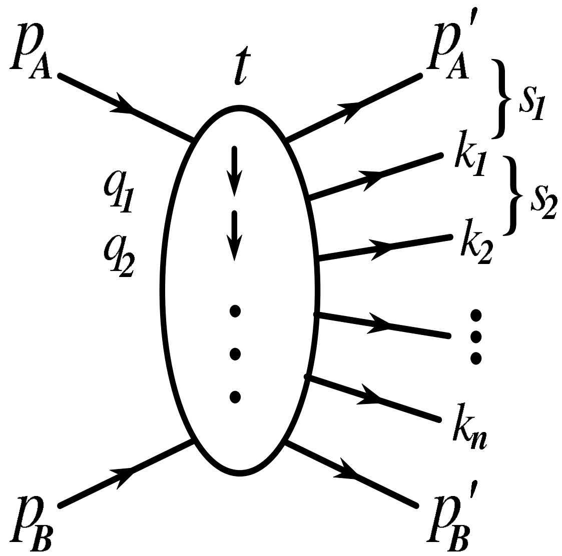
$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\,s^{\omega(t)},$$

$$\alpha_s \ln s \sim 1\,,\;\; \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$

5 Amplitudes in multi-Regge kinematics



$$M_{2\to 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... gT^{d_n}_{c_{n+1}c_n}\, C(q_{n+1},q_n)\, \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2}\,,$$

$$\omega_r = -\frac{\alpha_s N_c}{2\pi}\left(\ln\frac{|q_r^2|}{\mu^2} - \frac{1}{\epsilon}\right)\,,\ C(q_2,q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 2+n}|^2$$

6 Analyticity, unitarity and bootstrap

Steinmann relations for overlapping channels

$$\Delta_{s_r}\Delta_{s_{r+1}}M_{2\to 2+n}=0$$

Dispersion representation for $M_{2\to 3}$ in the Regge ansatz

$$M_{2\to 3}=c_1(-s)^{j(t_2)}(-s_1)^{j(t_1)-j(t_2)}+c_2(-s)^{j(t_1)}(-s_2)^{j(t_2)-j(t_1)}$$

Dispersion representation for $M_{2\to 4}$ in the Regge ansatz

$$M_{2\to 4}=d_1(-s)^{j_3}(-s_{012})^{j_2-j_3}(-s_1)^{j_1-j_2}+d_2(-s)^{j_1}(-s_{123})^{j_2-j_1}(-s_3)^{j_3-j_2}$$

$$+d_3(-s)^{j_3}(-s_{012})^{j_1-j_3}(-s_2)^{j_2-j_1}+d_4(-s)^{j_1}(-s_{123})^{j_3-j_1}(-s_2)^{j_2-j_3}$$

$$+d_5(-s)^{j_2}(-s_1)^{j_1-j_2}(-s_3)^{j_3-j_2}\,,\;\; j_r=j(t_r)$$

Bootstrap relation in LLA (BFKL (1975-1978))

$$\pi\omega(t_1)\,M_{2\to 2+n}=\sum_r \Im_{s_{0r}}M_{2\to 2+n}=\sum_t M_{2\to 2+t}M_{2+t\to 2+n}$$

7 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\;\sigma_t \sim s^{\Delta}\,,\;\;\Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

BFKL Hamiltonian

$$H_{12} = \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 + \ln\,|p_1 p_2|^2 - 4\psi(1)\,,$$

$$\rho_{12} = \rho_1 - \rho_2\,,\;\;\rho_r = x_r + iy_r\,,\;\;\Delta = 4\alpha N_c\,\ln 2\,/\pi$$

Möbius invariance and eigenvalues (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,\;\;m = \gamma + n/2\,,\;\;\widetilde{m} = \gamma - n/2\,,\;\;\gamma = 1/2 + i\nu\,,$$

$$E = \psi(m) + \psi(1-m) + \psi(\widetilde{m}) + \psi(1-\widetilde{m}) - 4\psi(1)$$

8 Non-linear screening effects

Balitskii-Kovchegov equation

$$\frac{dN(\vec{\rho}_1,\vec{\rho}_2)}{d\,y} =$$

$$\frac{\alpha_s N_c}{\pi^2}\int \frac{d^2\rho_0\,|\rho_{12}|^2}{|\rho_{10}|^2|\rho_{20}|^2}\left(N(\vec{\rho}_1,\vec{\rho}_0)+N(\vec{\rho}_2,\vec{\rho}_0)-N(\vec{\rho}_1,\vec{\rho}_2)-N(\vec{\rho}_1,\vec{\rho}_0)N(\vec{\rho}_2,\vec{\rho}_0)\right)$$

Triple Pomeron vertex and enhanced two parton GPD

$$\Phi(x,\vec{\rho}_1,\vec{\rho}_2;\vec{\rho}_0)\sim\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}d\nu\,\frac{|\rho_{12}|^2x^{-\omega(n,\nu)}}{|\rho_{10}|^2|\rho_{20}|^2}\left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^{\frac{1+n}{2}+i\nu}\left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}}\right)^{\frac{1-n}{2}+i\nu}$$

9 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = H\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n)\,,\;\; H = \sum_{k<l} \frac{\vec{T}_k \vec{T}_l}{-N_c}\,H_{kl}\,,\;\; \Delta = -\frac{\alpha_s N_c}{2\pi}\,E_0$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}(h+h^*)\,,\;\; h = \sum_{k=1}^{n}(\ln p_k p_{k+1} + \frac{1}{p_k}(\ln \rho_{k,k+1})\,p_k + \frac{1}{p_{k+1}}(\ln \rho_{k,k+1})\,p_{k+1} - 2\psi(1))$$

Holomorphic factorization of wave functions

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1, ..., \rho_n)\,\Psi_s(\rho_1^*, ..., \rho_n^*)$$

Representations of the Möbius group

$$M_a^2 \Psi_r = m(m-1)\Psi_r\,,\;\; M_a^{*2}\Psi_s = \widetilde{m}(\widetilde{m}-1)\Psi_s$$

10 Integrability at $N_c \to \infty$

Monodromy and transfer matrices (L. (1993))

$$t(u) = L_1 L_2 ... L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \; T(u) = A(u) + D(u)\,,$$

$$L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}, \; [T(u), h] = 0$$

Yang-Baxter equation (L. (1993))

$$t^{s_1}_{r'_1}(u)\, t^{s_2}_{r'_2}(v)\, l^{r'_1 r'_2}_{r_1 r_2}(v-u) = l^{s_1 s_2}_{s'_1 s'_2}(v-u)\, t^{s'_2}_{r_2}(v)\, t^{s'_1}_{r_1}(u)\,, \; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

Heisenberg spin model and duality symmetry (L. (1999))

$$p_r \to \rho_{r+1,r} \to p_{r+1}$$

11 BLV solution for Odderon

Eigenfunction of T for $\mu = 0\,,\ n = 2k+1$

$$\varphi^{(0)}_{m,\widetilde{m}} = 1 + (-x)^m(-x^*)^{\widetilde{m}} + (x-1)^m(x^*-1)^{\widetilde{m}}\,,\ \ \|\varphi\|_2^2 < \infty$$

BLV solution with $\mu = 0\,,\ n = 2k+1$

$$f_{m,\widetilde{m}} = \sum_{r=1}^{3} \delta^2(\rho_{r,r+1})|\rho_{r+1,r+2}|^2 f_{m,\widetilde{m}}(\vec{\rho}_{r+1}, \vec{\rho}_{r+2}; \vec{\rho}_0)$$

BLV solution and the zero mode of A ($\lambda = 0$)

$$\phi^{BLV+}_{1-m,1-\widetilde{m}} = \partial^{2-m}\partial^{*\,2-\widetilde{m}}\phi^{(0)-}_{1-m,1-\widetilde{m}} \sim \delta^2(x) + \delta^2(1-x)\,,$$

Recurrent relations

$$\phi^{\pm r}_{m.\widetilde{m}} = \mu \int \frac{d^2x'\,(x-x')^m(x^*-x'^*)^{\widetilde{m}}}{\pi}\,\phi^{\mp r-1}_{1-m.1-\widetilde{m}}\,,$$

12 μ^2-correction to the BLV Odderon

The first correction to the BLV solution

$$|x|^{-2}|1-x|^{-2}\,\mu^{-2}\Delta\phi^{BLV}(x) = -x^{*\,\widetilde{m}}\phi_{c1}(x) - x^{m}\phi_{c1}(x^*) - \phi_{c2}(x) - \phi_{c2}(x^*)$$

$$+(1-x^*)^{\widetilde{m}}\phi_{c1}(1-x) + (1-x)^{m}\phi_{c1}(1-x^*) + \phi_{c2}(1-x) + \phi_{c2}(1-x^*)$$

$$+(x-1)^{m}x^{\widetilde{m}}\phi_{c1}\left(-\frac{x}{1-x}\right) + (x^*-1)^{\widetilde{m}}x^{m}\phi_{c1}\left(-\frac{x^*}{1-x^*}\right)$$

$$+(x-1)^{m}(x^*-1)^{\widetilde{m}}\left(\phi_{c2}\left(-\frac{x}{1-x}\right) + \phi_{c2}\left(-\frac{x^*}{1-x^*}\right)\right)$$

Solutions of holomorphic duality equation

$$x(1-x)\partial^{1+m}\phi_{c_i}(x) = (1-x)^m\,,\;\; \phi_{c_1}(x) = x^m(R_m(x) + \ln x - \frac{1}{m})\,,$$

$$\phi_{c_2}(x) = R_{-m}(x)\,,\;\; R_m(x) = \sum_{k=1}^{\infty}\frac{m}{k(k+m)} = x\int_0^1 dt\,\frac{1-t^m}{1-xt}$$

13 Effective action approach

Locality in the rapidity space

$$y=\frac{1}{2}\ln\frac{\epsilon_k+|k|}{\epsilon_k-|k|}\,,\;\;|y-y_0|<\eta\,,\;\;\eta<<\ln\,s$$

Gluon and Reggeized gluon fields

$$v_\mu(x)=-iT^a v_\mu^a(x)\,,\;\;A_\pm(x)=-iT^a A_\pm^a(x)\,,\,\delta A_\pm(x)=0$$

Effective action for reggeized gluons (L., 1995)

$$S=\int d^4x\left(L_{QCD}+Tr(V_+\partial_\mu^2 A_- + V_-\partial_\mu^2 A_+)\right)\;,$$

$$V_+=-\frac{1}{g}\partial_+\,P\exp\left(-\frac{g}{2}\int_{-\infty}^{x^+}v_+(x')d(x')^+\right)=v_+-gv_+\frac{1}{\partial_+}v_+ + ...$$

14 Pomeron (and Odderon) in NLO

BFKL kernel in two loops (F., L. and C.,C. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N=4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M)+\rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+M}\left(\Psi'(k+1)\; - \;\frac{\Psi(k+1)-\Psi(1)}{k+M}\right)$$

15 Pomeron and reggeized graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2\,,\ \gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the energy-momentum conservation

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)$$

AdS/CFT relation for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ t = E^2/R^2\,,\ \alpha' = \frac{R^2}{2}\,\Delta$$

Large coupling asymptotics for Δ (KLOV, BPST)

$$\gamma = -\sqrt{2\pi(j-2)}\,\hat{a}^{1/4}\,,\ j = 2 - \Delta\,,\ \Delta = \frac{1}{2\pi}\,\hat{a}^{-1/2}$$

16 Effective action for gravity

Metric tensor and reggeized graviton fields

$$d^2 S = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu\, dx^\nu\;,\; \delta A^{++}(x) = \delta A^{--}(x) = 0$$

Effective action for the high energy gravity (L. 2011)

$$S = -\frac{1}{2\kappa}\int d^4x \left(\sqrt{-g}\, R + \frac{1}{2}\left(\partial_+ j^- \,\partial_\mu^2 A^{++} + \partial_- j^+ \,\partial_\mu^2 A^{--}\right)\right)$$

Hamilton-Jacobi equation for effective currents $j^\pm = 2x^\pm - \omega^\pm$

$$g^{\mu\nu}\partial_\mu \omega^\pm \partial_\nu \omega^\pm = 0\,,\;\; \partial_\pm j^\mp = h_{\pm\pm} - \left(h_{\rho\pm} - \frac{1}{2}\,\frac{\partial_\rho}{\partial_\pm} h_{\pm\pm}\right)^2 + ...$$

17 Double-logarithms in (super) gravity

Mellin representation for the scattering amplitude

$$A(s,t) = A_{Born}\, s^{-\alpha |q|^2 \,\ln \frac{|q|^2}{\lambda^2}}\, \Phi(\xi)\,,\;\; \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\,\omega}{2\pi i\,\omega} \left(\frac{s}{|q|^2}\right)^{\omega} f_\omega$$

Infrared evolution equation for supergravity (BLS (2012))

$$f_\omega = 1 + b\frac{d}{d\,\omega}\,\frac{f_\omega}{\omega} - b\,\frac{N-6}{2}\,\frac{f_\omega^2}{\omega^2}\,,\;\; b = \alpha |q|^2\,,\;\; \alpha = \frac{\kappa^2}{8\pi^2}\,,\;\; \xi = \alpha\,|q|^2\,\ln^2\frac{s}{|q|^2}$$

Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2}\,\frac{\xi}{2} + \frac{(N-4)(N-3)}{2}\,\frac{\xi^2}{4!} - \frac{N-4}{8}\,(5N^2 - 26N + 36)\,\frac{\xi^3}{6!} + \ldots$$

Solution in terms of the parabolic cylinder function

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N}\,\frac{1}{\sqrt{b}}\,\frac{d}{d\,x}\,\ln d^{(N)}(x)\,,\;\; d^{(N)}(x) = e^{\frac{x^2}{4}}\,D_{\frac{6-N}{2}}(x)\,,\;\; x = \frac{\omega}{\sqrt{b}}$$

18 Discussion

1. Locality of interactions in the rapidity space
2. Steinmann relations and bootstrap.
3. Integrability of the BFKL dynamics at large N_c.
4. Remarkable properties of NLLA in $N = 4$ SUSY.
5. NLO corrections to the Odderon and BKP equations.
6. Pomeron and the reggeized graviton in $N = 4$ SUSY.
7. Effective action for the reggeized gravitons.
8. Double-logarithmic approximation in gravity