From a complex scalar field to the hydrodynamics of dense quark matter

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dense matter in the QCD phase diagram

- r mode instability
  [e.g., N. Andersson, Astrophys. J. 502, 708 (1998)]
- pulsar glitches
  [e.g., B. Link, MNRAS 422, 1640 (2012)]
Superfluids in dense quark matter

- CFL breaks chiral symmetry and Baryon conservation:
  - octet of (pseudo) goldstone modes
    \[ SU(3)_C \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \rightarrow SU(3)_{L+R+C} \otimes Z(2) \]
  - CFL is a superfluid! - (exact) Goldstone mode due to U(1)_B

- Kaon condensation in CFL at high (but not asymptotically high) densities:
  - going down in density: \( m_s \) becomes non negligible
  - CFL reacts on stress on pairing pattern by developing a kaon condensate
    \[ U(1)_s \rightarrow 1 \]
  - on top of CFL breaking pattern: (not exact) Goldstone mode due to U(1)_s

- how many superfluid components are to expect in CFL with kaon condensation?
Landau’s model of superfluidity
Comparison: Landau vs microscopic model

<table>
<thead>
<tr>
<th>Landau model</th>
<th>Microscopic model</th>
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<tr>
<td>• Phenomenological model based on hydrodynamic equations such as:</td>
<td>• QFT model based on SSB and the existence of Goldstone modes</td>
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<tr>
<td>[ \vec{g} = \rho_s \vec{v}_s + \rho_n \vec{v}_n ]</td>
<td>• Condensate related to superfluid, elementary excitations to the normal fluid part</td>
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<tr>
<td>[ \epsilon = \epsilon_n + \epsilon_s + \frac{\rho_s v_s^2}{2} + \frac{\rho_n v_n^2}{2} ]</td>
<td>• Variables:</td>
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<td>• Fluid formally divided into superfluid and normal density (very successful describing superfluid He4)</td>
<td>- gradients of Goldstone fields: [ \partial_0 \psi, \vec{\nabla} \psi, T ] , ...</td>
</tr>
<tr>
<td>• Variables: ( \rho, \vec{v}_s, T ) , ...</td>
<td>• Variables:</td>
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</table>

\[ T = \begin{cases} 0 & \rho = \rho_s \\ > T_C & \rho = \rho_n \end{cases} \]
SSB in a $\varphi^4$ model

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 |\varphi|^2 - \lambda |\varphi|^4$$

- from effective theory for mesons:
  $$\lambda_{eff} = \frac{4\mu^2 K^0 - m_{K^0}^2}{6f^2_\pi} \quad \mu K^0 = \frac{m_{s}^2 - m_{d}^2}{2\mu}$$

- ansatz for the condensate:
  $$\varphi(x) \rightarrow \frac{\rho(x)}{\sqrt{2}} e^{i\psi(x)} + \text{fluctuations}$$

$$\mathcal{L} = -U + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)}$$

$$U = -\frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{\rho^2}{2} (\sigma^2 - m^2) + \frac{\lambda}{4} \rho^4$$

$$\sigma^2 = \partial_\mu \psi \partial^\mu \psi$$

- assumption:
  $$\rho, \partial_\mu \psi = \text{cons}$$

homogeneous superflow
Comparison: Landau vs microscopic model

### Hydrodynamics

- **super current**
  \[ j^\mu = n_s v^\mu \]
- **stress-energy tensor**
  \[ T^{\mu\nu} = (\epsilon_s + P_s) v^\mu v^\nu - g^{\mu\nu} P_s \]
- **superfluid density**
  \[ n_s = \sqrt{j^\mu j_\mu} = v^\mu j_\mu \]
- **superfluid velocity**
  \[ v^\mu = \gamma (1, \vec{v}_s) \]
- **energy density**
  \[ \epsilon_s = v_\mu v_\nu T^{\mu\nu} \]
- **pressure**
  \[ P_s = -\frac{1}{3} (g_{\mu\nu} - v_\mu v_\nu) T^{\mu\nu} \]

### Microscopic Model

- **Noether current**
  \[ j^\mu = \partial \mathcal{L} / \partial (\partial_\mu \psi) = \partial^\mu \psi (\sigma^2 - m^2) / \lambda \]
- **stress-energy tensor**
  \[ T^{\mu\nu} = \partial^\mu \psi \partial^\nu \psi (\sigma^2 - m^2) / \lambda - g^{\mu\nu} \mathcal{L} \]
- **superfluid density**
  \[ n_s = \sigma (\sigma^2 - m^2) / \lambda \]
- **superfluid velocity**
  \[ v^\mu = \partial^\mu \psi / \sigma \]
- **energy density**
  \[ \epsilon_s = v_\mu \partial^\mu \psi n_s - \mathcal{L} \]
- **pressure:**
  \[ P_s = \mathcal{L} + (v_\mu \partial^\mu \psi - \sigma) n_s \]
Superfluidity from $\varphi^4$ model (T=0)

- connection to thermodynamics:
  \[
  \epsilon_s + P_s = \mu_s n_s \rightarrow \mu_s = \sigma = \nu \mu \partial_\mu \psi
  \]

→ chemical potential and flow velocity of the superfluid are both determined in terms of the phase of the condensate:

- rotations of the phase around the U(1) circle generate the chem. pot.
- number of rotations/unit length gives rise to the superflow velocity

→ Lorentz factor in $\sigma$:
  \[
  \sigma = \sqrt{(\partial_0 \psi)^2 - (\nabla \psi)^2} = \partial_0 \psi \sqrt{1 - \left(\frac{\nabla \psi}{\partial_0 \psi}\right)^2} = \mu \sqrt{1 - v^2}
  \]
finite temperature

- use pressure as effective action:
  \[ \Gamma_{eff}[\Phi, S] = -U(\Phi) - \frac{1}{2} Tr \ln S^{-1} \]
  \[ S^{-1}(k) = \begin{pmatrix} -k^2 + 2(\sigma^2 - m^2) & 2ik \cdot \partial \psi \\ 2ik \cdot \partial \psi & -k^2 \end{pmatrix} \]

- Anisotropic dispersions (Goldstone + massive):
  \[ \epsilon_{1,k} = \sqrt{\frac{\sigma^2 - m^2}{3\sigma^2 - m^2}} \zeta(\cos \theta) |k| + \mathcal{O}(|k|^3) \]
  \[ \epsilon_{2,k} = \sqrt{2} \sqrt{3\sigma^2 - m^2 + 2(\nabla \psi)^2} + \mathcal{O}(|k|) \]
  \[ \zeta(\theta) = \frac{\sqrt{1 - v_s^2} \sqrt{1 - \frac{v_s^2}{3} (1 + 2 \cos^2 \theta) + \frac{2|v_s|}{\sqrt{3}} \cos \theta}}{1 - \frac{v_s^2}{3}} \]

- Anisotropic pressure: \[ P\delta_{ij} \to \{ P_\perp, P_\parallel \} \]

- low T approximation: use linear and cubic terms in \( k \) in the dispersions.
- to go (numerically) up to \( T_c \): use 2 particle irreducible formalism (CJT)
finite temperature

• two major difficulties to be resolved:

→ How to define thermodynamics in a relativistic two fluid system?

[I. M. Khalatnikov and V.V. Lebedev, Phys. Lett. 91A, 70 (1982)]
[V.V. Lebedev and I. M. Khalatnikov, Sov. Phys. JETP 56, 923 (1982)]
[N. Andersson and G. Comer, Living Rev. Relativity 10, 1 (2005)]

→ In which frame of reference are the microscopic calculations performed?
Carter`s canonical two fluid formalism
[B. Carter and I. M. Khalatnikov, PRD 45, 4536 (1992)]

• generalization of TD relation (generalized Pressure and Energy):
  \[ \epsilon + P = \mu n + Ts \rightarrow \Lambda + \Psi = \partial \psi \cdot j + \theta \cdot s \]
  \[ d\Lambda = \partial_{\mu} \psi \, dj^{\mu} + \Theta_{\mu} ds^{\mu} \quad d\Psi = j_{\mu} d(\partial^{\mu} \psi) + s_{\mu} d\Theta^{\mu} \quad \partial_{\mu} j^{\mu} = 0 \]
  \[ \Lambda = \Lambda[j^2, s^2, s \cdot j] \quad \Psi = \Psi[(\partial \psi)^2, \theta^2, \partial \psi \cdot \theta] \quad \partial_{\mu} s^{\mu} = 0 \]
  \[ \rightarrow \text{hydro description either in terms of conserved currents or their conjugated momenta!} \]
  (this set of variables differs from \( \partial_{\mu} j^{\mu} \neq 0 \) and \( \partial_{\mu} j^{\mu} \neq 0 \))

• definition of coefficients (A, B, C), A “entrainment coefficient”
  \[ \partial^{\mu} \psi = \frac{\partial \Lambda}{\partial j_{\mu}} = B j^{\mu} + A s^{\mu}, \Theta^{\mu} = \frac{\partial \Lambda}{\partial s_{\mu}} = A j^{\mu} + C s^{\mu} \]

• stress energy tensor:
  \[ T^{\mu\nu} = -g^{\mu\nu} \Psi + j^{\mu} \partial^{\nu} \psi + s^{\mu} \Theta^{\nu} \quad \partial_{\mu} T^{\mu\nu} = 0 \]
  \[ T^{\mu\nu} = -g^{\mu\nu} \Psi + B j^{\mu} j^{\nu} + C s^{\mu} s^{\nu} + A (j^{\mu} s^{\nu} + s^{\mu} j^{\nu}) \]
relativistic two fluid formalism

- relation to formalism of Landau\Khalatnikov:
  [e.g., D. T. Son, Int. J. Mod. Phys. A 16S1C, 1284 (2001)]

\[ T^{\mu\nu} = (\epsilon_n + P_n)u_n^\mu u_n^\nu + P_n g^{\mu\nu} + (\epsilon_s + P_s)u_s^\mu u_s^\nu + P_s g^{\mu\nu} \]

\[ j^\mu = n_n u^\mu + n_s \frac{\partial^\mu \psi}{\sigma} \quad s^\mu = su^\mu \]

- expressible through coefficients:

\[ A = -\frac{\sigma n_n}{s n_s}, \quad B = \frac{\sigma}{n_s}, \quad C = \frac{\sigma n_n^2}{s^2 n_s} + \frac{\mu n_n + s T}{s^2} \]

- the microscopic calculations are performed in the normal fluid restframe (restframe of the heat bath), in this frame, we can identify:

\[ \Psi = \frac{1}{3} (g^{\mu\nu} - u^\mu u^\nu) (j_\mu \partial_\nu \psi - T_{\mu\nu}) \rightarrow \frac{T}{V} \Gamma = \Psi = T_\perp \quad u^\mu = (1, \vec{0}) \]

\[ \Lambda + \Psi = \partial \psi \cdot j + \theta \cdot s \quad s^0 = \frac{\partial \Psi}{\partial T} \rightarrow \theta^0 = T \]
relation to microscopic theory

• coefficients:

\[
A = -\frac{\partial^0 \psi}{s^0 \nabla \psi} \left( \bar{v}_s j^0 \partial^0 \psi - \vec{j} \cdot \nabla \psi \right) \quad B = -\frac{\left( \nabla \psi \right)^2}{\nabla \psi \cdot \vec{j}}
\]

\[
C = \frac{j^0 \partial^0 \psi \left( \bar{v}_s j^0 \partial^0 \psi - \vec{j} \cdot \nabla \psi \right) + s^0 \partial^0 \left( \vec{j} \cdot \nabla \psi \right)}{(s^0)^2 \left( \vec{j} \cdot \nabla \psi \right)}
\]

• some exemplary results:

\[
\frac{T}{V} \Gamma_{eff} \simeq \frac{\mu^4}{4 \lambda} (1 - v_s^2)^2 + \frac{\pi^2 T^4}{10 \sqrt{3}} \frac{(1 - v_s^2)^2}{(1 - 3v_s^2)^2} - \frac{4\pi^4 T^6}{105 \sqrt{3}} \frac{(1 - v_s^2)^2}{(1 - 3v_s^2)^5} (5 + 30v_s^2 + 9v_s^4)
\]

\[
n_s \simeq \frac{\mu^3}{\lambda} (1 - v_s^2) - \frac{4\pi^2 T^4}{5 \sqrt{3} \mu} \frac{1 - v_s^2}{(1 - 3v_s^2)^3} + \frac{8\pi^4 T^6}{105 \sqrt{3}} \frac{1 - v_2^2}{(1 - 3v_s^2)^6} (95 + 243v_s^2 - 135v_s^4 - 27v_s^6)
\]

\[
n_n \simeq \frac{4\pi^2 T^4}{5 \sqrt{3} \mu} \frac{(1 - v_s^2)^2}{(1 - 3v_s^2)^3} - \frac{16\pi^4 T^6}{35 \sqrt{3} \mu^3} \frac{(1 - v_s^2)^2}{(1 - 3v_s^2)^6} (15 + 38v_s^2 - 9v_s^4)
\]
relation to analog gravity (linear dispersion)

relation of sonic metric and superfluidity in the cold limit
[Barcelo, Liberati, Visser; Living Rev. Rel. 8:12, 2005] fluids as laboratory for black hole physics

- sonic metric: \( G^{\mu\nu} = g^{\mu\nu} + \left( \frac{1}{u_s^2} - 1 \right) v_s^\mu v_s^\nu \)

- dispersions: \( G^{\mu\nu} k_\mu k_\nu = 0 \)

\( \Rightarrow \) eliminate \( v_s, T, \mu \):
\[ \sigma^2 = \mu^2 (1 - v_s^2), \quad \partial_\psi \cdot \theta = \mu T - \frac{A}{B} (\nabla \psi)^2 = \frac{1 - v_s^2}{1 - 3v_s^2} \mu T + O(T^3) \]
\[ \theta^2 = T^2 - \frac{A^2}{B^2} (\nabla \psi)^2 = \frac{(1 - v_s^2)(1 - 9v_s^2)}{(1 - 3v_s^2)^2} T^2 + O(T^4) \]
\( \Rightarrow \) \( \Psi(\partial_\psi^2, \theta^2, \partial_\psi \cdot \theta) = \frac{\sigma^4}{4\lambda} + \frac{\pi^2}{90\sqrt{3}} \left[ \theta^2 + 2 \frac{(\partial_\psi \cdot \theta)^2}{\sigma^2} \right]^2 = \frac{\sigma^4}{4\lambda} + \frac{\pi^2}{90\sqrt{3}} [G^{\mu\nu} \theta_\mu \theta_\nu]^2 \)
first and second sound

- two fluid system allows for two sound modes:

  - **first sound**: normal and super fluid densities oscillate in phase, pressure wave, weak temperature dependence

  - **second sound**: normal and super fluid densities oscillate out of phase, temperature wave

figure: [R.J. Donnelly, Physics Today 62, 10 (2009)]
first and second sound- limit cases

- the general form of the 2 equations which determine the sound velocities is complicated, discussion of limit cases:

$\rightarrow T=0:$

\[ 0 \simeq (g^{\mu\nu} + 2v_s^{\mu}v_s^{\nu}) \partial_\mu \partial_\nu \delta\mu(\vec{x}, t) = G^{\mu\nu} k_\mu k_\nu \delta\mu(\vec{x}, t) \]

we recover the term linear in $k$ of the dispersion relation:

\[ \epsilon_k = u_1 k + O(k^3) \]

$\rightarrow v_s = 0:$

\[ 0 = w \left( \partial_s \frac{\partial n}{\partial \mu} - \frac{\partial n}{\partial \mu} \frac{\partial s}{\partial T} \right) \omega^2 - n_s s^2 \vec{k}^4 \]

\[ + \left[ s^2 \mu \frac{\partial n}{\partial \mu} + (\mu n_n^2 + wn_s) \frac{\partial s}{\partial T} - 2\mu sn_n \frac{\partial s}{\partial \mu} \right] \omega^2 \vec{k}^2 \]

we recover the (velocity independent) results for first and second sound given for example in: [Herzog, Kovtun, Son; Phys.Rev. D79, 066002 (2009)]
first and second sound: full results

\[ T = 0 \]

\[ T/\mu = 0.02 \]

\[ T/\mu = 0.04 \]
what we have discussed:

- The microscopic physics of CFL and kaon condensation can be translated into a multifluid hydrodynamic system.

- We provide a translation in between existing descriptions of superfluid hydrodynamics.

- We show how the hydro parameters emerge from an underlying microscopic field theory.

- We study the (anisotropic) first and second sound modes.
what is left to do:

- towards a complete hydro description of CFL and kaon condensation:

  - use the CJT formalism to numerically go up to $T_c$
    [work in progress; Mark G. Alford, S. Kumar Mallavarapu, Andreas Schmitt, Stephan Stetina]
  - include effect of weak interactions (include a $U(1)_s$ breaking term into the effective Lagrangian)
    [work in progress; Denis Parganlija, Andreas Schmitt]
  - repeat the analysis for a fermionic system of Cooper pairs
Thank you!