Mesons in a Bethe-Salpeter approach in Rainbow-Ladder

Milan Vujinović, Richard Williams, Reinhard Alkofer

Department of Physics, University of Graz

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Outline of the research

- Investigate bound states in strongly interacting theories (e.g. QCD).
- Work in non-perturbative regime $\rightarrow$ appropriate tools
- Chiral limit $\rightarrow$ heavy quarks
- Concentrate on continuum approaches.

$\Rightarrow$ Dyson-Schwinger and Bethe-Salpeter equations (DS/BS approach).
Dyson-Schwinger equations

- Equations of motion for Greens functions of a QFT.
- Give an exact, non-perturbative, continuous formulation of a QFT.
- An infinite tower of coupled integral equations.
- Truncations required to close the system (QCD modeling).
- Basic guide for truncations - preservation of symmetries.
Mesons (\( q\bar{q} \) bound states) described within DS/BS approach

First investigations concerned properties of light mesons (\( \pi, K, \rho \ldots \) )


Baryons first treated in the quark-diquark BSE approach


Fairly recently baryons within the covariant three-body Fadeev approach

G. Eichmann, PRD 84, 014014 (2011)
The quark DSE

\[
S^{-1}(p) = S_0^{-1}(p) + \int_q \gamma_\mu S(q) \Gamma_\nu(q,p) D_{\mu\nu}(p-q)
\]

(1)

The quark Dyson-Schwinger equation.
Fundamental objects in a DSE

Dressed quark propagator $S(p) = [\phi A(p^2) + B(p^2)]^{-1}$

Dressed gluon propagator $D_{\mu\nu}(k) = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2})Z_2(k^2)$.

Dressed quark-gluon vertex $\Gamma_\nu$
Inhomogeneous BSE

- Consider a general meson-$q\bar{q}$ vertex

\[ \Gamma(P, p) = \Gamma_0 + \Gamma(P, q) \]

\[ K[2](P, p, q) \]

\[ S^a(q_+) \]

\[ S^b(q_-) \]

- Inhomogeneous Bethe-Salpeter equation for the BS amplitude $\Gamma$.

\[ \Gamma^M(p, P) = \Gamma_0^M(p, P) + \int_q K(p, q, P) S(q^+) \Gamma^M(q, P) S(q^-) \]
Homogeneous BSE

- Bound states appear as poles in $\Gamma \Rightarrow$ On-shell properties are given by the residues
- Homogeneous Bethe-Salpeter equation:

$$\Gamma^M(q, P) = \int_k K(k, q, P)S(k^+)\Gamma^M(k, P)S(k^-)$$

\[ \Gamma_{[h]}(q, P) = \Gamma_{[h]}(k, P) \]

\[ S^a(k_+) \]

\[ S^b(k_-) \]

\[ K(k, q, P) \]
Symmetries

- Want our theories to respect symmetries (gauge, chiral, CPT...)
- In this investigation we focus on the chiral symmetry.
- Constraints of the $\chi$ symm. expressed in the axWTI.
- axWTI connects the quark-gluon vertex to the kernel of the BSE.

\[-iP_{\mu}\Gamma_{\mu}^{5} = S_{F}^{-1}(p_{+})\gamma_{5} + \gamma_{5}S_{F}^{-1}(p_{-}) - 2m_{R}\Gamma_{5}^{5}(p; P)\] (2)

The diagrammatic representation of axWTI
Dichotomy of the pion

- Pion much less massive than it 'should be':
  1. $m_\rho/2 \approx 350$ MeV
  2. $m_N/3 \approx 350$ MeV
  3. $m_\pi/2 \approx 70$ MeV

- How can one describe this consistently without fine-tuning?
- Theory must recognize double nature of pseudoscalar mesons: both $q\bar{q}$ bound states and Pseudo-Goldstone bosons of $\chi_{SB}$.
- This would ensure that Gell-Man-Oakes-Renner relation is satisfied:

$$m_\pi^2 f_\pi^2 = 2m_{curr}\langle q\bar{q}\rangle_0$$ (3)
The Rainbow-Ladder Truncation

- Rainbow approximation in the quark DSE:
  \[
  \Gamma_\nu(q, p) \rightarrow \gamma_\nu
  \]  

- The ladder approximation in the meson BSE:
  \[
  K(p, q; P) \rightarrow G_{eff}(k^2) \cdot \left( \frac{\lambda^a}{2} \gamma_\mu \right) \otimes \left( \frac{\lambda^a}{2} \gamma_\mu \right)
  \]

Models for $G_{eff}(k)$:
- R. Alkofer, C. Fischer, R. Williams, EPJ A38 (2008)
Quark DSE in Rainbow-Ladder

\[ S(p)^{-1} = [\phi A(p^2) + B(p^2)] , \quad M(p^2) = B(p^2)/A(p^2) \]  

(6)

- Project out the vector part \( A \) and scalar part \( B \).
- Two coupled nonlinear integral equations, solve by iteration.

Quark DSE in Rainbow approximation.
Solution for quark on the real axis

MT interact.

\[ m_{\text{curr}} = 5 \text{ MeV} \]

\[ m_{\text{curr}} = 0 \]
Euclidean space

- Wick rotation: poles moved from $P^2 = M^2$ to $P^2 = -M^2$.
- Thus $P$ is imaginary, in the bound-state rest frame:

$$P^\mu = iM(0, 0, 0, 1)$$  \hspace{1cm} (7)

- BSE requires knowledge of quark propagator in the complex plane.

$$S(q^\pm) = S(q \pm \eta P) = S(q \pm i\eta\sqrt{M^2}) , \quad \eta \in (0, 1)$$  \hspace{1cm} (8)

- $\eta$ - momentum partitioning factor.
- $(q^\pm)^2$ define parabolas in the complex plane.
- How to obtain the quark propagator along these parabolas?
Analytic continuation of the gluon

- If gluon $D_{\mu\nu}(k)$ is defined in complex plane, solution is easy.

\[
S^{-1}(p) = S_0^{-1}(p) + \int_q \gamma_\mu S(q) \gamma_\nu D_{\mu\nu}(p-q) , \quad p \in \mathbb{C}, q \in \mathbb{R}
\]  

(9)

$(p - q)^2$ sweeps the interior of the parabola.
Inverse onion method

- If knowledge of gluon is confined to real spacelike momenta:

\[
S^{-1}(p) = S_0^{-1}(p) + \int_k \gamma_\mu S(p-k) \gamma_\nu D_{\mu\nu}(k), \quad p \in \mathbb{C}, k \in \mathbb{R}
\]  

(10)

- Use inverse onion method.
Inverse onion method II

- Divide the process into several steps (layers) - better for interpolation.
- Solve the quark DSE layer by layer.
- Iterate outwards from real axis to the desired parabola.
Inverse onion results

Watson int.
m_{curr} = 5 MeV
Inverse onion

Re[B] GeV

Re[\rho^2] GeV^2

Im[\rho^2] GeV^2
The quark-meson amplitude

- With $S(p)$ provided, all is set to solve BSE in RL:

$$\Gamma(k, P) = \int_q G_{\text{eff}}(k - q)\gamma_\mu S(q^+) \Gamma(q, P) S(q^-) \gamma_\nu$$  \hspace{1cm} (11)

- $\Gamma(k, P)$ expanded in covariants which reflect $J^{PC}$ of the bound state:

$$\Gamma(k; P) = \sum_{i=1}^{N} T_i(k; P) \left[ \begin{array}{c} \text{Covariants} \\ \text{Coefficients} \end{array} \right] F_i^i(k^2, P^2, k \cdot P)$$  \hspace{1cm} (12)

- Project the components of $\Gamma$ with appropriately constructed projectors.

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Solving the BSE

- BSE as a matrix equation

\[ \Gamma_i(k, P, k \cdot P) = \sum_j M_{ij} \Gamma_j(q, P, q \cdot P) \]  \hspace{1cm} (13)

- Solutions exist for discrete values of \( P^2 \).
- For general \( P^2 \), formulate as an eigenvalue problem

\[ \lambda(P^2) \Gamma = M(P^2) \Gamma \]  \hspace{1cm} (14)

- On-shell solution(s) of the BSE correspond to \( P^2 \) for which \( \lambda(P^2) = 1 \).
Results for the pion

\[ \lambda_0 \]

Watson inter.
Inter. strength = 16 GeV²
Inter. width = 0.5 GeV
\( m_{u \text{-quark}} = 5 \text{ MeV} \)
\( M_\pi \approx 140 \text{ MeV} \)
Checking the GMOR relation for pion

\[ M\pi^2 \, [\text{GeV}^2] \]

GMOR for pion: \( M\pi^2 \sim m_{\text{u-quark}} \)

\[ \text{m}_{\text{u-quark}} \, [\text{MeV}] \]
Normalisation

- $\Gamma(k, P)$ can be used to calculate physical quantities (e.g. decay constants, form factors...).

\[
\left( \frac{d \ln(\lambda)}{dP^2} \right)^{-1} = \text{tr} \int_k \Gamma(k, -P)S(k_+)\Gamma(k, P)S(k_-)
\]  

\[\text{(15)}\]

C. S. Fischer, R. Williams, Phys Rev Lett, 103, 122001 (2009)
Calculate $f_\pi$ by coupling the pion to point axial field:

$$f_\pi \propto \text{tr} \int \frac{d^4k}{(2\pi)^4} \left( S(k) \Gamma_\pi(k, P) \gamma_5 \slashed{P} \right) S(k)$$

(16)
Results

RL results for $\pi$ and $\rho$ mass, and $\pi$ decay constant. All units are GeV.

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>D</th>
<th>$m_\pi$</th>
<th>$m_\rho$</th>
<th>$f_\pi$</th>
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<tbody>
<tr>
<td>0.50</td>
<td>16</td>
<td>138</td>
<td>758</td>
<td>0.133</td>
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<tr>
<td>Experiment</td>
<td>/</td>
<td>/</td>
<td>135</td>
<td>776</td>
</tr>
</tbody>
</table>
Summary of RL results

- **Mesons** $q\bar{q}$
  - Reasonable estimates for pseudoscalar, vector and tensor mesons

- **Baryons** $qqq$
  - Nucleon/Delta
    - G. Eichmann, PRD 84, 014014 (2011)

- **Applied to** $qq\bar{q}\bar{q}$

- **Shortcomings:** axial vectors/no flavor mixing/no decay channels.
Outlook

We want to go ’Beyond the Rainbow’, several ways to do this.

Unquenching the quark

C. S. Fischer, R. Williams, Phys Rev D. 78, 074006 (2008)

Unquenching the gluon

Use input from quark-gluon vertex DSE

C. S. Fischer, R. Williams, Phys Rev Lett 103, 122001 (2009)
See talk by M. Hopfer

Applications of these methods to Technicolor theories?

THANK YOU FOR YOUR ATTENTION!
Inverse onion method III

Need propagator at this point

Inverse map
In this diagram, the variables $m^2$ and $t^2$ are plotted on the vertical and horizontal axes, respectively. The points $m_1$ and $m_2$ are marked on the $m^2$ axis. The region $m_1$ is shaded, and there is an arrow pointing from $m_1$ to $t^2 = 0$. The text within the diagram reads: "Get propagator at these points with spline interpolation."
Inverse onion method V

Linear interpolation to get propagator at this point.
Inverse onion method VI

\[ m^2 \]

\[ m_1 \]

\[ 0 \]

\[ -m_1 \]

\[ t^2 \]

*Triple-point interpolation along this line*

*Complex conjugation to get propagator at this point*