Differentiating between Δ and Y-string confinement: Can one see the difference in baryon spectra?

Excited QCD2013,
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Outline:

- **Strings in QCD and baryon spectroscopy**
- **The Δ and Y-string potentials in hyperspherical coordinates**
- **Dynamical O(2) symmetry of the Y-string**
- **Low-lying baryon spectra in 3 dimensions**
- **Complete algebraization of the 3-body problem and spectra in 2 dimensions**
- **Conclusions, Outlook, Addenda and Curiosum**

QCD flux-tubes in baryons

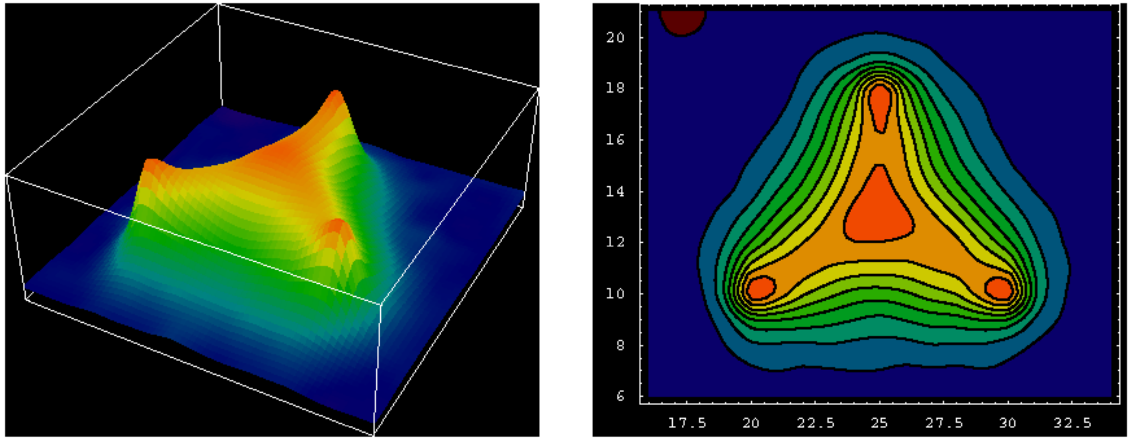
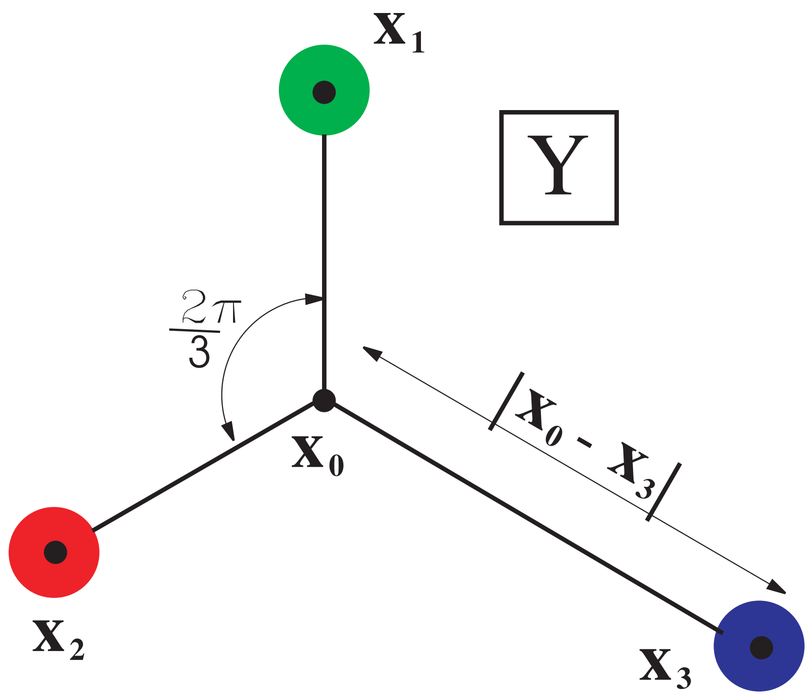


Figure 2. The flux-tube profile in the spatially-fixed 3Q system, in the MA projected QCD.[6] The distance between the junction and each quark is about 0.5 fm.

- Color flux-tube profiles from Takahashi, Ichie and Suganuma, (“Wako 2003, Color confinement and hadrons in quantum chromodynamics” p. 470-474)
- Support from lattice QCD Takahashi,Matsufuru, Nemoto and Suganuma, PRL86,18(’01); PRD65, 11409 (’02)

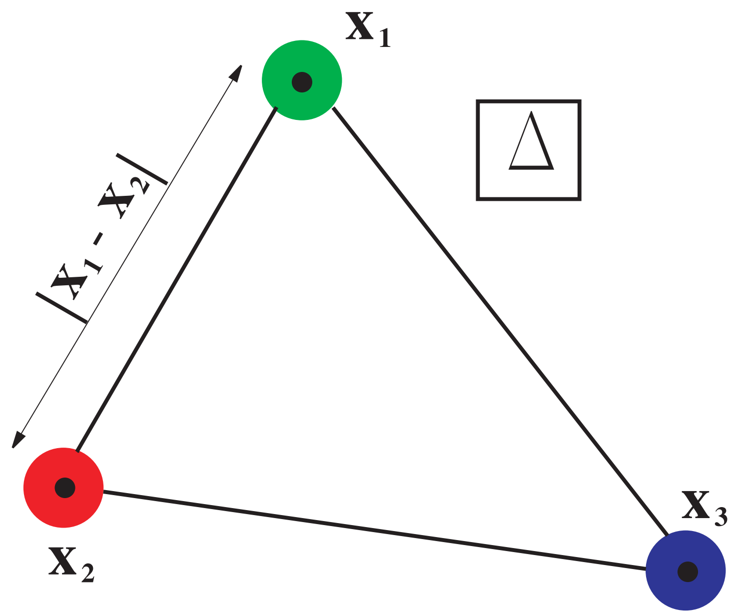
The Y-string

- Defined as the shortest sum of string lengths; this means that the strings pointing to the three quarks form 120 degree angles at the juncture (Fermat-Torricelli-Steiner point)
- Support from lattice QCD Takahashi,Matsufuru, Nemoto and Suganuma, PRL86,18('01); PRD65, 11409 ('02).



The Δ-string

- Sum of two-body potentials
- Lattice QCD support from Alexandrou, deForcrand Tsapalis, PRD65, 054503 ('02)
- Bonn group (Metsch-Petry) EJPA 10 ('01) claim equivalence of Δ and Y-strings !?!
- Can one distinguish between these two kinds of string potentials using (only) the baryon spectra?



The Y *vs.* Δ-string?

- Bonn group (Metsch, Petry et al.) EJPA 10 ('01) claim equivalence of Δ and Y-strings !?!
- Can one distinguish between these two kinds of string potentials using (only) the baryon spectra?
- We shall show that, yes, there are clear differences, but only at K=3 and higher.
- These differences are related to the dynamical symmetry of the Y-string.

We would like to note at this stage that we have tested the various radial dependencies (3), (4) and (6) in our Salpeter model. Our investigations, however, clearly showed that the structure of resulting spectra depends only slightly on the various radial dependencies chosen. It turned out that the slope parameter b can always be appropriately rescaled (as, *e.g.*, in eq. (5) with the factor f) to obtain almost the same spectrum for all three choices. We therefore prefer for our model the Δ-shape string potential rising linearly with $r_\Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \sum_{i<j} |\mathbf{x}_i - \mathbf{x}_j|$ which, on the one hand, is favored by the most recent lattice studies anyway and, on the other hand, is also much easier to handle numerically. We found, however, that the structure of the resulting spectra depends much more on the Dirac structure chosen, which we shall consider next.

The gross features of the baryon resonances seem to

The Y-string potential

- The minimal Y-string length (potential) (Suganuma&Takahashi) contains two square-roots: classical e.o.m. are feasible, but
- How do you solve this problem in quantum mechanics?

$$L_{\min}=\left[\frac{1}{2}(a^2+b^2+c^2)+\frac{\sqrt{3}}{2}\times\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\right]^{1/2},$$

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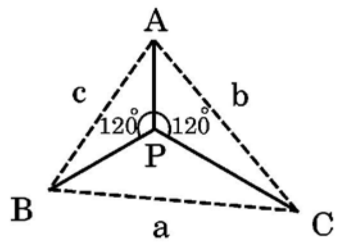


FIG. 1. The flux-tube configuration of the 3Q system with the minimal value of the total flux-tube length. There appears a physical junction linking the three flux tubes at the Fermat point P.

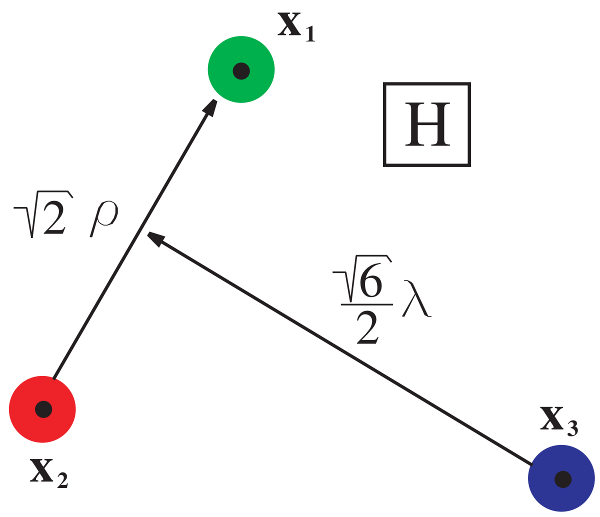
Jacobi and hyper-spherical variables

- Jacobi vectors $(\vec{\rho}, \vec{\lambda})$
- The hyper-radius R and hyper-angles

$$R^2 = \vec{\rho}^{\,2} + \vec{\lambda}^2$$

$$2\chi = \tan^{-1}\left(\frac{2\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^{\,2} - \vec{\lambda}^2}\right)$$

$$\theta = \cos^{-1}\left(\frac{\vec{\rho}\cdot\vec{\lambda}}{\rho\lambda}\right)$$



$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)$$

The Y-string in hyperspherical coordinates

- The Y-string potential in the hyperspherical coordinates

$$V_Y(R,\chi,\theta) = \sigma R\sqrt{\frac{3}{2}\left(1+\sin(2\chi)\left|\sin\theta\right|\right)}$$

- It is linear in the hyper-radius R, with a dependence on both hyperangles
- Solve the Schrodinger eq. with linear hyper-radial potential and this hyperangular dependence:

Schroedinger eqn. in hyperspherical coordinates

$$\Psi = \frac{1}{\rho^{5/2}} \sum_{K\gamma} Y^{l_x l_y}_{KLM}(\Omega^{\kappa}_5) \sum_{K'\gamma'} Y^{l'_x l'_y}_{K'L'M'}(\Omega_5)\, f_{K\gamma,K'\gamma'}(\kappa\rho)$$

$$K = l_x + l_y + 2\nu$$

$$\left\langle Y^{l_x l_y}_{KL}(\Omega_5) \middle| \sum_{i=1}^{3} \sum_{j>i}^{3} V_{ij}(\rho,\Omega_5) \middle| Y^{l''_x l''_y}_{K''L''}(\Omega_5) \right\rangle$$

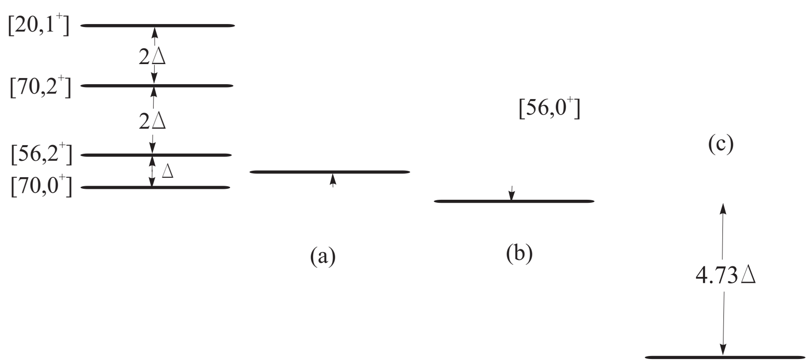
$$\left(-\frac{\hbar^2}{2m} \left[\frac{d^2}{d\rho^2} + \frac{(K+3/2)(K+5/2)}{\rho^2} \right] - E \right) f_{K\gamma,K'\gamma'}(\rho) = -\sum_{K''\gamma''} V_{K''\gamma'',K\gamma}(\rho) f_{K''\gamma'',K'\gamma'}(\rho)$$

- Solve the hyper-radial eqn with linear potential for various values of “grand angular momentum” K=0,1,2

Rescaled Δ vs. Y- string results

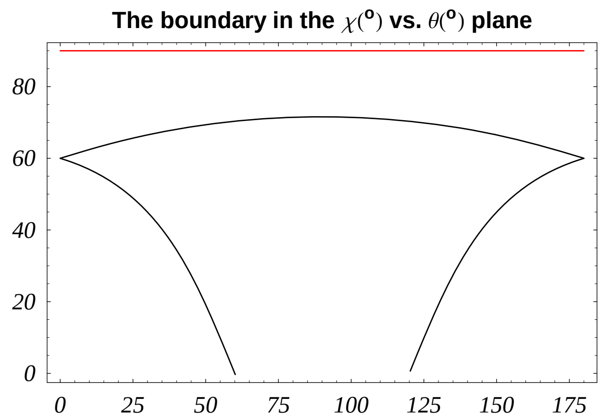
- We did the integrals numerically and solved the Schrodinger eq. for K=0,1,2 (Eur. J. Phys. C 62, 383 (2009))
- Compare with the rescaled Δ string potential (J.M. Richard and Taxil, NPB 329, 310 (1990)):
- The three lowest states agree exactly! (agrees with Bonn group's claim)
- The splittings among the N=2 states have similar ratios (why?), but not the same absolute values: the Bonn group are approximately right!
- The Y-string K=3,4,… states not solved for, yet.

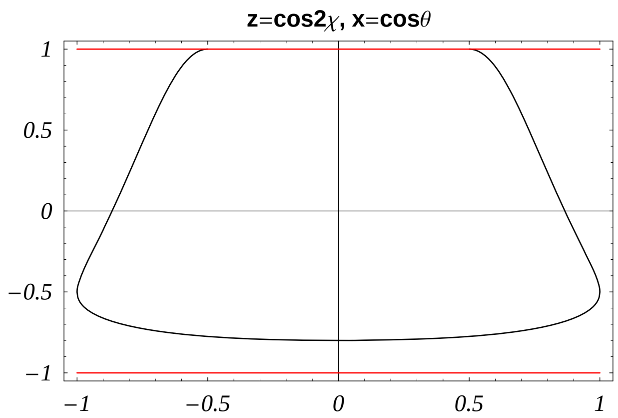
K	N_K	$[SU(6), L^P]$	$E^{(Y)}_{N_K,K,L}$	$f^{2/3}E^{(\Delta)}_{N_K,K,L}$
0	0	$[56, 0^+]$	5.1761	5.1761
1	0	$[70, 1^-]$	6.3160	6.3160
0	1	$[56, 0^+]$	7.1360	7.1360
2	0	$[70, 0^+]$	7.1733	7.2832
2	0	$[56, 2^+]$	7.2437	7.3143
2	0	$[70, 2^+]$	7.3968	7.3767
2	0	$[20, 1^+]$	7.5550	7.4390



The Y-string hyperangular potential

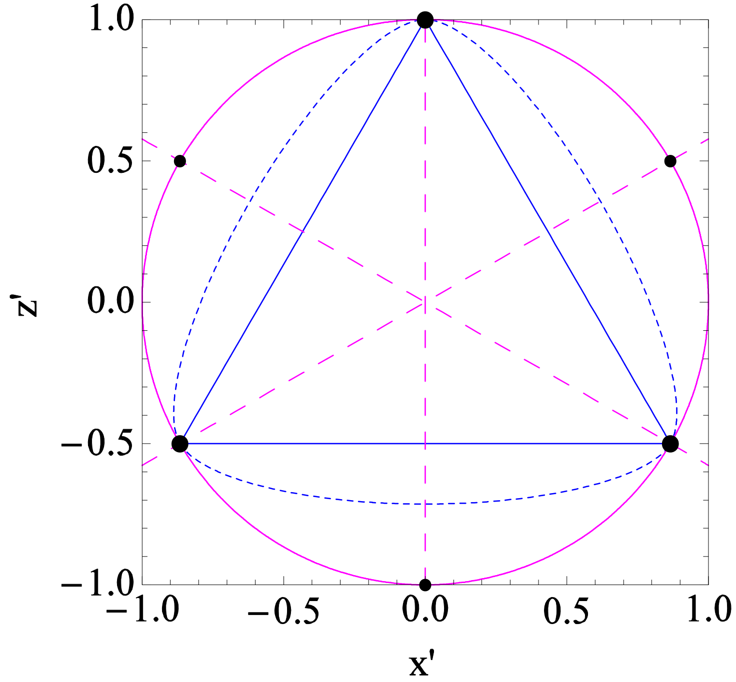
- The exact Y-string potential contains 4 parts: the Y-string is “in the middle” and 3 different forms “in the corners”, the boundary between the three regions is the black line.
- Note the absence of manifest permutation symmetry.
- Cosines of hyper-angles make the boundaries convex, but still not permutation symmetric. Change variables?





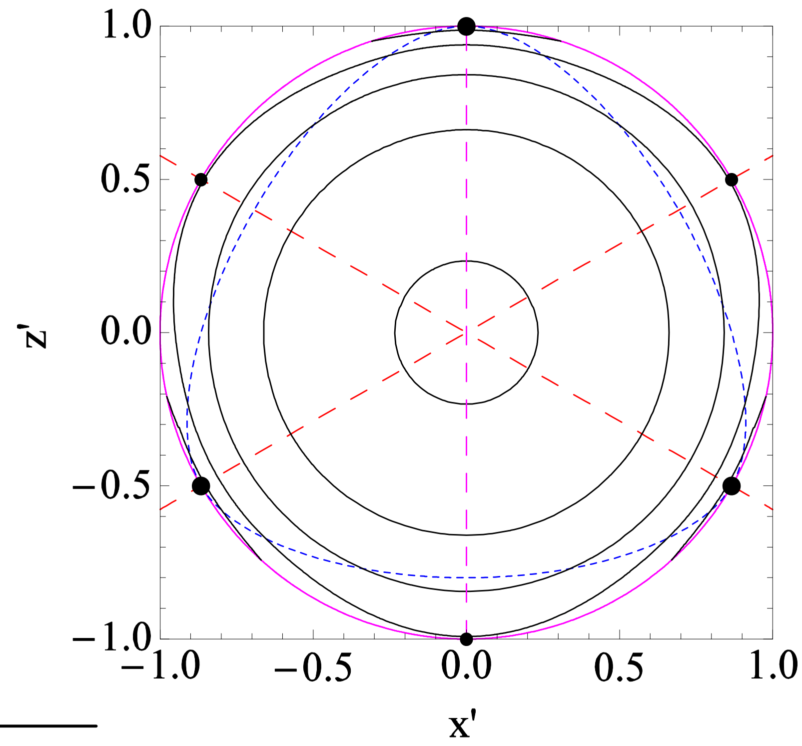
Permutation-symmetric hyper-angles

- Define the new (permutation-symmetric) hyper-angles
$$z' = z, x' = x\sqrt{1-z^2}$$
- The defining domain changes from the square into a circle (solid pink)
- The convex boundary between the three regions is closed.
- The discrete symmetry group consisting of three reflections (about the vertical (solid black) and the two slanted (red dashes) axes) and two rotations through $2\pi/3$ that correspond to the permutation group S3 of three quarks.



The Y-string in terms of new hyper-angles

- The contour plot of the Y-string potential consists of concentric circles (solid black)
- The Y-string potential is axially symmetric under rotations: not a function of the new hyperangle ϕ



$$V_Y(R,\alpha,\phi) = \sigma R \sqrt{\frac{3}{2}\left(1+\left|\cos\alpha\right|\right)}$$

Dynamical O(2) symmetry of the Y-string

- Consequently there is a new constant of motion G ("hyper-angular momentum") associated with "hyper-rotations" in the $(\vec{\rho},\vec{\lambda})$ plane
- This constant G is the conjugate momentum to the new hyper-angle ϕ

$$G = \boldsymbol{\lambda} \cdot \mathbf{p}_\rho - \boldsymbol{\rho} \cdot \mathbf{p}_\lambda,$$

$$\begin{aligned} \delta\boldsymbol{\rho} &= \varepsilon\boldsymbol{\lambda} \\ \delta\boldsymbol{\lambda} &= -\varepsilon\boldsymbol{\rho}. \end{aligned}$$

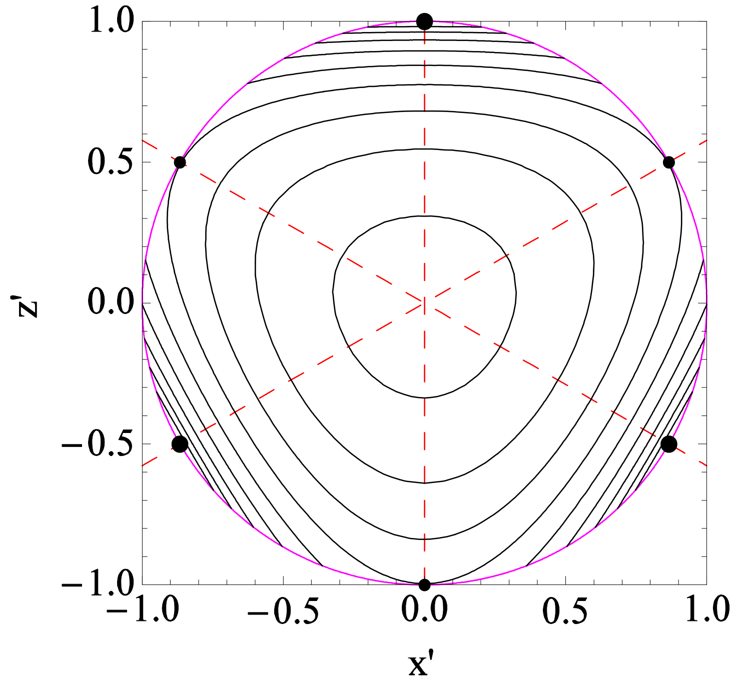
$$G = \frac{L}{2}\cos\alpha - \frac{m}{4}\left(R\sin\alpha\right)^2\dot{\phi}$$

$$\phi = \tan^{-1}\left(\frac{2\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^{\,2}-\vec{\lambda}^2}\right)$$

$$\alpha = \cos^{-1}\left(\frac{2\left(\vec{\rho}\times\vec{\lambda}\right)_z}{\vec{\rho}^{\,2}+\vec{\lambda}^2}\right)$$

The Δ-string potential in terms of new hyper-angles

- This O(2) symmetry is not shared by sums of two-body potentials, like the Δ string.
- The Δ-string potential has a different form in the new permutation-symmetric hyper-angles
- Other (sums of) two-body potentials have similar contour plots
- There must be a difference between the Δ and Y-strings, but where in the spectrum is it?

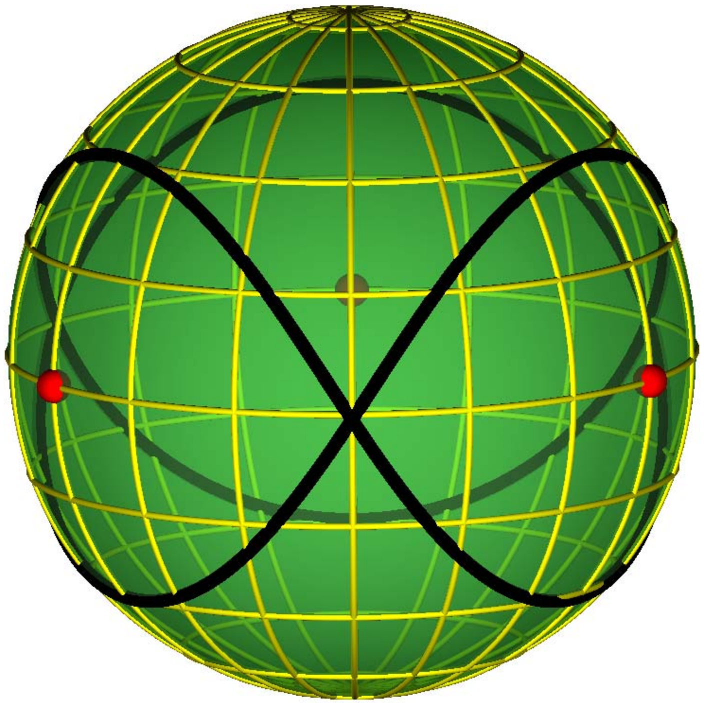


The shape space sphere

$$X = \left(\frac{2\vec{\rho}\cdot\vec{\lambda}}{\vec{\rho}^{\,2}+\vec{\lambda}^{\,2}} \right)$$

$$Y = \left(\frac{\vec{\rho}^{\,2}-\vec{\lambda}^{\,2}}{\vec{\rho}^{\,2}+\vec{\lambda}^{\,2}} \right)$$

$$Z = \left(\frac{2\left(\vec{\rho}\times\vec{\lambda}\right)_z}{\vec{\rho}^{\,2}+\vec{\lambda}^{\,2}} \right)$$



- Define a unit sphere with (X,Y,Z) coordinates
- What we showed before was a projection from infinity “above” the North Pole.
- Red points correspond to 2-body collisions

The O(4) approach to the three-body problem in 2 dimensions

- The effective potential is

$$V_{\text{eff.}}(R)C_{[K'],[K]} = V(R)\left(\delta_{[K'],[K]}\frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}} + \sqrt{\frac{\pi}{2}}\sum_{J>0,M}^{\infty} v_{JM}^{3-\text{body}}\langle \mathcal{Y}_{[K']}(\alpha,\phi,\Phi)|\mathcal{Y}_{0M}^{J}(\alpha,\phi,\Phi)|\mathcal{Y}_{[K]}(\alpha,\phi,\Phi)\rangle\right) \quad (13)$$

- It is entirely determined by ordinary O(3) spherical harmonics expansion coefficients v_{JM} and O(4) hyper-spherical harmonics matrix elements

$$V_{\Delta}(R,\alpha,\phi) = V_{\Delta}(R)V_{\Delta}(\alpha,\phi) = V_{\Delta}(R)\sum_{J,M}^{\infty} v_{JM}^{\Delta} Y_{JM}(\alpha,\phi)$$

$$v_{JM}^{\Delta} = \int_0^{2\pi} d\phi \int_0^{\pi} V_{\Delta}(\alpha,\phi)\, Y_{JM}^{*}(\alpha,\phi)\sin(\alpha)\, d\alpha.$$

- The Δ and Y-string potentials have different dynamical symmetries in the shape space. The Y-string does not depend on hyperangle ϕ so it has an "extra" O(2) symmetry.

The O(4) approach to the three-body problem in 2 dimensions

- O(4) hyper-angular matrix elements can be reduced to O(4)=O(3)xO(3) C.G. coefficients times the potential expansion coefficients

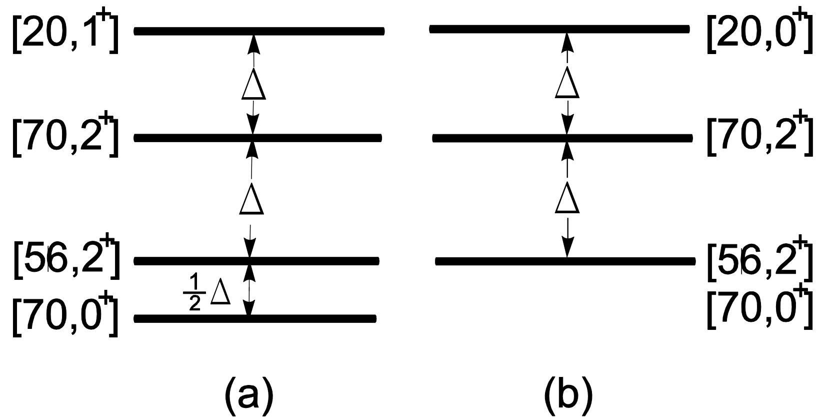
$$C_{[K'],[K]} = \delta_{[K'],[K]} + \sum_{J>0,M}^{\infty} \left(\frac{v_{JM}^{3-\mathrm{body}}}{v_{00}^{3-\mathrm{body}}} \right) \sqrt{\frac{(K'+1)(2J+1)}{(K+1)}} \, C_{J0,\frac{K'}{2}\frac{L'}{2}}^{\frac{K}{2}\frac{L}{2}} \, C_{JM,\frac{K'}{2}G_3'}^{\frac{K}{2}G_3}$$

- The hyper-radial equation with linear potential yields the Airy functions: problem solved !
- Look at the spectra: splitting of K-bands
- K=0,1 trivial – only one SU(6) multiplet
- K=2,3 interesting

K	a_0	a_1	a_2
0	3.82	5.26	6.54
1	4.66	5.99	7.19
2	5.43	6.67	7.81

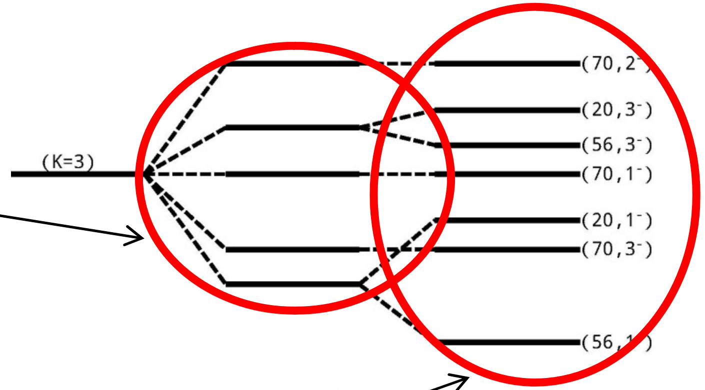
The O(4) algebra of the three-quark confinement in 2 dimensions

- There must be a difference between the Δ and Y-string spectra, but where? Compare 2D and 3D results for the Δ-string!
- At K=2 there is only one (minor) difference: [56,2] and [70,0] are degenerate in 2D

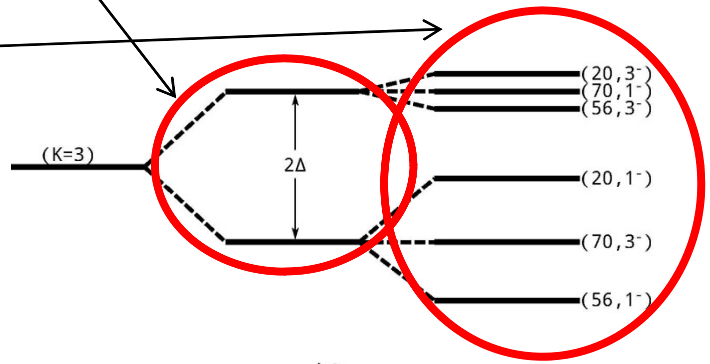


K=3 band of states in 2 and 3 dimensions

- At K=3 there are more differences between 2D and 3D, but the patterns are still (very) similar: with only v_{20} non-zero coefficient, i.e. for the Y-string
- The [70,1] state is not degenerate with [20,3] and [56,3] in 3D, nor is [70,3] degenerate with [20,1] and [56,1]
- with both v_{20}, v_{33} non-zero coefficients, i.e. for the Δ-string there is complete lifting of all degeneracies



a)



b)

Summary

- The Y-string potential has a hidden dynamical O(2) symmetry that distinguishes it from the Δ string
- This is the first systematic study of the O(2) symmetry in K=0,1,2,3 band states with the Y-string potential in 2D baryons.
- We compared with known 3D results: many similarities indicate importance of O(2) dynamical symmetry in 3D.

Igor Salom

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Publications

- **V.D., T. Sato and M. Šuvakov, Eur. J. Phys. C 62, 383 (2009)**
- **V.D., T. Sato and M. Šuvakov, Phys. Rev. D 80, 054501 (2009)**
- **M. Šuvakov and V.D., Phys. Rev. E 83, 056603 (2011)**
- **I. Salom and V.D., submitted to Eur. J. Phys. C (2013)**

Supplementary slides

Open problems

- O(6) algebraic approach to 3D three-body problem?
- Connection with the “collective” quark models, a la Bijker, Iachello & Leviatan?
- Could the confinement be due to a purely area-dependent three-body potential? (so as to lower the Roper mass) [Takahashi-Hosaka-Toki (1999)]
- Or to a sum of two- and three-body potentials ? (d-coupling SU(3) color dependence of the Y-string potential for color saturation)
- The Y-string Luescher term?

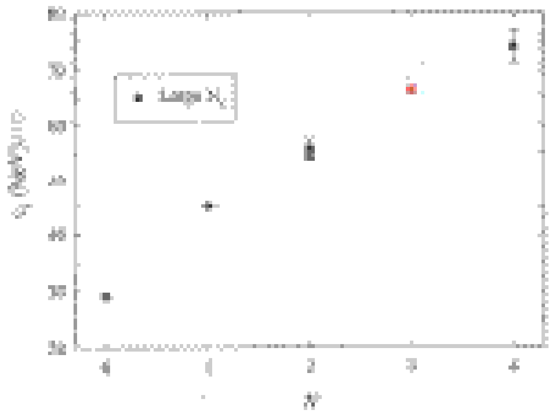
Large N_c expansion for baryons

- What do we compare our results with? There are no baryons w/o strong HFIs.
- The large N_c analysis yields the leading-N_c coefficient c_1 related to the baryon masses
- This is an (almost) straight line! Harmonic oscillator?!?
- Must look more carefully: no N_c^0 term: the splitting of states with equal N=K was ignored!
- This is tacit assumption of hyper-radial confining potential

$$M = c_1 N_c + c_4 \frac{1}{N_c} S^2 + \mathcal{O}\left(\frac{1}{N_c^2}\right)$$

Fit to N and Δ: c_1 = 289 MeV, c_4 = 292 MeV

c_1 ---> free mass term + kinetic + confinement



Large Nc expansion for baryons

Large N_c operator analysis. Observables:

$$\mathcal{O} = \sum_i c_i \, O_i + \sum_i b_i \, \overline{B}_i$$

An n-body operator acts on n quark lines

$$O_i = \frac{1}{N_c^{n-1}} O_\ell^{(k)} \cdot O_{SF}^{(k)}$$

Generators of O(3) (ℓ_i) and of SU(6) (S_i, T_a, G_{ia})

c_i, b_i **: reduced matrix elements which encode QCD dynamics, fitted from data**

N. B. Matrix elements of O_{SF} can carry nontrivial N_c dependence,
$\overline{B}_i$ are SU(3)- flavour breaking

Widths: Generic large N_c counting rules $\Gamma \sim O(N_c^0)$

The $[56,4^+]$ baryon masses in the $1/N_c$ expansion

N.Matagne and F.S. PRD71:014010(2005)

$$M = \sum_{i=1}^{3} c_i O_i + \sum_{i=1}^{3} b_i \bar{B}_i$$

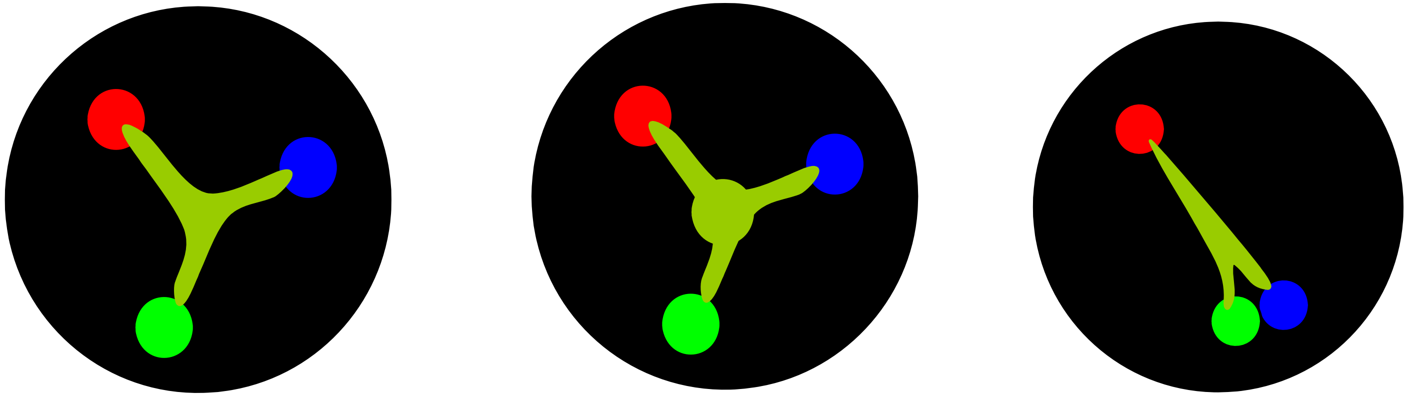
N.B. $\langle \bar{B}_i \rangle = 0$ for nonstrange baryons. Fit to 5 masses.

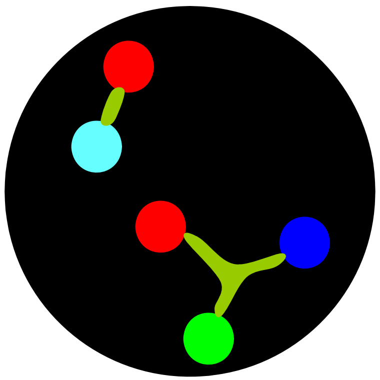
Operator	Fitted coef. (MeV)			
$O_1 = N_c \mathbb{1}$	$c_1 =$	736	$\pm$	30
$O_2 = \frac{1}{N_c} l_i S_i$	$c_2 =$	4	$\pm$	40
$O_3 = \frac{1}{N_c} S_i S_i$	$c_3 =$	135	$\pm$	90
$\bar{B}_1 = -\mathcal{S}$	$b_1 =$	110	$\pm$	67
$\bar{B}_2 = \frac{1}{N_c} l_i G_{i8} - \frac{1}{2\sqrt{3}} O_2$				
$\bar{B}_3 = \frac{1}{N_c} S_i G_{i8} - \frac{1}{2\sqrt{3}} O_3$				

$\chi^2_{\rm dof} = 0.26$ $\qquad \frac{\langle O_2 \rangle}{\langle O_3 \rangle} = \frac{\langle \bar{B}_2 \rangle}{\langle \bar{B}_3 \rangle}$ for all states

$$N_c^0$$

Strings as the confinement mechanism?

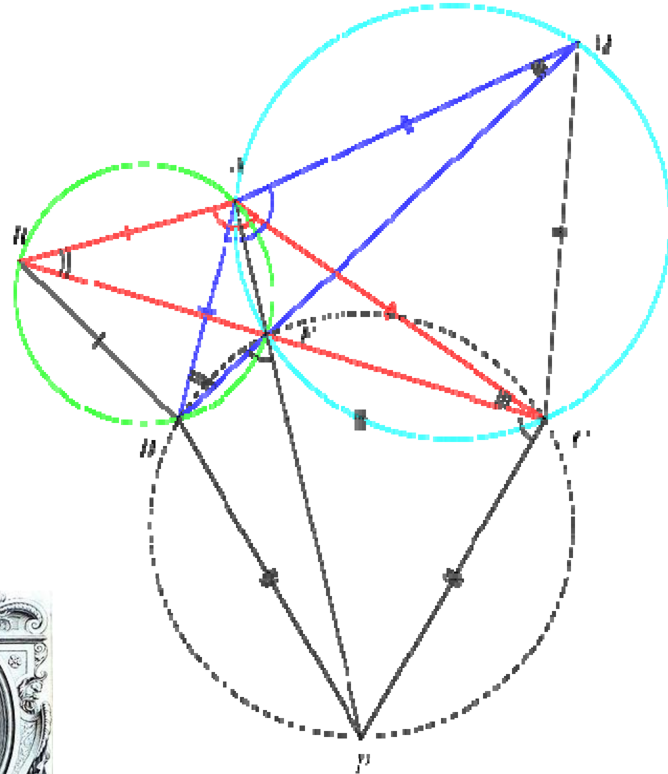




$v^{\Delta}_{3\pm 3} = 0.141232$ ir

Construction of the minimal Y-string

- "Simson's line" geometric construction allows straightforward evaluation of the (minimal) string length without determining the Fermat-Torricelli-Steiner point explicitly!



The Y-string potential II

- Rewrite in terms of Jacobi coordinates $(\vec{\rho}, \vec{\lambda})$
- Three "special" cases:
- At first take the first term only (the "true" Y-string potential)

$$
\begin{aligned}
V_{\mathrm{Y}} &= \sigma\sqrt{\frac{3}{2}(\rho^2+\boldsymbol{\lambda}^2+2|\boldsymbol{\rho}\times\boldsymbol{\lambda}|)},\\
&\quad \text{when} \quad 2\rho^2-\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\rho^2+3\boldsymbol{\lambda}^2-2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}\\
&\quad \text{and} \quad 2\rho^2+\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\rho^2+3\boldsymbol{\lambda}^2+2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}\\
&\quad \text{and} \quad 3\boldsymbol{\lambda}^2-\rho^2 \geq -\frac{1}{2}\sqrt{(\rho^2+3\boldsymbol{\lambda}^2)^2-12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2}
\end{aligned}
$$

$$
\begin{aligned}
V_V &= \sigma\left(\sqrt{\frac{1}{2}(\rho^2+3\boldsymbol{\lambda}^2+2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}+\sqrt{\frac{1}{2}(\rho^2+3\boldsymbol{\lambda}^2-2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right)\\
&\quad \text{when} \quad 3\boldsymbol{\lambda}^2-\rho^2 \leq -\frac{1}{2}\sqrt{(\rho^2+3\boldsymbol{\lambda}^2)^2-12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2}
\end{aligned}
$$

$$
\begin{aligned}
V_V &= \sigma\left(\sqrt{2}\rho+\sqrt{\frac{1}{2}(\rho^2+3\boldsymbol{\lambda}^2+2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right)\\
&\quad \text{when} \quad 2\rho^2+\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\rho^2+3\boldsymbol{\lambda}^2+2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}
\end{aligned}
$$

$$
\begin{aligned}
V_V &= \sigma\left(\sqrt{2}\rho+\sqrt{\frac{1}{2}(\rho^2+3\boldsymbol{\lambda}^2-2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda})}\right)\\
&\quad \text{when} \quad 2\rho^2-\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \leq -\rho\sqrt{\rho^2+3\boldsymbol{\lambda}^2-2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}
\end{aligned}
$$

Some analytic results

- Capstick and Isgur ('86) reduced this 4-fold integral to a sum over CG and Racah cofficients and one last (numerical) integral
- we find numerical agreement with conjectured exact result to at least five decimal places; the challenge is to prove the exact result.

where $|LM\rangle$ is the orbital angular momentum of both $|\alpha\rangle$ and $|\beta\rangle$, and we have written all Clebsch-Gordan coefficients in terms of the 3-j coefficients for clarity. The next step is to identify the sum over the magnetic quantum numbers as a Racah coefficient and restore the radial integrations which results in (after some algebra)

$$\begin{aligned}\langle\alpha|V_{3b}|\beta\rangle = {} & \delta_{S_\alpha S_\beta}\delta_{L_\alpha L_\beta}[(2l_{\rho_\alpha}+1)(2l_{\rho_\beta}+1)(2l_{\lambda_\alpha}+1)(2l_{\lambda_\beta}+1)]^{1/2}(-1)^{L+l_{\lambda_\alpha}+l_{\lambda_\beta}} \\ & \times\sum_{L_\rho} C(l_{\rho_\alpha}0l_{\rho_\beta}0;L_\rho 0)C(l_{\lambda_\alpha}0l_{\lambda_\beta}0;L_\rho 0)W(l_{\rho_\alpha}l_{\rho_\beta}l_{\lambda_\alpha}l_{\lambda_\beta};L_\rho L) \\ & \times\langle n_{\rho_\alpha}l_{\rho_\alpha}|\langle n_{\lambda_\alpha}l_{\lambda_\alpha}|V_{L_\rho}(\rho,\lambda)|n_{\lambda_\beta}l_{\lambda_\beta}\rangle|n_{\rho_\beta}l_{\rho_\beta}\rangle\ ,\end{aligned} \tag{B30}$$

where the last line is the obvious generalization of the radial integral in (B2). The $\cos\theta$ integrals in (B28) and the ρ and λ integrals in (B30) can be carried out numerically.

Summary of 3D Results

- The difference between the highest and lowest states' energies is reduced as we improve the approximation: this series of approximations converges
- Compare with the Delta string potential (J.M. Richard and Taxil, NPB 329, 310 (1990): large discrepancy even between the lowest states!
- Need to rescale the string tension to compare with Delta string

K	N_K	$[SU(6), L^P]$	$E^{(0)}_{N_K,K}$	$E^{(1)}_{N_K,K,L}$	$E^{(2)}_{N_K,K,L}$	$E^{(Y)}_{N_K,K,L}$	$E^{(\Delta)}_{N_K,K,L}$
0	0	$[56, 0^+]$	3.8175	4.6658	4.5182	4.5218	5.3592
1	0	$[70, 1^-]$	4.6582	5.6934	5.5132	5.5176	6.5395
0	1	$[56, 0^+]$	5.2630	6.4326	6.2290	6.2340	7.3885
2	0	$[70, 0^+]$	5.4290	6.3942	6.2493	6.2665	7.5409
2	0	$[56, 2^+]$	5.4290	6.4907	6.3199	6.3279	7.5731
2	0	$[70, 2^+]$	5.4290	6.6837	6.4604	6.4617	7.6377
2	0	$[20, 1^+]$	5.4290	6.8767	6.5993	6.5999	7.7022

The O(4) algebra of the three-body problem in 2 dimensions

- The 2D n.r. kinetic term has an O(4) symmetry.

$$T = \frac{m}{2}\left(\dot{\rho}^2 + \dot{\boldsymbol{\lambda}}^2\right) = \frac{m}{2}\left(\dot{\boldsymbol{X}}_\mu\right)^2 = \frac{1}{2m}\left(\mathrm{p}_\rho^2 + \mathrm{p}_\lambda^2\right) = \frac{1}{2m}\left(\boldsymbol{P}_\mu\right)^2 \ .$$

- It can be written as a function of hyper-radial kin. en. and the “grand angular momentum” tensor squared

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}$$

$$\begin{aligned} K_{\mu\nu} &= m\left(\mathbf{X}_\mu \dot{\mathbf{X}}_\nu - \mathbf{X}_\nu \dot{\mathbf{X}}_\mu\right) \\ &= \left(\mathbf{X}_\mu \mathbf{P}_\nu - \mathbf{X}_\nu \mathbf{P}_\mu\right) \end{aligned}$$

- The 3-body problem in 2D can be completely O(4) “algebrized”; i.e. reduced to a set of hyper-radial Schrodinger equation with a different effective potential for each state

Shape sphere expansion coefficients

- O(3) spherical harmonics expansion coefficients v_{JM} for the Y and Δ strings

$$\sum_{JM}^{\infty} v^{\Delta}_{JM} Y_{JM}(\alpha,\phi) = \sum_{J=0,2,\ldots}^{\infty} v^{\Delta}_{J0} Y_{J0}(\alpha,\phi) + \sum_{J,M=\pm 3}^{\infty} v^{\Delta}_{JM} Y_{JM}(\alpha,\phi) + \sum_{J,M=\pm 6}^{\infty} v^{\Delta}_{JM} Y_{JM}(\alpha,\phi) + \ldots$$

$$\sum_{J,M=\pm 3}^{\infty} v^{\Delta}_{JM} Y_{JM}(\alpha,\phi) = \sum_{J=3,5,\ldots}^{\infty} v^{\Delta}_{J\pm 3} Y_{J\pm 3}(\alpha,\phi).$$

$$v^{\Delta}_{3\pm 3} = 0.141232 \text{ ir}$$

a_n^Y	a_n^Y	$\sqrt{\frac{3}{2}} v_n^Y$
$\frac{2}{3}\left(-1+2\sqrt{2}\right)$	1.21895	5.29221
$\frac{1}{105}\left(11-4\sqrt{2}\right)$	0.0508871	0.494019
$-\frac{1}{2772}\left(5+16\sqrt{2}\right)$	-0.0099666	-0.129813
$\frac{1}{17160}\left(111-32\sqrt{2}\right)$	0.0038313	0.0599748
$-\frac{1}{310080}\left(237+256\sqrt{2}\right)$	-0.00193188	-0.0345825
$\frac{1}{1174656}\left(2053-512\sqrt{2}\right)$	0.00113133	0.0225086

n	v_n^{Δ}	$\sqrt{\frac{3}{2}} v_n^Y$
0	10.0265	5.29221
2	0.320285	0.494019
4	0.232132	-0.129813
6	0.0158003	0.0599748
8	-0.00699939	-0.0345825
10	0.00369641	0.0225086