Simulation of Gauge-Higgs models using the worm algorithm

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Der Wissenschaftsfonds.

Motivation: Sign problem of QCD

- Expectation value of observables:

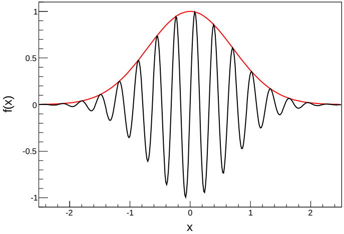
$$\langle O \rangle = \frac{1}{Z} \int D[\phi, \overline{\phi}, U] \; O[U, \phi, \overline{\phi}] \; e^{-S[\phi,\overline{\phi},U]}$$

- Use Monte-Carlo method.
  Generate configurations with probability:

$$\frac{1}{Z} e^{-S[\phi,\overline{\phi},U]}$$

- At finite density: $e^{-S(\mu)}$ is complex for $\mu > 0$

$$e^{-S(\mu)} = |e^{-S(\mu)}| e^{i\theta} \quad \Longrightarrow \quad \text{sign problem}$$



Motivation: Sign problem of QCD

- Way out:
  - Taylor expansion in terms of μ/T (not exact).
  - Complex Langevin (exact). [G. Aarts, F.A. James]
  - Rewrite the partion sum using new variables: **Dual representation** (exact).

P. de Forcrand (2010) (review LQCD and sign problem)

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- Way out:
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  - Rewrite the partion sum using new variables: **Dual representation** (exact).

- Test new methods using full QCD is too complicated!!

- Use “toy models”:
  - SU(3) spin model (last excited QCD) [Y. D., C. Gattringer (2012)]
  - Relativistic Bose gas [C. Gattringer, T. Kloiber (2013)]
  - $\mathbb{Z}_3$ gauge-Higgs model [C. Gattringer, A. Schmidt (2012)]
  - U(1) gauge-Higgs model (this talk) [Y .D., C. Gattringer, A. Schmidt (2013)

P. de Forcrand (2010) (review LQCD and sign problem)

- In the continuum

$$S = \int d^4x \{-\phi(x)^*[\partial_\nu + iA_\nu(x)][\partial_\nu + iA_\nu(x)]\phi(x) + [m^2 - \mu^2]|\phi(x)|^2\} + i\mu N + \int d^4x \frac{1}{4} F_{\mu\nu}F_{\mu\nu}$$

- On the lattice

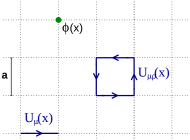
$$S_G = -\frac{\beta}{2}\sum_x \sum_{\nu<\rho} [U_{x,\nu\rho} + U^*_{x,\nu\rho}]$$

$$S_H = \kappa \sum_x |\phi_x|^2 - \sum_{x,\nu} [e^{-\mu\delta_{\nu 4}} \phi^*_x U_{x,\nu}\phi_{x+\hat{\nu}} + e^{\mu\delta_{\nu 4}} \phi^*_x U^*_{x-\hat{\nu},\nu}\phi_{x+\hat{\nu}}]$$

$$\phi_x \in \mathbb{C}$$

$$U_{x,\nu} = e^{iA_\nu} \in U(1),\ A_\nu \in [-\pi, \pi]$$

$$U_{x,\nu\rho} = U_{x,\nu}U_{x+\hat{\nu},\rho}U^*_{x+\hat{\rho},\nu}U^*_{x,\rho}$$



Dual representation-1

Rewrite terms of partition sum:

- A single nearest neighbor term:

$$e^{e^{-\mu\delta_{\nu 4}}\phi_x^* U_{x,\nu}\phi_{x+\hat{\nu}}} = \sum_{l_{x,\nu}} \frac{\left(e^{-\mu\delta_{\nu 4}}\right)^{l_{x,\nu}}}{l_{x,\nu}!} (U_{x,\nu})^{l_{x,\nu}} (\phi_x^*)^{l_{x,\nu}} (\phi_{x+\hat{\nu}})^{l_{x,\nu}}$$

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- A single plaquette term:

$$e^{\beta U_{x,\nu}U_{x+\hat{\nu},\rho}U^*_{x+\hat{\rho},\nu}U^*_{x,\rho}} = \sum_{p_{x,\nu\rho}} \frac{\beta^{p_{x,\nu\rho}}}{p_{x,\nu\rho}!} \left[U_{x,\nu}U_{x+\hat{\nu},\rho}U^*_{x+\hat{\rho},\nu}U^*_{x,\rho}\right]^{p_{x,\nu\rho}}$$

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Partition sum:

$$Z = \sum_{\{p,l\}} \left[\prod_{x,\nu\rho} \frac{\left(e^{-\mu\delta_{\nu 4}}\right)^{l_{x,\nu}} \beta^{p_{x,\nu\rho}}}{l_{x,\nu}! p_{x,\nu\rho}!} \int dU_{x,\nu} d\phi_x d\phi_x^* F(U, \phi, \phi^*, l_{x,\nu}, p_{x,\nu\rho}, \kappa)\right]$$

Dual representation-2

- Integrate out U(1) fields $\rightarrow$ new degrees of freedom:
  - Links: $l_{x,\nu} \ \in \ \mathbb{Z}$
  - Plaquettes: $p_{x\nu\rho} \ \in \ \mathbb{Z}$

- New partion sum:

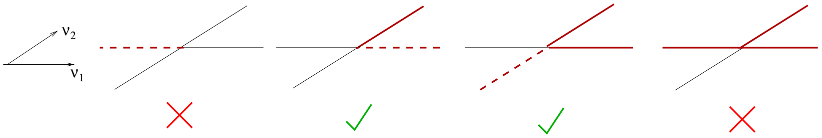
$$Z \ \propto \sum_{\{p,l\}} \mathcal{W}[p,l] \, \mathcal{C}_S[l] \, \mathcal{C}_L[p,l]$$

  - $\mathcal{W}[p,l]$: positive weight factor (sign problem solved).
  - $\mathcal{C}_S[l]$: site constraint $\rightarrow$ matter loops.
  - $\mathcal{C}_L[p,l]$: link constraint $\rightarrow$ gauge surfaces.

Site constraint

- Site constraint (matter loops):

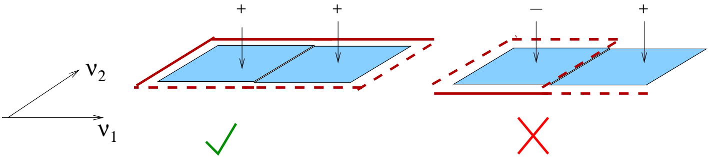
$$\mathcal{C}_S[l] = \prod_x \delta\left(\sum_{\nu=1}^{4}[l_{x,\nu} - l_{x-\hat{\nu},\nu}]\right)$$



Link constraint

- Link constraint (gauge surfaces):

$$\mathcal{C}_L[p,l] = \prod_x \prod_{\nu=1}^{4} \delta\left(\sum_{\rho:\nu<\rho} [p_{x,\nu\rho} - p_{x-\hat{\rho},\nu\rho}] - \sum_{\rho:\nu>\rho} [p_{x,\rho\nu} - p_{x-\hat{\rho},\rho\nu}] + l_{x,\nu} \right)$$



MC simulation

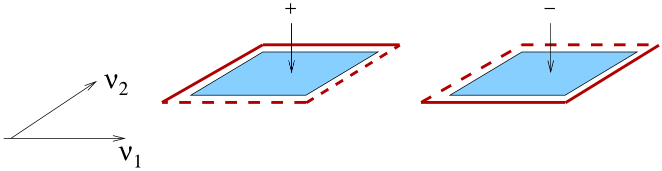
- We used two algorithms:
  - Local Metropolis update
  - **Worm algorithm**

- **Advantages of the worm algorithm:**
  - Most suitable algorithm (for constrained variables).
  - Local updates of the configurations.
  - Smaller autocorrelation time in critical regions.

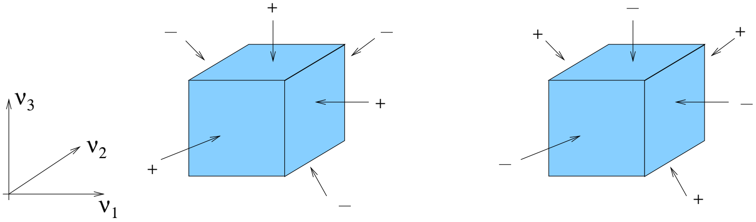
N. Prokof'ev and B. Svistunov (2001),
Y. Deng, T. M. Garoni and A. D. Sokal (2007).

Local Metropolis Update

- Plaquette update:

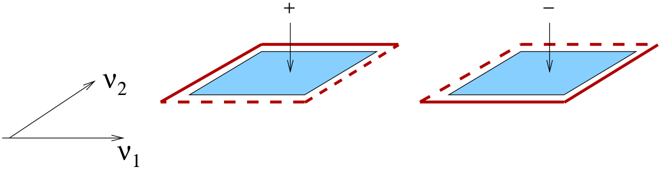


- Cube update:

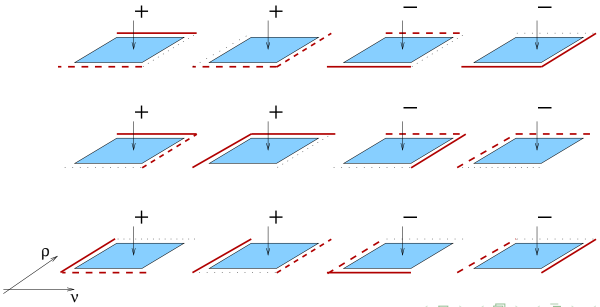


Elements of the WA

- Take smallest unit of the local update:

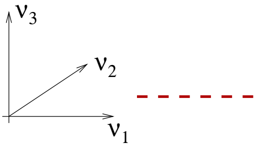


- Relax the constraints in 2 elements → segments



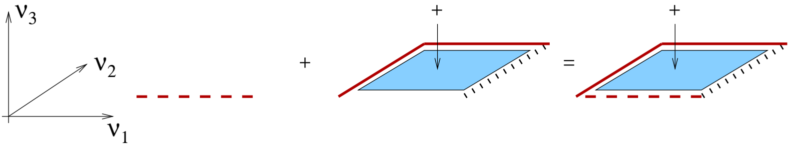
Updating scheme

1. One link is inserted at a random position of the lattice L_0.



Updating scheme

1. One link is inserted at a random position of the lattice L_0.
2. The worm may insert a new segment at L_v, healing the constraints at this position and then move to one of other three links of the segment.



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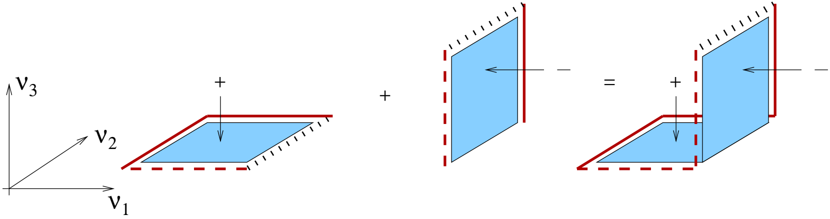

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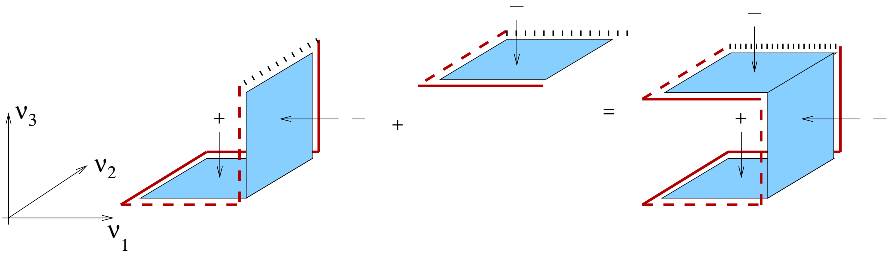

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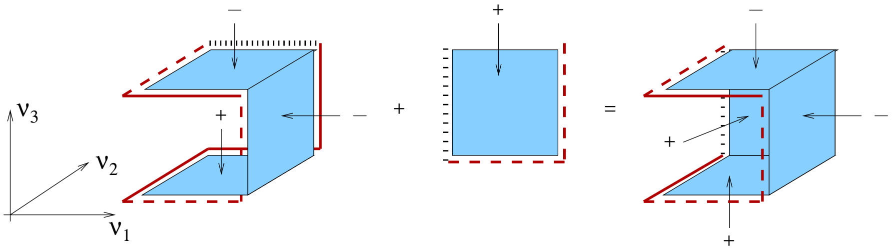
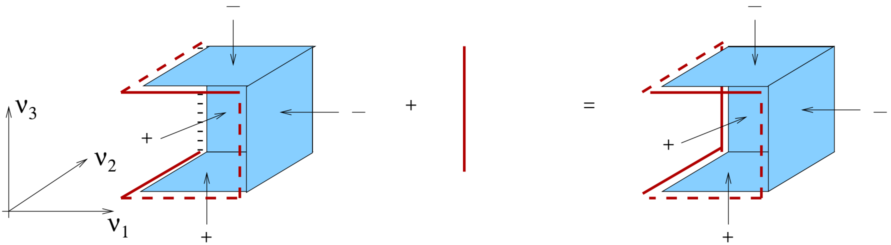


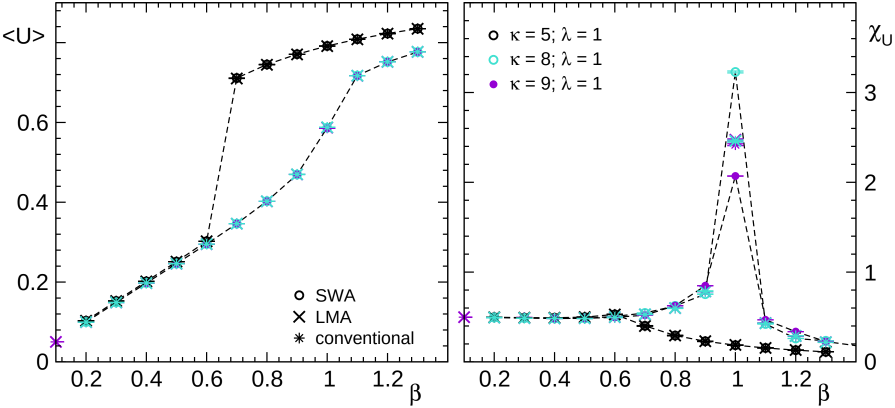
Updating scheme

1. One link is inserted at a random position of the lattice L_0.
2. The worm may insert a new segment at L_v, healing the constraints at this position and then move to one of other three links of the segment.
3. The worm ends modifying the link occupation number at L_v.



WA vs. LMA

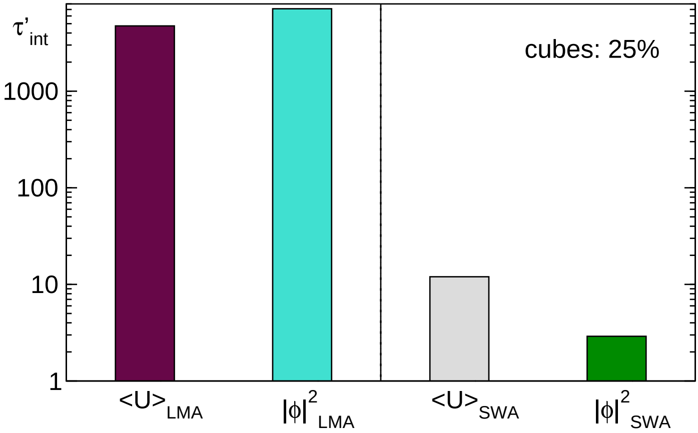
Verify correctness of WA ($\mu = 0$ case).



WA vs. LMA: $\kappa = 5, \beta = 0.65, V = 8^4$

Close to the 1st order transition

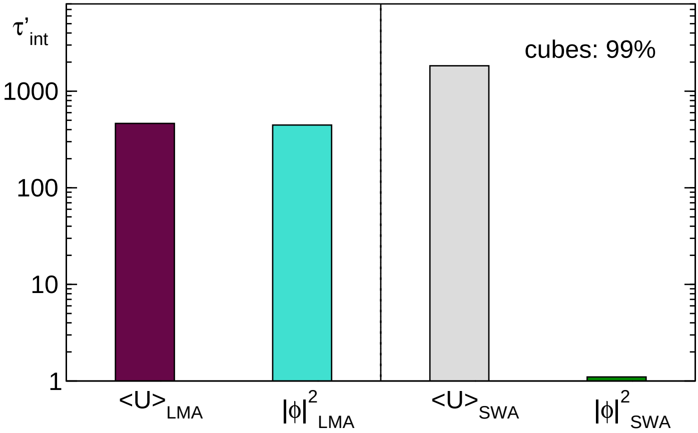
- $\langle U \rangle$ function of plaquettes.
- $\langle |\phi|^2 \rangle$ function of links.



WA vs. LMA: $\kappa = 8, \beta = 1.1, V = 8^4$

Configurations dominated by closed surfaces (links are expensive).

- $\langle U \rangle$ function of plaquettes.
- $\langle |\phi|^2 \rangle$ function of links.



Scalar electrodynamics with two flavors

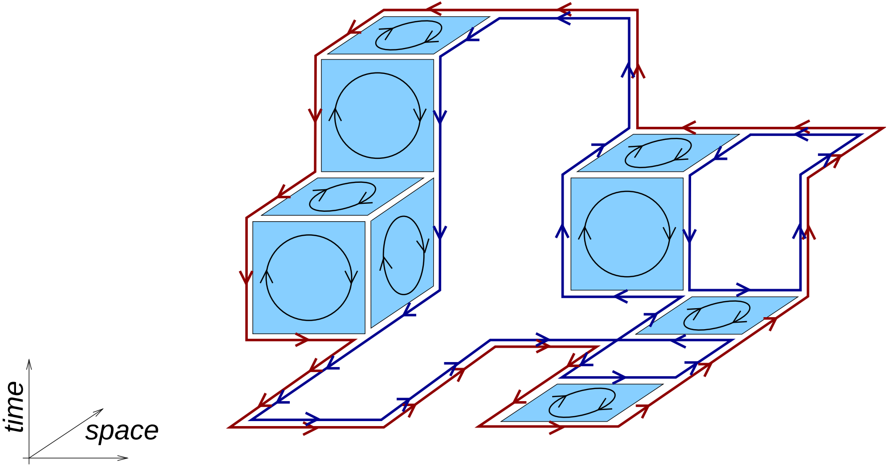
- **Conventional action on the lattice:**

$$
\begin{aligned}
S_G &= -\beta \sum_x \sum_{\mu<\nu} Re\, U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^\star_{x+\hat{\nu},\mu}\, U^\star_{x,\nu} \\
S_H &= \sum_x \left[\kappa^1\, |\phi^1_x|^2 + \kappa^2\, |\phi^2_x|^2 \right] \\
&- \sum_x \left[\sum_\mu \left(e^{\delta_{\mu 4}\mu^1}\, \phi^{1\,\star}_x\, U_{x,\mu}\, \phi^1_{x+\widehat{\mu}} + e^{-\delta_{\mu 4}\mu^1}\, \phi^{1\,\star}_x\, U^\star_{x-\widehat{\mu},\mu}\, \phi^1_{x-\widehat{\mu}} \right) \right] \\
&- \sum_x \left[\sum_\mu \left(e^{\delta_{\mu 4}\mu^2}\, \phi^{2\,\star}_x\, U^\star_{x,\mu}\, \phi^2_{x+\widehat{\mu}} + e^{-\delta_{\mu 4}\mu^2}\, \phi^{2\,\star}_x\, U_{x-\widehat{\mu},\mu}\, \phi^2_{x-\widehat{\mu}} \right) \right]
\end{aligned}
$$

- **Dual form of the partition sum:**

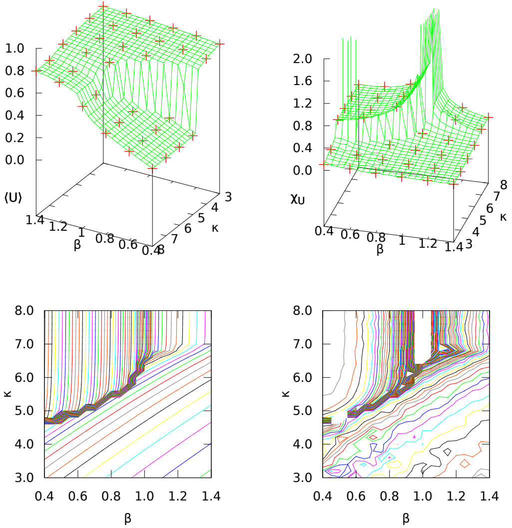
$$
Z = \sum_{\{p,l^1,l^2\}} \mathcal{W}(p,l^1,l^2)\, \mathcal{C}_L(p,l^1,l^2)\, \mathcal{C}_S(l^1,\, l^2)
$$

An admissible configuration



Chemical potential favors flux forward in time.

Phase diagram at $\mu = 0$, $\kappa^1 = \kappa^2$



A. Schmidt.

Summary

- Considerable progress was made towards rewriting several systems in the dual representation, where the sign problem is solved.
- We have proposed an extension of the worm algorithm to simulate abelian gauge theories.
- Outlook:
  - Implement the worm algorithm for the 2 flavor case.
  - Phase diagram at finite density.
  - Dual representation of non-abelian theories??

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Thank you for your attention!