Single Perturbative Splitting Diagrams in Double Parton Scattering

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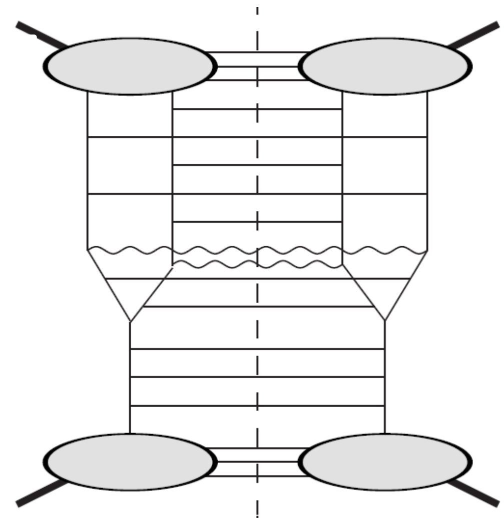
Based on [arXiv:1207.0480]

Outline

- Will talk in detail here about the contribution to the LO pp DPS cross section from graphs in which two nonperturbatively generated (NP) ladders interact with one that has split perturbatively into two – 2v1 graphs.



- First will outline some known results regarding to contribution to the DPS cross section from 'diagonal' 2v1 graphs.
- Will then point out that diagrams in which the two NP ladders exchange partons contribute to the LO DPS cross section, provided this 'crosstalk' occurs at lower scales than the perturbative 1→2 branching on the other side.
- Will discuss the issue of colour in relation to the crosstalk interactions.

Total and Differential Cross Section for DPS

Assuming only the factorisation of the two hard processes A and B, we can write the cross section for proton-proton DPS as follows:

Two-parton GPDs

Parton-level cross sections

$$\sigma^D_{(A,B)} \propto \sum_{i,j,k,l} \int \Gamma_{ij}(x_1, x_2, \boldsymbol{b}; Q_A^2, Q_B^2)\hat{\sigma}_{ik\to A}(\hat{s} = x_1 x_3 s)\hat{\sigma}_{jl\to B}(\hat{s} = x_2 x_4 s) \times \Gamma_{kl}(x_3, x_4, \boldsymbol{b}; Q_A^2, Q_B^2) dx_1 dx_2 dx_3 dx_4 d^2\boldsymbol{b} \tag{1}$$

Transverse parton pair separation

$$\int d^2\mathbf{b} \propto \Lambda^2, \hat{\sigma} \propto 1/Q^2 \Rightarrow \boxed{\sigma^D \propto \Lambda^2/Q^4}$$

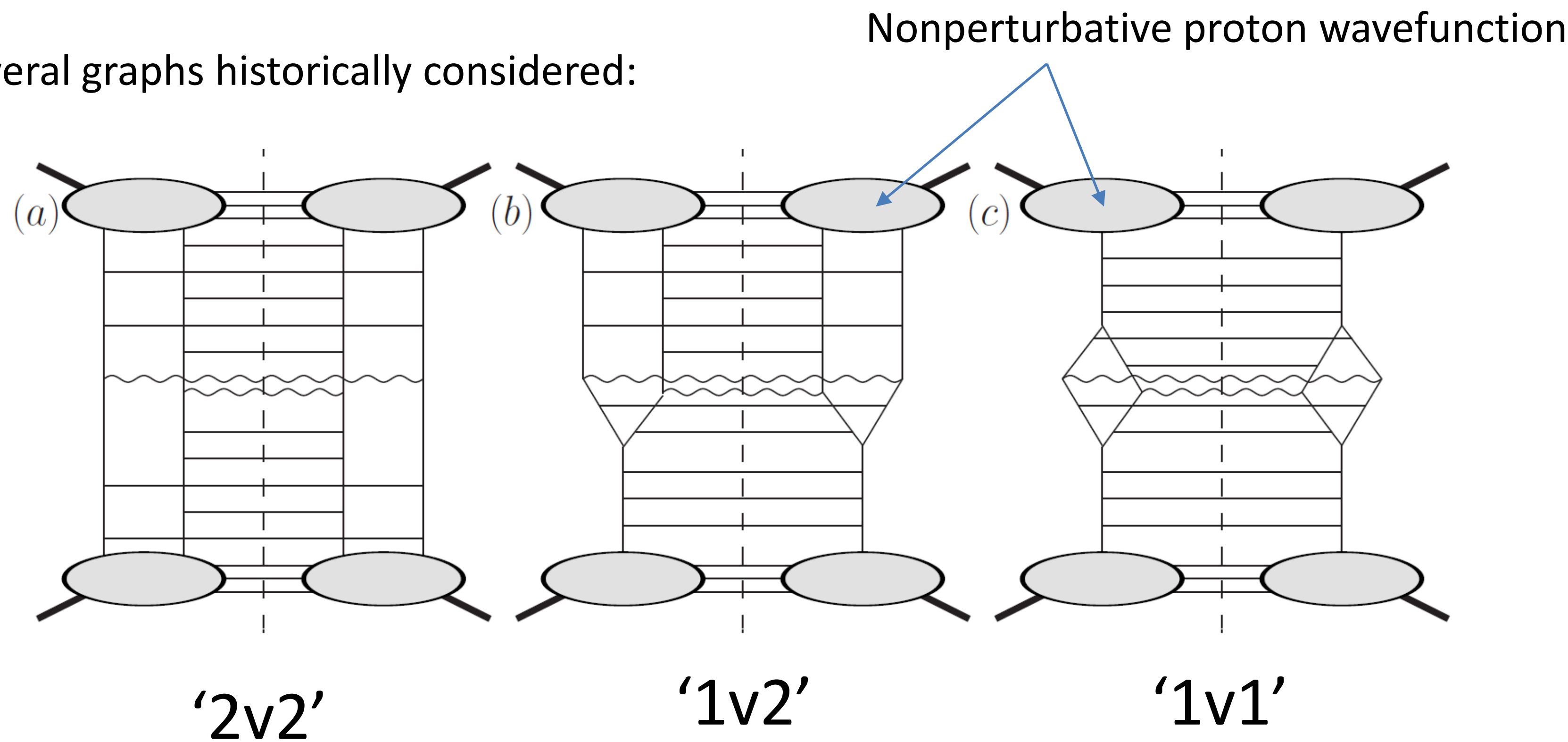
Total DPS cross section power suppressed wrt SPS cross section!

Differential DPS cross section $d^4\sigma^D_{(A,B)}/d^2\mathbf{q}_A d^2\mathbf{q}_B$ is comparable to the corresponding SPS quantity in the back-to-back region $\mathbf{q}_A^2, \mathbf{q}_B^2 \ll Q^2$ – this may be a more relevant quantity for experiments. For differential cross section require 'TMD 2pGPDs' $F(x_1, x_2, \mathbf{b}, \mathbf{k_1}, \mathbf{k_2}, Q^2)$ rather than collinear 2pGPDs.

However, for $\Lambda^2 \ll \mathbf{q}_A^2, \mathbf{q}_B^2 \ll Q^2$ it should be possible to relate TMD 2pGPDs to collinear 2pGPDs, and there is a collinear part of the differential cross section whose structure mirrors (1) – it is with this in mind that we will only discuss the total cross section.

Graphs potentially contributing to DPS

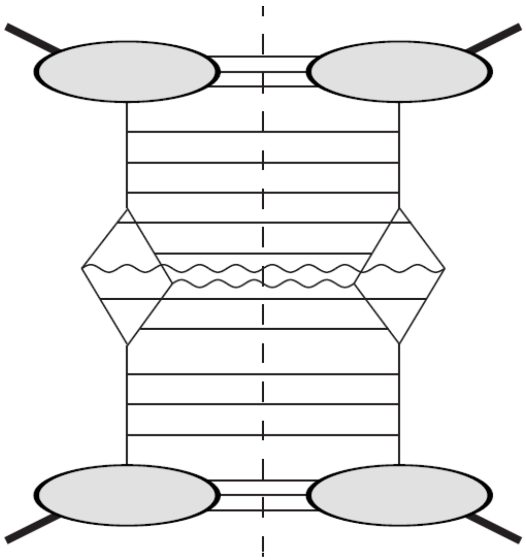
Several graphs historically considered:



In a long-established framework for calculating DPS cross sections developed by Snigirev et al., there is a contribution from each type of graph in the DPS cross section.

Shelest, Snigirev, Zinovjev, Phys. Lett. B113 (1982) 325
Shelest, Snigirev, Zinovjev, Theor. Math. Phys. 51 (1982) 523–528,
Snigirev, Phys. Rev. D68 (2003) 114012

1v1 graphs in DPS



In order for this graph to be included in LO DPS cross section, there should be a part of it proportional to:

$$\left[\alpha_s \log\left(Q^2/\Lambda^2\right)\right]^n \times \left(\frac{\Lambda^2}{Q^4}\right)$$

Transverse momentum log for every α_S (LO)

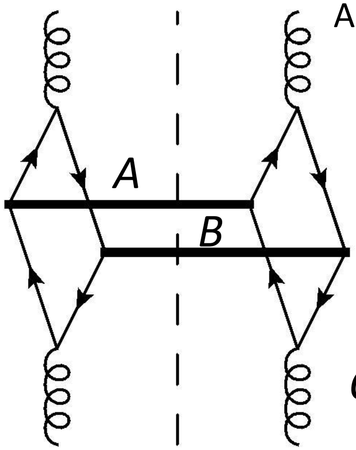
Power suppression (DPS)

n = number of QCD branchings in amplitude/conjugate

1v1 graphs in DPS

We checked this for the simplest possible graph of the '1v1' type – namely this graph (with n = 2)

JG and Stirling, JHEP 1106 048 (2011)



A,B of arbitrary type

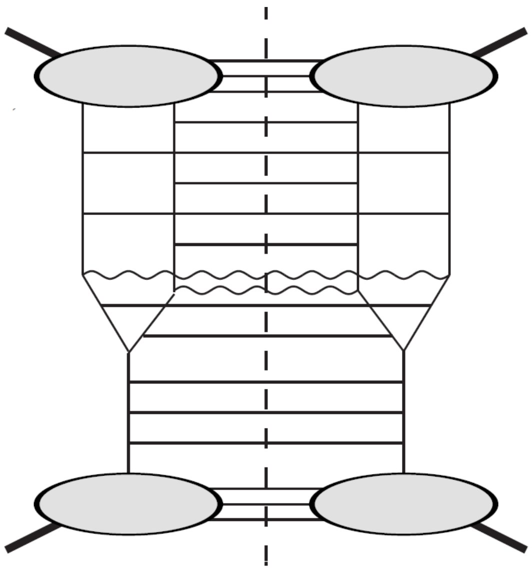
$Q_A^2, Q_B^2 > 0$

No natural part of the graph of the required type – maybe we should not include any part of this graph as DPS, and regard entirety of '1v1' graphs as SPS?

This suggestion has been made in other recent papers, for fundamentally similar reasons:

Blok, Dokshitzer, Frankfurt, Strikman, Eur.Phys.J. C72 (2012) 1963
Manohar, Waalewijn, Phys.Lett. B713 (2012) 196–201

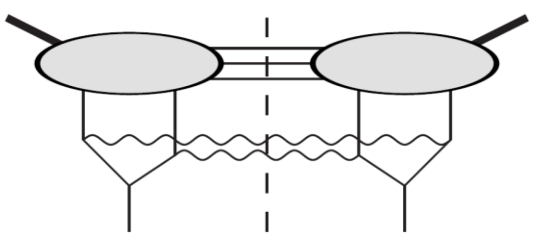
2v1 graphs in DPS



In light of this discovery, a careful re-analysis of other classes of graph that can potentially contribute to the LO DPS cross section would seem appropriate – the most interesting choice of graph to analyse is the '2v1' graph.

Take a similar approach as we did for the 1v1 graphs. Look at the simplest graph in which a single parton splits and then interacts with two 'nonperturbatively generated' partons from a proton, and see if there is a structure in the cross section formula $\propto \left(\frac{\Lambda^2}{Q^4}\right)\left(\alpha_s \log\left(\frac{Q^2}{\Lambda^2}\right)\right)$

Need to use a wavefunction on the side with the two nonperturbative partons to represent the fact that the two partons are tied together in the same proton. I used formalism of Paver and Treleani (Nuovo Cim. A70 (1982) 215).



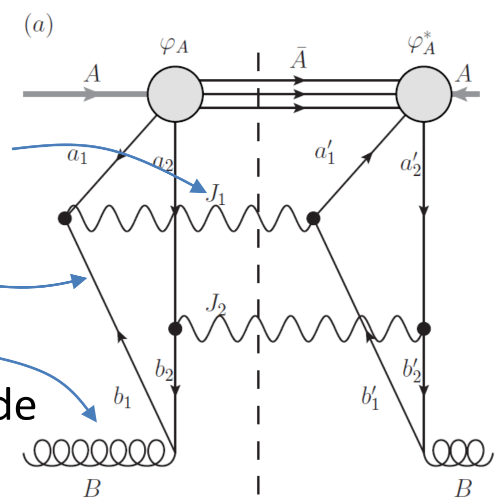
Simplest 2v1 Graph - Calculation

Process explicitly considered:

Off shell photons ($Q^2 > 0$) produced

Quarks in loop

Initiated by a gluon on one side



Result:

$$\frac{1}{Q^2} \quad \frac{1}{Q^2}$$

$$\sigma_{1v2}(s) = \sum_{s_i s_i' t_i t_i'} \int dx_1 dx_2 dy_1 dy_2 \hat{\sigma}_{\bar{q}q\to\gamma*}^{s_1,t_1;s_1',t_1';\mu_1}(\hat{s} = x_1 y_1 s)\hat{\sigma}_{q\bar{q}\to\gamma*}^{s_2,t_2;s_2',t_2';\mu_2}(\hat{s} = x_2 y_2 s)$$

$$\times \quad \Gamma_A^{s_1 s_2, s_1' s_2'}(x_1, x_2; \boldsymbol{b} = \boldsymbol{0}) \quad \left[\frac{\alpha_s}{2\pi} P_{g\to q\bar{q}}^{\lambda\to t_2 t_1, t_2' t_1'}(y_2)\,\delta(1 - y_1 - y_2) \int_{\Lambda^2}^{Q^2} \frac{d\boldsymbol{J_1}^2}{\boldsymbol{J_1}^2}\right]$$

2pGPD of nonperturbatively generated parton pair evaluated at **b** = **0** $\propto \Lambda^2$

$1 \longrightarrow 2$ splitting function

Gives required logarithm

Suggests 2v1 graphs should be included in DPS cross section!

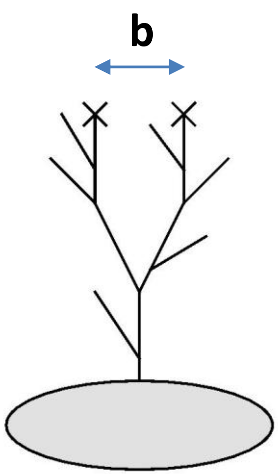
Total contribution from diagonal 2v1 graphs

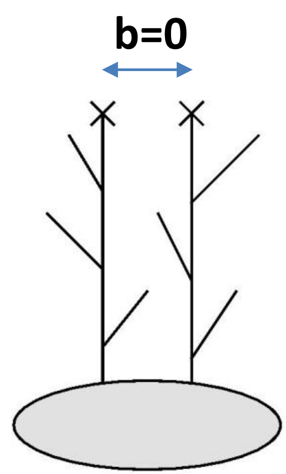
Assuming that only diagonal 2v1 diagrams contribute to the DPS cross section at leading logarithmic order, then summing up the contributions from all of the 2v1 diagrams yields the following:

$$\sigma^{D,1v2}_{(A,B)}(s) = 2 \times \frac{m}{2} \int dx_1 dx_2 dy_1 dy_2 \hat{\sigma}_{ik\to A}(\hat{s} = x_1 y_1 s)\hat{\sigma}_{jl\to B}(\hat{s} = x_2 y_2 s) \times \breve{D}^{ij}_{p}(x_1, x_2; Q^2)\Gamma^{kl}_{p,indep}(y_1, y_2, \boldsymbol{b} = \boldsymbol{0}; Q^2)$$

‘sPDF feed part of dPDF’

$$\int d^2\mathbf{b}$$





‘Independent branching’ 2pGPD

Comments on the formula for σ^{1v2}

The critical requirement for the validity of the derivation on the previous page is that parton pairs connected only via nonperturbative interactions should have an ***r*** distribution that is cut off at values of order Λ_{QCD} (or a ***b*** distribution that is smooth on scales of size $\ll R_p$). That is, the ***r*** profile of $\Gamma^{kl}_{p,indep}(y_1, y_2, \mathbf{\Delta}; Q^2)$ should have a width of order Λ_{QCD} .

The results of the previous slide are potentially misleading, in that they appear to indicate that 2v1 contribution to DPS probes independent branching 2pGPDs at zero parton separation. In fact, the results correspond to a broad logarithmic integral over values of $\mathbf{b}^2$ that are $\ll R_p^2$ but $\gg 1/Q^2$.

Comments on the formula for σ^{1v2}

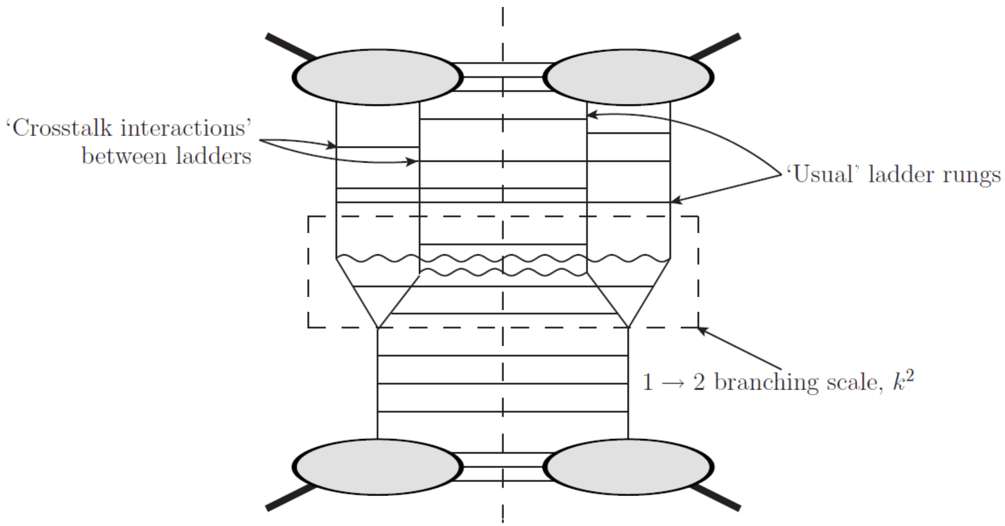
If we assume $\Gamma^{ij}_{p,indep}(x_1, x_2, \boldsymbol{b}; Q^2) = \tilde{D}^{ij}(x_1, x_2; Q^2) F(\boldsymbol{b})$ then 2v1 contribution to DPS cross section is similar to that predicted by the framework of Snigirev et al., except with a different geometrical prefactor:

$$\frac{1}{\sigma_{eff,2v2}} \equiv \int d^2\boldsymbol{b} [F(\boldsymbol{b})]^2$$

$$\frac{1}{\sigma_{eff,1v2}} \equiv F(\boldsymbol{b} = \boldsymbol{0})$$

Gaussian for F(**b**) gives factor of two geometrical enhancement for each 2v1 contribution. In [arXiv:1206.5594] Blok et al. argue that F(**b**) should be equal to a sum of two Gaussians – gives a similar enhancement.

Crosstalk in 2v1 graphs



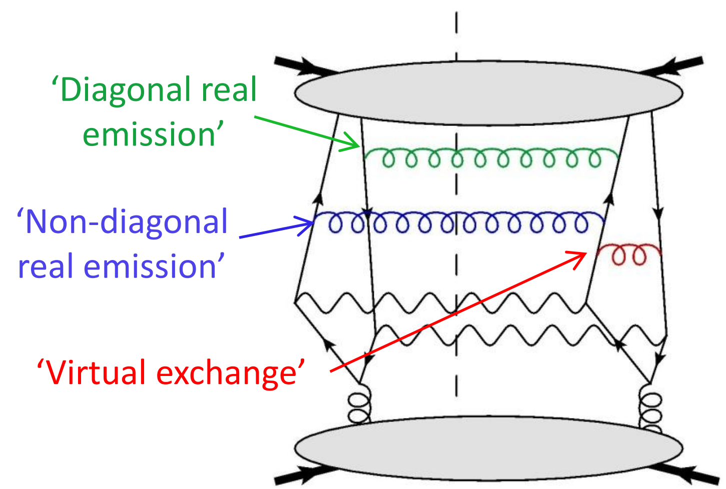
The contribution from diagonal 2v1 graphs to the DPS cross section has previously been written down in:

Ryskin and Snigirev, Phys.Rev. D83 (2011) 114047)
Blok et al., Eur. Phys. J. C72 (2012) 1963 (eq 13b)

We have discovered that there is an additional contribution to the LO 2v1 DPS cross section, associated with non-diagonal interactions ('crosstalk') on the side with two NP ladders.

Contributes at LO provided that the crosstalk on the 'two NP ladder' side occurs at lower scales than the 1→2 branching on the other side.

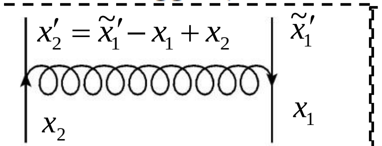
I demonstrated this by studying the simplest type of 2v1 graph containing a crosstalk interaction. Two types of crosstalk – I chose to look in detail at a diagram containing a non-diagonal real emission.

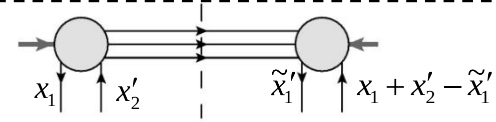


Crosstalk 2v1 graph - calculation

$$\frac{1}{Q^2}$$

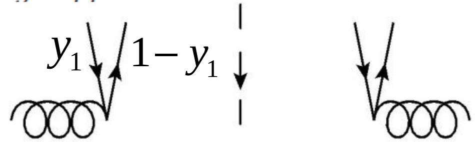
$$\sigma_{XT}(s) = \sum_{s_i \tilde{s}_i t_i \tilde{t}_i \tilde{s}'_1 s'_2} \int dx_1 dx_2 dy_1 dy_2 \hat{\sigma}^{s_1,t_1;\tilde{s}_1,\tilde{t}_1;\mu_1}_{\bar{q}q\to\gamma*}(\hat{s} = x_1 y_1 s)\hat{\sigma}^{s_2,t_2;\tilde{s}_2,\tilde{t}_2;\mu_2}_{q\bar{q}\to\gamma*}(\hat{s} = x_2 y_2 s)$$



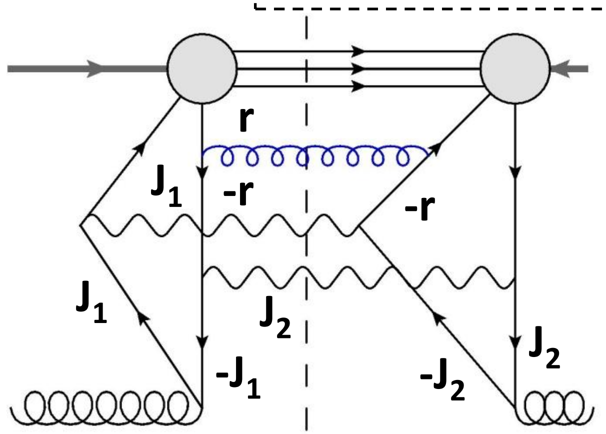


$$\left[\frac{\alpha_s}{2\pi}\int_{x_1}^{1-x_2} d\tilde{x}'_1 V^{\tilde{s}'_1 s'_2 \to \tilde{s}_1 s_2;\mu_3}_{I,q\to q}(x_1,\tilde{x}'_1,x'_2)\Gamma^{s_1 s'_2,\tilde{s}'_1 \tilde{s}_2}_{p;q\bar{q}}(x_1,x'_2,\tilde{x}'_1)\right] \propto \Lambda^2$$

$$\left[\frac{\alpha_s}{2\pi}P^{\lambda\to t_2 t_1,\tilde{t}_2\tilde{t}_1}_{g\to q\bar{q}}(y_2)\,\delta(1-y_1-y_2)\right]\int d\boldsymbol{J_1}^2 d\boldsymbol{r}^2 \frac{2\epsilon_\lambda\cdot\boldsymbol{J}_1\epsilon^*_\lambda\cdot(\boldsymbol{J}_1+\boldsymbol{r})}{\boldsymbol{r}^2\boldsymbol{J}_1^2(\boldsymbol{J}_1+\boldsymbol{r})^2}$$



$$\boldsymbol{r}^2 \ll \boldsymbol{J}_1^2, \boldsymbol{J}_2^2$$



$J_2 = -J_1 - r$

$$\int_{\Lambda^2}^{Q^2}\frac{d\boldsymbol{J_1}^2}{\boldsymbol{J_1}^2}\int_{\Lambda^2}^{\boldsymbol{J_1}^2}\frac{d\boldsymbol{r}^2}{\boldsymbol{r}^2} = \boxed{\log^2\left(\frac{Q^2}{\Lambda^2}\right)}$$

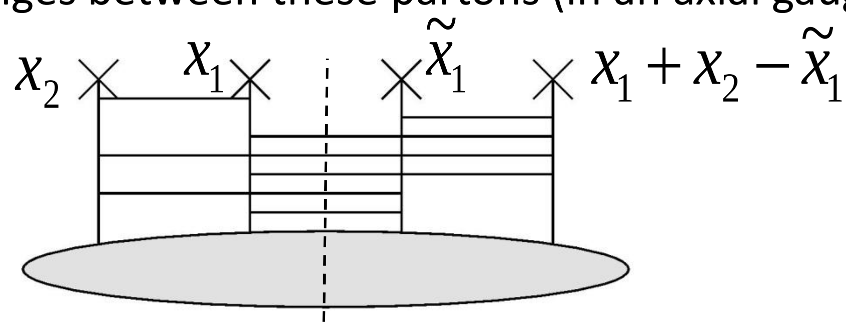
Scale of crosstalk interaction ($\mathbf{r}^2$) must be smaller than that of 1→2 splitting ($\mathbf{J_1}^2$) for leading log contribution.

2v1 cross section including crosstalk

Summing up the leading logarithmic contributions from all 2v1 graphs, including those with crosstalk on the side with two ‘nonperturbatively generated’ partons:

$$
\begin{aligned}
\sigma^{D,2v1}_{(A,B)}(s) =& 2 \times \frac{m}{2} \sum_{l i_i j_i i'_i j'_i} \int_{\Lambda^2}^{Q^2} dk^2 \frac{\alpha_s(k^2)}{2\pi k^2} \int dx_1 dx_2 dy_1 dy_2 \frac{dx'_1}{x'_1} \frac{dx'_2}{x'_2} \frac{dy'_1}{y'_1} \frac{dy'_2}{y'_2} \\
& \times \hat{\sigma}_{i_1 j_1 \to A}(\hat{s} = x_1 y_1 s) \hat{\sigma}_{i_2 j_2 \to B}(\hat{s} = x_2 y_2 s) \\
& \times \frac{D^l_h(y'_1 + y'_2, k^2)}{y'_1 + y'_2} P_{l \to j'_1 j'_2} \left(\frac{y'_1}{y'_1 + y'_2} \right) D^{j_1}_{j'_1} \left(\frac{y_1}{y'_1}; k^2, Q^2 \right) D^{j_2}_{j'_2} \left(\frac{y_2}{y'_2}; k^2, Q^2 \right) \\
& \times D^{i_1}_{i'_1} \left(\frac{x_1}{x'_1}; k^2, Q^2 \right) D^{i_2}_{i'_2} \left(\frac{x_2}{x'_2}; k^2, Q^2 \right) \Gamma^{i'_1 i'_2}_h (x'_1, x'_2; x'_1, k^2)
\end{aligned}
$$

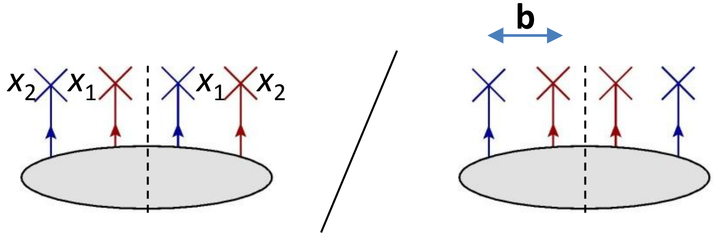
$\Gamma^{i'_1 i'_2}_h (x_1, x_2; \tilde{x}_1, \mu^2)$ is a four-parton ‘twist 4’ matrix element whose evolution involves all possible exchanges between these partons (in an axial gauge).



Colour in the evolution of $\Gamma(x_1, x_2; \tilde{x}_1)$

Recall that for the 2pGPD with finite **b**, every distribution which does not have the partons with the same lightcone momentum fractions paired up into colour singlets is Sudakov suppressed:

M. Mekhfi and X. Artru, Phys.Rev. D37 (1988) 2618–2622
Diehl, Ostermeier and Schafer (JHEP 1203 (2012) 089)
A. V. Manohar and W. J. Waalewijn, Phys.Rev. D85 (2012) 114009



$$\sim \exp\left(\frac{\alpha_s}{2\pi} \left(C_R^I - C_V^I\right) \ln^2\left(\mathbf{b}^2 Q^2\right) \right)$$

$\left(C_R^I - C_V^I\right) = -\frac{1}{2} C_A$ for the two quark case considered here.

In axial gauge: arises from incomplete cancellation between real emission diagrams and virtual self-energy corrections in colour interference distributions.

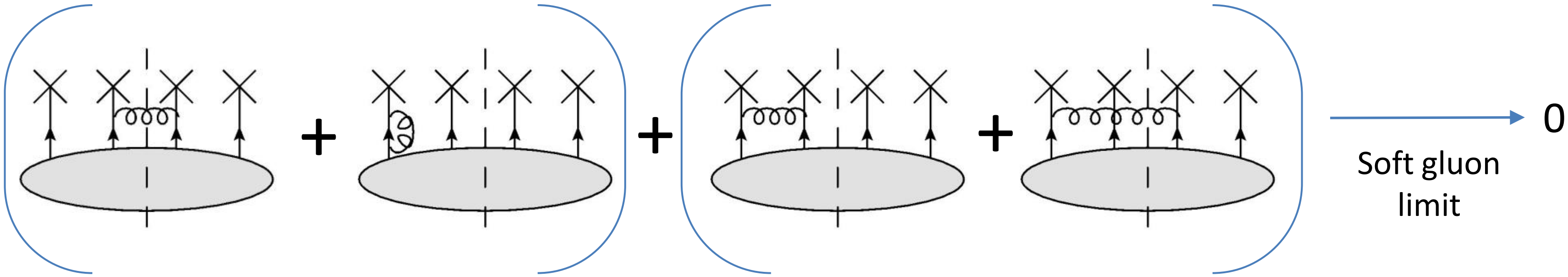
M. Mekhfi and X. Artru, Phys.Rev. D37 (1988) 2618–2622

Physically: occurs because these distributions involve a movement of colour by the large transverse distance **b** in the hadron. A. V. Manohar and W. J. Waalewijn, Phys.Rev. D85 (2012) 114009

Colour in the evolution of $\Gamma(x_1, x_2; \tilde{x}_1)$

There is no such Sudakov suppression of colour interference distributions for the twist-four distribution Γ. The extra 'crosstalk' diagrams that are allowed in the evolution of this distribution provide extra soft-gluon divergences that cancel those from 'diagonal' real emission and virtual self energy corrections.

Schematically:



Physically: All four partons in this distribution are close together in transverse space (much closer together than $1/\mu$ in transverse space). Soft longwave gluons can only resolve total colour of all four partons, which is constrained to be 0 since the proton is colourless).

Levin, Ryskin, Shuvaev, Nucl.Phys. B387 (1992) 589–616
Gribov, Dokshitzer, Troyan, Khoze Sov. Phys. JETP 88 (1988) 1303

Colour in crosstalk and small x

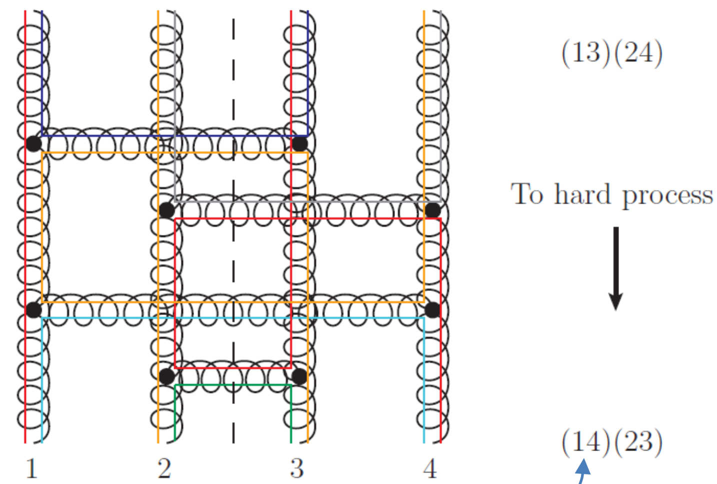
Now we focus our attention on the region of small x (perhaps most relevant region for DPS processes at the LHC). Gluons dominant → we'll consider only these partons.

Though colour interference twist-4 distributions are not Sudakov suppressed, in the low x region distributions in which two pairs of gluons are in colour singlet configurations win out under evolution (anomalous dimension is larger).

Levin, Ryskin, Shuvaev, Nucl.Phys. B387 (1992) 589–616
Diehl, Ostermeier and Schafer (JHEP 1203 (2012) 089)

At the scale of the 1→2 splitting in the 2v1 diagram, k^2, the nonperturbatively generated partons with identical x fractions should be in a colour singlet state to avoid Sudakov suppression in subsequent evolution.

Using two off-diagonal + two diagonal real emissions, can alter the way in which the legs are grouped into colour singlets at scales < k^2 - 'colour recombination'.



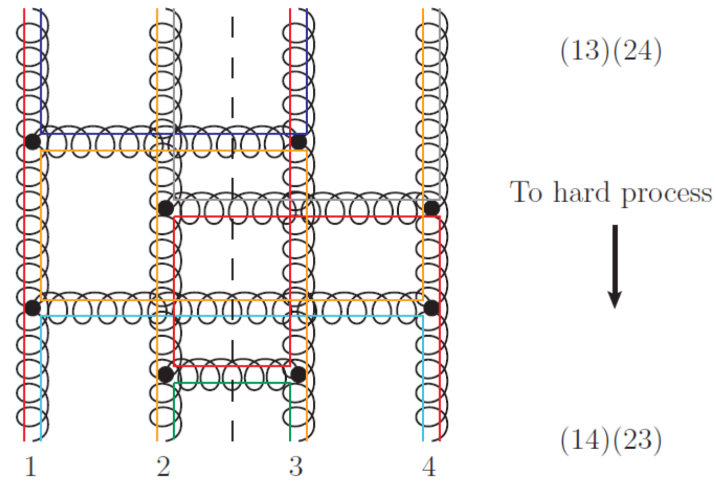
Colour singlet pairings

(Could also achieve a colour recombination using virtual exchanges rather than off-diagonal emissions – but this is subleading at low x).

Colour in crosstalk and small *x*

The colour recombination process is non-planar – therefore colour recombination vertex is suppressed by $\boxed{1/\left(N_c^2-1\right)}$

J. Bartels, Phys.Lett. B298 (1993) 204–210
J. Bartels, Z.Phys. C60 (1993) 471–488
J. Bartels and M. Ryskin, Z.Phys. C60 (1993) 751–756
J. Bartels and M. Ryskin, arXiv:1105.1638.



Given this suppression, do crosstalk processes and recombination effects have any significant numerical impact on the 2v1 DPS cross section?

Colour in crosstalk and small *x*

Can perhaps get a clue from a study by Bartels and Ryskin (J. Bartels and M. Ryskin, Z. Phys. C60 (1993) 751–756) into the four-gluon twist 4 matrix element in the context of shadowing corrections to DIS. They found that inclusion of recombination effects in evolution increased Γ by the following 'K factor':

$$K_2 = 1 + 2\sqrt{\pi\delta}\left(4\frac{N_c\alpha_s}{\pi}\ln\left(\frac{1}{x}\right)\ln\left(\frac{Q^2}{Q_0^2}\right)\right)^{1/4} \qquad \delta \sim 1/N_c^4 \qquad Q_0^2 \text{ = evolution start scale}$$

For $x \approx 10^{-3}, t \equiv \log\left(Q^2/Q_0^2\right) \approx 3$ (values considered that paper relevant to the HERA experiment), K_2 = 1.63 - a significant enhancement despite the colour suppression!

K_2 would be even larger for x and t values relevant to the LHC. We must bear in mind that 2v1 process probes twist-4 matrix element at a range of scales < Q^2 and x values > x_i → numerical estimates of the size of the recombination effects in the context of 2v1 contributions to DPS at the LHC are needed. This is ongoing work.

Summary

- We have closely studied the contribution to the LO pp DPS cross section from 2v1 diagrams.
- We found:
  - That diagonal 2v1 graphs contribute to the LO DPS cross section in the way originally written down by Ryskin and Snigirev, and Blok et al.
  - That 2v1 graphs in which the two NP ladders exchange partons with one another contribute to the DPS cross section, provided that the ‘crosstalk’ takes place at lower scales than the perturbative 1→2 splitting.
- Crosstalk interactions between the two NP ladders are suppressed by colour effects. At low *x* the most likely sort of crosstalk interaction is a ‘colour recombination’ – this is suppressed by $1/(N_c^2 - 1)$.
- Despite the colour suppression of the recombination vertex, crosstalk effects might have an appreciable impact on the 2v1 DPS cross section – more detailed numerical calculations are required.

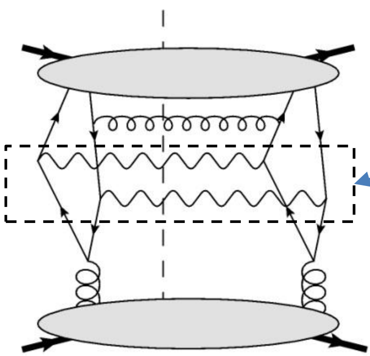
Backup Slides

2v1 contribution at NLO

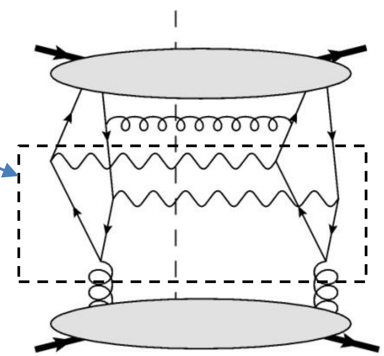
At the next to leading logarithmic (or NLO) level, one needs to append an extra term to the expression on the previous slide:

$$\int dx_1 dx_2 d\tilde{x}_1 dy D_h^k(y, Q^2)\Gamma_h^{ij}(x_1, x_2; \tilde{x}_1, Q^2)\hat{\sigma}_{ijk\to AB}(x_1, x_2, \tilde{x}_1, y)$$

This is the conventional twist 4 contribution to the pp → AB + X production cross section. At the level of total cross sections, this cannot be distinguished from DPS – the two should just be included together as power suppressed contributions to the pp → AB + X cross section.



Hard part (i.e. virtualities of order Q^2)



$$\propto \left(\frac{\Lambda^2}{Q^4}\right)\left(\alpha_s^2 \log^2\left(\frac{Q^2}{\Lambda^2}\right)\right)$$ LO

$$\propto \left(\frac{\Lambda^2}{Q^4}\right)\left(\alpha_s^2 \log\left(\frac{Q^2}{\Lambda^2}\right)\right)$$ NLO

The DPS Cross Section

Combining suggestions for 1v1 and 2v1 graphs, we obtain the following formula for the DPS cross section:

$$\sigma^{D}_{(A,B)}(s) = \sigma^{D,2v2}_{(A,B)}(s) + \sigma^{D,1v2}_{(A,B)}(s)$$

$$\begin{aligned}\sigma^{D,2v2}_{(A,B)}(s) =& \frac{m}{2}\int dx_1 dx_2 dy_1 dy_2 \hat{\sigma}_{ik\to A}(\hat{s} = x_1 y_1 s)\hat{\sigma}_{jl\to B}(\hat{s} = x_2 y_2 s) \\ &\times \int d^2\boldsymbol{b}\Gamma^{ij}_{p,indep}(x_1, x_2, \boldsymbol{b}; Q^2)\Gamma^{kl}_{p,indep}(y_1, y_2, \boldsymbol{b}; Q^2)\end{aligned}$$

$$\begin{aligned}\sigma^{D,2v1}_{(A,B)}(s) =& 2\times\frac{m}{2}\sum_{li_ij_ii'_ij'_i}\int_{\Lambda^2}^{Q^2} dk^2 \frac{\alpha_s(k^2)}{2\pi k^2}\int dx_1 dx_2 dy_1 dy_2 \frac{dx'_1}{x'_1}\frac{dx'_2}{x'_2}\frac{dy'_1}{y'_1}\frac{dy'_2}{y'_2} \qquad (3.14) \\ &\times \hat{\sigma}_{i_1 j_1\to A}(\hat{s} = x_1 y_1 s)\hat{\sigma}_{i_2 j_2\to B}(\hat{s} = x_2 y_2 s) \\ &\times \frac{D^l_h(y'_1 + y'_2, k^2)}{y'_1 + y'_2} P_{l\to j'_1 j'_2}\left(\frac{y'_1}{y'_1 + y'_2}\right) D^{j_1}_{j'_1}\left(\frac{y_1}{y'_1}; k^2, Q^2\right) D^{j_2}_{j'_2}\left(\frac{y_2}{y'_2}; k^2, Q^2\right) \\ &\times D^{i_1}_{i'_1}\left(\frac{x_1}{x'_1}; k^2, Q^2\right) D^{i_2}_{i'_2}\left(\frac{x_2}{x'_2}; k^2, Q^2\right)\Gamma^{i'_1 i'_2}_h(x'_1, x'_2; x'_1, k^2)\end{aligned}$$

The DPS Cross Section

Comments on this formula:

1. We were originally expecting to get a formula for the DPS cross section looking something like:

$$\sigma^D_{(A,B)} \propto \sum_{i,j,k,l} \int \Gamma_{ij}(x_1, x_2, \boldsymbol{b}; Q_A^2, Q_B^2) \hat{\sigma}_{ik\to A}(\hat{s} = x_1 x_3 s) \hat{\sigma}_{jl\to B}(\hat{s} = x_2 x_4 s)$$
$$\times \Gamma_{kl}(x_3, x_4, \boldsymbol{b}; Q_A^2, Q_B^2) dx_1 dx_2 dx_3 dx_4 d^2\boldsymbol{b}$$

with the 2pGPDs being expressible in terms of hadronic matrix elements. What we have got does not seem to look like this.

2. In this formula, we have made a sharp distinction between perturbatively and nonperturbatively generated parton pairs. Is there some scale at which we can regard all parton pairs in the proton as being 'nonperturbatively generated', and if so what is the appropriate choice for this scale? (Presumably something rather close to Λ_{QCD}).

3. We have ignored all of the interesting correlated parton and interference contributions pointed out by Mekhfi (Phys.Rev. D32 (1985) 2380, Phys.Rev. D32 (1985) 2371), and Diehl, Ostermeier and Schafer (arXiv:1111.0910).