$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature$^a$

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Introduction and motivation

RHIC experiments show that hot QCD matter has very intricate properties = a big challenge to understand. But it is clear that:

- Hot QCD matter may be called "QGP", but this cannot be perturbatively interacting quark-gluon gas (as once expected) until much higher $T$

- Still no direct "smoking gun" signal of deconfinement, etc ... compelling signals of new form of matter sought:

  e.g., a change in symmetries obeyed by the strong interaction: the restoration of the $[SU_A(3) \text{ flavor}]$ chiral symmetry and $U_A(1)$ symmetry ⇒ a good understanding of the light-quark pseudoscalar nonet is needed - especially $\eta$, $\eta'$.
Hot hadrons = important for understanding hot QCD matter

- especially since lattice (and other, e.g., see Gossiaux’s talk) show: $J/\Psi$ and $\eta_c$ stay bound till $\sim 2T_{\text{cri}}$, maybe higher ... + similar indications about light-quark mesons = motivation to:

- explore validity of meson relations, e.g., **WV relation**:

\[ M_{\eta'}^2 + M_\eta^2 - 2M_K^2 = \frac{2N_f}{f_\pi^2} \chi_{\text{YM}} \left( +O\left(\frac{1}{N_c}\right) \right) \]

(... and test validity of various $T$-rescaling procedures).

- use, even at high $T$, **bound-state equations** ... here, **Dyson-Schwinger approach** (by Zagreb group: Horvatić et al. PRD76 (2007) 096009) for non-anomalous sector, but results of Benić et al., Phys. Rev. D84 (2011) 016006, for $U_A(1)$-anomalous sector of $\eta-\eta'$ complex at $T > 0$. 

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature\(^\text{a}\) – p. 3/49
**Introduction and motivation**

$U_A(1)$ symmetry is broken by the nonabelian ("gluon") axial anomaly: even in the chiral limit (ChLim, where $m_q \to 0$),

$$\partial_\alpha \bar{\psi}(x) \gamma^\alpha \gamma^5 \frac{\lambda^0}{2} \psi(x) \propto F^a(x) \cdot \tilde{F}^a(x) \equiv \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F^{a\rho\sigma}(x) \neq 0.$$ 

This breaks the $U_A(1)$ symmetry of QCD and precludes the 9$^{th}$ Goldstone pseudoscalar meson $\Rightarrow$ very massive $\eta'$:

even in ChLim, where $m_\pi, m_K, m_\eta \to 0$, still (‘ChLim WVR’)

$$0 \neq \Delta M_{\eta_0}^2 = \Delta M_{\eta'}^2 = \frac{(A = \text{qty.dim.mass})^4}{("f_{\eta'}")^2} = \frac{6 \chi_{YM}}{f_\pi^2} + O\left(\frac{1}{N_c}\right)$$

... but uncertain fate of $U_A(1)$ breaking as $T$ grows towards $T_{YM}$ and $T_{Ch}$, where $\chi_{YM}(T)$ and $f_\pi(T)$ strongly drop/vanish!

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature^a \quad p. 4/49
Experimental observation of in-medium $\eta'$ mass reduction

- High E heavy ion collisions $\Rightarrow$ hot and dense medium

- $\sqrt{s} = 200$ GeV Au+Au collide at RHIC $\Rightarrow$ enhanced $\eta'$ abundance = 1st exp. signature of a partial $U_A(1)$ sym. restoration ... is it before chiral symmetry restoration?!?!

- Combined STAR & PHENIX data analyzed robustly through six popular models for multiplicities (ALCOR, FRITIOF, ...) $\Rightarrow$ at 99.9% confidence level, $\eta'$ mass is reduced by at least 200 MeV inside fireball.

  $M_{\eta'}^* = 340^{+50}_{-60}(\text{statist.})^{+280}_{-140}(\text{model}) \pm 42(\text{system.})$ MeV


  = “The return of the prodigal Goldstone boson!”

  What are implications for the WV relation at $T > 0$?
Dyson-Schwinger approach to quark-hadron physics

= the bound state approach which is nonpertubative, covariant and **chirally well-behaved**.

a) direct contact with QCD through *ab initio* calculations

b) phenomenological modeling of hadrons as quark bound states (used also here, for example)

coupled system of integral equations for Green functions of QCD

... but ... equation for n-point function calls (n+1)-point function ... → cannot solve in full the growing tower of DS equations

→ various degrees of truncations, approximations and modeling is unavoidable (more so in phenomenological modeling of hadrons, as here)

η' multiplicity and Witten-Veneziano relation at finite temperature
For the present purposes, the most important advantage of DS approach is that it is **chirally well-behaved**: **non-anomalous parts** of the masses of the light pseudoscalar $q\bar{q}'$ mesons (i.e., all parts except $\Delta M_{\eta_0}$) behave as 

$$M_{q\bar{q}'}^2 = \text{const} \left(m_q + m_{q'}\right) , \quad (q, q' = u, d, s) .$$

⇒ non-anomalous parts of the masses in WVR cancel:

$$M_{\eta'}^2 + M_{\eta}^2 - 2 M_K^2 \approx \Delta M_{\eta_0}$$

⇒ already ChLim WVR reveals the essence of the influence of the gluon anomaly on the masses in $\eta$-$\eta'$ complex.

= **IMPORTANT**, since it shows almost model-independently that the WVR containing $\chi_{YM}(T)$ implies $M_{\eta'}(T)$ in conflict with experiment

⇒ Model dependence of our discussion is minimal as everything boils down to the ratio $\chi_{YM}(T)/f_\pi(T)^2$ ...
Dyson-Schwinger approach to quark-hadron physics

- Gap equation for propagator $S_q$ of dressed quark $q$

\[
\frac{\lambda}{2} \gamma^\mu S_q(l) \Gamma^\alpha_{\nu}(l, p) = \gamma^\mu S_q(l) \Gamma^\alpha_{\nu}(l, p)
\]

- Homogeneous Bethe-Salpeter (BS) equation for a $M$eson $q\bar{q}$ bound state vertex $\Gamma_{q\bar{q}}$

\[
\Gamma_{q\bar{q}} = \Gamma_{q\bar{q}}
\]

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature
Gap and BS equations in ladder truncation

\[ S_q(p)^{-1} = i\gamma \cdot p + \tilde{m}_q + \frac{4}{3} \int \frac{d^4 \ell}{(2\pi)^4} g^2 G^{\text{eff}}_{\mu\nu}(p - \ell) \gamma_{\mu} S_q(\ell) \gamma_{\nu} \]

\[ \rightarrow S_q(p) = \frac{1}{i\not{p} A_q(p^2) + B_q(p^2)} = \frac{-i\not{p} A_q(p^2) + B_q(p^2)}{p^2 A_q(p^2)^2 + B_q(p^2)^2} = \frac{1}{A_q(p^2)} \frac{-i\not{p} + m_q(p^2)}{p^2 + m_q(p^2)^2} \]

\[ \Gamma_{qq'}(p, P) = -\frac{4}{3} \int \frac{d^4 \ell}{(2\pi)^4} g^2 G^{\text{eff}}_{\mu\nu}(p - \ell) \gamma_{\mu} S_q(\ell + \frac{P}{2}) \Gamma_{qq'}(\ell, P) S_q(\ell - \frac{P}{2}) \gamma_{\nu} \]

- Euclidean space: \( \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \gamma_\mu^\dagger = \gamma_\mu, a \cdot b = \sum_{i=1}^{4} a_i b_i \)
- \( P \) is the total momentum, \( M^2 = -P^2 \) meson mass
- \( G^{\text{eff}}_{\mu\nu}(k) \) an “effective gluon propagator” - modeled!

\( \gamma' \) multiplicity and Witten-Veneziano relation at finite temperature - p. 9/49
From the gap and BS equations ...

- solutions of the gap equation $\rightarrow$ the **dressed** quark mass function

$$m_q(p^2) = \frac{B_q(p^2)}{A_q(p^2)}$$

- propagator solutions $A_q(p^2)$ and $B_q(p^2)$ pertain to **confined** quarks if

$$m_q^2(p^2) \neq -p^2 \text{ for real } p^2$$

- The BS solutions $\Gamma_{q\bar{q}'}$ enable the calculation of the properties of $q\bar{q}$ bound states, such as the decay constants of pseudoscalar mesons:

$$f_{PS} P_{\mu} = \langle 0 | \bar{q} \frac{\chi^{PS}_\mu}{2} \gamma_5 q | \Phi_{PS}(P) \rangle$$

$$\rightarrow f_\pi P_{\mu} = N_c \text{ tr}_s \int \frac{d^4\ell}{(2\pi)^4} \gamma_5 \gamma_\mu S(\ell + P/2) \Gamma_\pi(\ell; P) S(\ell - P/2)$$
Renormalization-group improved interactions

Landau gauge gluon propagator: 
\[ g^2 G_{\mu\nu}^{\text{eff}}(k) = G(-k^2)(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}) , \]

\[ G(Q^2) \equiv 4\pi \frac{\alpha_s^{\text{eff}}(Q^2)}{Q^2} = G_{\text{UV}}(Q^2) + G_{\text{IR}}(Q^2) , \quad Q^2 \equiv -k^2 . \]

\[ G_{\text{UV}}(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \approx \frac{4\pi^2 d}{Q^2 \ln(x_0 + \frac{Q^2}{\Lambda_{\text{QCD}}})} \left\{ 1 + b \frac{\ln[\ln(x_0 + \frac{Q^2}{\Lambda_{\text{QCD}}})]}{\ln(x_0 + \frac{Q^2}{\Lambda_{\text{QCD}}})} \right\} , \]

but modeled non-perturbative part, e.g., Jain & Munczek:

\[ G_{\text{IR}}(Q^2) = G_{\text{non-pert}}(Q^2) = 4\pi^2 a Q^2 \exp(-\mu Q^2) \quad \text{(similar: Maris, Roberts...)} \]

or, the dressed propagator with dim. 2 gluon condensate
\[ \langle A^2 \rangle \text{-induced dynamical gluon mass (Kekez & Klabučar)}: \]

\[ G'(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \left( \frac{Q^2}{Q^2 - M^2_{\text{gluon}} + \frac{c_{\text{ghost}}}{Q^2}} \right)^2 \frac{Q^2}{Q^2 + M^2_{\text{gluon}} + \frac{c_{\text{gluon}}}{Q^2}} . \]
Some effective strong couplings $\alpha_s^{\text{eff}}(Q^2) \equiv \frac{Q^2 G(Q^2)}{4\pi}$


**Important:** integrated IR strength must be sufficient for DChSB!
Separable model = good, + easier at $T > 0$

- Calculations simplify with the separable Ansatz for $G_{\mu\nu}^{\text{eff}}$:

$$G_{\mu\nu}^{\text{eff}}(p - q) \rightarrow \delta_{\mu\nu} G(p^2, q^2, p \cdot q)$$

$$G(p^2, q^2, p \cdot q) = D_0 \ f_0(p^2) f_0(q^2) + D_1 \ f_1(p^2)(p \cdot q) f_1(q^2)$$

- Two strength parameters $D_0, D_1$, and corresponding form factors $f_i(p^2)$. In the separable model, gap equation yields

$$B_f(p^2) = \tilde{m}_f + \frac{16}{3} \int \frac{d^4 q}{(2\pi)^4} G(p^2, q^2, p \cdot q) \frac{B_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)}$$

$$[A_f(p^2) - 1] p^2 = \frac{8}{3} \int \frac{d^4 q}{(2\pi)^4} G(p^2, q^2, p \cdot q) \frac{(p \cdot q) A_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)}.$$ 

- This gives $B_f(p^2) = \tilde{m}_f + b_f \ f_0(p^2)$ and $A_f(p^2) = 1 + a_f \ f_1(p^2)$, reducing to nonlinear equations for constants $b_f$ and $a_f$. 

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature
A simple choice for ‘interaction form factors’ of the separable model:

\[
\begin{align*}
    f_0(p^2) &= \exp\left(-\frac{p^2}{\Lambda_0^2}\right) \\
    f_1(p^2) &= \frac{1 + \exp\left(-\frac{p_0^2}{\Lambda_1^2}\right)}{1 + \exp\left((p^2 - p_0^2)\right)/\Lambda_1^2}
\end{align*}
\]

gives good description of pseudoscalar properties if the interaction is strong enough for realistic DChSB, when \(m_{u,d}(p^2 \sim \text{small}) \sim \text{the typical constituent quark mass scale} \sim \frac{m_\rho}{2} \sim \frac{m_N}{3}\).
Nonperturbative dynamical propagator dressing

Dynamical Chiral Symmetry Breaking (DChSB)

\[ A_f, B_f [\text{GeV}] \]

\[ A_{u,d}, A_s, B_s, B_{u,d} \]

\[ p^2 [\text{GeV}^2] \]

\[ \eta' \] multiplicity and Witten-Veneziano relation at finite temperature

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DChSB = nonperturb. generation of large quark masses ...

... even in the chiral limit ($\tilde{m}_f \to 0$), where the octet pseudoscalar mesons are Goldstone bosons of DChSB!

\[ m_f [\text{GeV}] \]

\[ m_s \]

\[ m_{u,d} \]

\[ p^2 [\text{GeV}^2] \]
At $T = 0$, good DS results; e.g., “non-anomalous”:

- Separable model parameter values reproducing experimental data:
  \[ \tilde{m}_{u,d} = 5.5 \text{ MeV}, \Lambda_0 = 758 \text{ MeV}, \Lambda_1 = 961 \text{ MeV}, p_0 = 600 \text{ MeV}, \]
  \[ D_0 \Lambda_0^2 = 219, D_1 \Lambda_1^4 = 40 \text{ (fixed by fitting } M_\pi, f_\pi, M_\rho, g_{\rho \pi^+ \pi^-}, g_{\rho e^+ e^-} \text{ – pertinent predictions } a_{u,d} = 0.672, b_{u,d} = 660 \text{ MeV, i.e., } m_{u,d}(p^2), \langle \bar{u}u \rangle \text{)} \]

- \[ \tilde{m}_s = 115 \text{ MeV (fixed by fitting } M_K \rightarrow \text{ predictions } a_s = 0.657, b_s = 998 \text{ MeV, i.e., } m_s(p^2), \langle \bar{s}s \rangle, M_{s\bar{s}}, f_K, f_{s\bar{s}} \text{)} \]

- Summary of results (all in GeV) for $q = u, d, s$ and pseudoscalar mesons without the influence of gluon anomaly:

<table>
<thead>
<tr>
<th>$PS$</th>
<th>$M_{PS}$</th>
<th>$M_{PS}^{exp}$</th>
<th>$f_{PS}$</th>
<th>$f_{PS}^{exp}$</th>
<th>$m_q(0)$</th>
<th>$-\langle q\bar{q} \rangle^{1/3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>0.140</td>
<td>0.1396</td>
<td>0.092</td>
<td>0.0924 ± 0.0003</td>
<td>0.398</td>
<td>0.217</td>
</tr>
<tr>
<td>$K$</td>
<td>0.495</td>
<td>0.4937</td>
<td>0.110</td>
<td>0.1130 ± 0.0010</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s\bar{s}$</td>
<td>0.685</td>
<td>0.119</td>
<td></td>
<td></td>
<td>0.672</td>
<td></td>
</tr>
</tbody>
</table>

\[ \eta'$ multiplicity and Witten-Veneziano relation at finite temperature\]
At $T > 0$, good and less good DS results

E.g., chiral symmetry restoration qualitatively good, but $T_{Ch}$ lower than lattice (maybe up to 35%, and even more for ‘more realistic’ DS models unless they contain $\delta$-function):

$B_{f}(0), -\langle q\bar{q}\rangle_0^{1/3}[GeV]$
Same with pseudoscalar decay constants $f_P(T)$:

Both crossover and Ch-limit behavior OK, but $T_{Ch} = 128$ MeV

... but this is cured by introducing Polyakov loop (PL)
Similarly with the $T$-dependence of $\pi, K, s\bar{s}, \sigma$ masses:

‘Deconfinement’ $T_{d,q}$ from $S_q$ pole - very different $T_{d,u}, T_{d,s}$ ... also cured/synchronized with $T_{Ch}(=T_{cri})$ by PL

\[ T_{d,u} \quad T_{Ch} \quad T_{d,s} \]

\[ M_{\pi}, M_K, M_{\sigma}, M_{s\bar{s}} \]

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature
Anomaly and mixing in $\eta-\eta'$ complex

- present approach yields mass$^2$ eigenvalues
  
  \[ M_{u\bar{d}}^2 = M_{\pi^+}^2, M_{u\bar{s}}^2 = M_K^2, \ldots, \hat{M}_{NA}^2 = \text{diag}(M_{u\bar{u}}, M_{d\bar{d}}, M_{s\bar{s}}) \]

- $|u\bar{d}\rangle = |\pi^+\rangle$, $|u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles (at $T = 0$ at least!), although in the isospin limit (adopted from now on)
  
  \[ M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_\pi. \ I \text{ is a good quantum number!} \]

- $\Rightarrow$ recouple into "more physical" $I_3 = 0$ octet-singlet basis

\[
I = 1 \quad |\pi^0\rangle = \frac{1}{\sqrt{2}}( |u\bar{u}\rangle - |d\bar{d}\rangle ) ,
\]

\[
I = 0 \quad |\eta_8\rangle = \frac{1}{\sqrt{6}}( |u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle ) ,
\]

\[
I = 0 \quad |\eta_0\rangle = \frac{1}{\sqrt{3}}( |u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle ) .
\]
Anomaly and mixing in $\eta$-$\eta'$ complex

- the “non-anomalous” (chiral-limit-vanishing!) part of the mass-squared matrix of $\pi^0$ and $\eta$'s is (in $\pi^0$-$\eta_8$-$\eta_0$ basis)

$$\hat{M}_{NA}^2 = \begin{pmatrix} M_{\pi}^2 & 0 & 0 \\ 0 & M_{88}^2 & M_{80}^2 \\ 0 & M_{08}^2 & M_{00}^2 \end{pmatrix}$$

$$M_{88}^2 \equiv \langle \eta_8 | \hat{M}_{NA}^2 | \eta_8 \rangle = \frac{2}{3} (M_{s\bar{s}}^2 + \frac{1}{2} M_{\pi}^2),$$

$$M_{80}^2 \equiv \langle \eta_8 | \hat{M}_{NA}^2 | \eta_0 \rangle = M_{08}^2 = \frac{\sqrt{2}}{3} (M_{\pi}^2 - M_{s\bar{s}}^2)$$

$$M_{00}^2 \equiv \langle \eta_0 | \hat{M}_{NA}^2 | \eta_0 \rangle = \frac{2}{3} (\frac{1}{2} M_{s\bar{s}}^2 + M_{\pi}^2),$$

- Not enough! In order to avoid the $U_A(1)$ problem, one must break the $U_A(1)$ symmetry (as it is destroyed by the gluon anomaly) at least at the level of the masses.
Gluon anomaly is not accessible to ladder approximation!

**Diamond graph**: an example of a transition $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ ($q, q' = u, d, s[...]$), contributing to the anomalous masses in the $\eta$-$\eta'$ complex, but not included in the interaction kernel in the ladder approximation.
Anomaly and mixing in $\eta$-$\eta'$ complex

- All masses in $\hat{M}_{NA}^2$ are calculated in the ladder approx., which cannot include the gluon anomaly contributions.

- Large $N_c$: the gluon anomaly suppressed as $1/N_c! \rightarrow$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the $\eta_0$ (squared) mass by anomalous contribution $3\beta$.

- Complete mass matrix is then $\hat{M}^2 = \hat{M}_{NA}^2 + \hat{M}_A^2$ where

$$
\hat{M}_A^2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 3\beta
\end{pmatrix}
$$

does not vanish in the chiral limit.

$3\beta = \Delta M_{\eta_0}^2$ = the anomalous mass$^2$ of $\eta_0$ [in SU(3) limit incl. ChLim] is related to the YM topological susceptibility. Fixed by phenomenology or (here) taken from the lattice.
Anomaly and mixing in $\eta$-$\eta'$ complex

we can also rewrite $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$

\[
\hat{M}_A^2 = \beta \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

We introduced the effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ ($q, q' = u, d, s$). $s\bar{s}$ transition suppression estimated by $X \approx f_\pi / f_{s\bar{s}}$.

Then, $\hat{M}_A^2$ in the octet-singlet basis is modified to

\[
\hat{M}_A^2 = \beta \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{2}{3}(1 - X)^2 & \frac{\sqrt{2}}{3}(2 - X - X^2) \\
0 & \frac{\sqrt{2}}{3}(2 - X - X^2) & \frac{1}{3}(2 + X)^2
\end{pmatrix}
\]

→ In the isospin limit, one can always restrict to $2 \times 2$ submatrix of etas ($I = 0$), as $\pi^0$ ($I = 1$) is decoupled then.
Anomaly and mixing in $\eta$-$\eta'$ complex

- nonstrange (NS) – strange (S) basis

$$|\eta_{NS}\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle,$$

$$|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle.$$

- the $\eta$–$\eta'$ matrix in this basis is

$$\hat{M}^2 = \begin{pmatrix} M_{\eta_{NS}\eta_{NS}}^2 & M_{\eta_{NS}\eta_S}^2 \\ M_{\eta_{NS}\eta_S}^2 & M_{\eta_S}^2 \end{pmatrix} = \begin{pmatrix} M_{u\bar{u}}^2 + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M_{s\bar{s}}^2 + \beta X^2 \end{pmatrix} \phi \begin{pmatrix} m_{\eta}^2 & 0 \\ 0 & m_{\eta'}^2 \end{pmatrix}$$

- NS–S mixing relations

$$|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle, \quad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle.$$

$$\theta = \phi - \arctan\sqrt{2}$$
Anomaly and mixing in $\eta$-$\eta'$ complex

Let lowercase $m_M$’s denote the empirical mass of meson $M$. From our calculated, model mass matrix in $\mathcal{NS}$–$\mathcal{S}$ basis, we form its empirical counterpart $\hat{m}_{\text{exp}}^2$ by

i) obvious substitutions $M_{u\bar{u}} \equiv M_\pi \to m_\pi$, $M_{s\bar{s}} \to m_{s\bar{s}}$

ii) by noting that $m_{s\bar{s}}$, the “empirical” mass of the unphysical $s\bar{s}$ pseudoscalar bound state, is given in terms of masses of physical particles as $m_{s\bar{s}}^2 \approx 2m_K^2 - m_\pi^2$. Then,

$$
\hat{m}_{\text{exp}}^2 = \begin{bmatrix}
m_\pi^2 + 2\beta & \sqrt{2}\beta X \\
\sqrt{2}\beta X & 2m_K^2 - m_\pi^2 + \beta X^2
\end{bmatrix} \rightarrow \phi_{\text{exp}} \begin{bmatrix}
m_\eta^2 & 0 \\
0 & m_{\eta'}^2
\end{bmatrix}.
$$
Finally, fix anomalous contribution to $\eta$-$\eta'$:

- the trace of the empirical $\hat{m}_{\text{exp}}^2$ demands the 1\textsuperscript{st} equality in

$$\beta(2+X^2) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{2N_f}{f_\pi^2} \chi_{\text{YM}} \quad (2\text{nd equality} = \text{WV relation})$$

- requiring that the experimental trace $(m_\eta^2 + m_{\eta'}^2)_{\text{exp}} \approx 1.22$ GeV\(^2\) be reproduced by the theoretical $\hat{M}^2$, yields

$$\beta_{\text{fit}} = \frac{1}{2+X^2} [(m_\eta^2 + m_{\eta'}^2)_{\text{exp}} - (M_{uu}^2 + M_{ss}^2)]$$

- Or, get $\beta$ from lattice $\chi_{\text{YM}}$! Then no free parameters!

- then, can calculate the $NS$-$S$ mixing angle $\phi$

$$\tan 2\phi = \frac{2 M_{\eta S}^2 \eta_{NS}}{M_{\eta S}^2 - M_{\eta NS}^2} = \frac{2 \sqrt{2} \beta X}{M_{\eta S}^2 - M_{\eta NS}^2}$$

and

$$M_{\eta NS}^2 = M_{uu}^2 + 2\beta = M_\pi^2 + 2\beta, \quad M_{\eta S}^2 = M_{ss}^2 + \beta X^2 = M_{ss}^2 + \beta \frac{f_\pi^2}{f_{ss}^2}$$

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Anomaly and mixing in $\eta$-$\eta'$ complex

The diagonalization of the $NS$-$S$ mass matrix then finally gives us the calculated $\eta$ and $\eta'$ masses:

\[
M_{\eta}^2 = \cos^2 \phi M_{\eta NS}^2 - \sqrt{2} \beta X \sin 2\phi + \sin^2 \phi M_{\eta S}^2 \\
M_{\eta'}^2 = \sin^2 \phi M_{\eta NS}^2 + \sqrt{2} \beta X \sin 2\phi + \cos^2 \phi M_{\eta S}^2
\]

Equivalently, from the secular determinant,

\[
M_{\eta}^2 = \frac{1}{2} \left[ M_{\eta NS}^2 + M_{\eta S}^2 - \sqrt{(M_{\eta NS}^2 - M_{\eta S}^2)^2 + 8\beta^2 X^2} \right] \\
= \frac{1}{2} \left[ M_{\pi}^2 + M_{ss}^2 + \beta(2+X^2) - \sqrt{(M_{\pi}^2+2\beta-M_{ss}^2-\beta X^2)^2 + 8\beta^2 X^2} \right]
\]

\[
M_{\eta'}^2 = \frac{1}{2} \left[ M_{\eta NS}^2 + M_{\eta S}^2 + \sqrt{(M_{\eta NS}^2 - M_{\eta S}^2)^2 + 8\beta^2 X^2} \right] \\
= \frac{1}{2} \left[ M_{\pi}^2 + M_{ss}^2 + \beta(2+X^2) + \sqrt{(M_{\pi}^2+2\beta-M_{ss}^2-\beta X^2)^2 + 8\beta^2 X^2} \right]
\]
Separable model results on $\eta$ and $\eta'$ mesons (at $T = 0$)

<table>
<thead>
<tr>
<th></th>
<th>$\beta_{\text{fit}}$</th>
<th>$\beta_{\text{latt.}}$</th>
<th>Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>-12.22°</td>
<td>-13.92°</td>
<td></td>
</tr>
<tr>
<td>$M_\eta$</td>
<td>548.9</td>
<td>543.1</td>
<td>547.75</td>
</tr>
<tr>
<td>$M_{\eta'}$</td>
<td>958.5</td>
<td>932.5</td>
<td>957.78</td>
</tr>
<tr>
<td>$X$</td>
<td>0.772</td>
<td>0.772</td>
<td></td>
</tr>
<tr>
<td>$3\beta$</td>
<td>0.845</td>
<td>0.781</td>
<td></td>
</tr>
</tbody>
</table>

- Masses are in units of MeV, $3\beta$ in units of GeV$^2$ and the mixing angles are dimensionless.
- $\beta_{\text{latt.}}$ was obtained from $\chi_{\text{YM}}(T = 0) = (175.7 \text{ MeV})^4$.
- $X = f_\pi/f_{s\bar{s}}$ as well as the whole $\hat{M}_{NA}^2$ (consisting of $M_\pi$ and $M_{s\bar{s}}$) are calculated model quantities.
For three DS models: summary of $T = 0$ results from WV

<table>
<thead>
<tr>
<th>from Ref.</th>
<th>J-M&amp;WV</th>
<th>$A^2$ &amp;WV</th>
<th>separab&amp;WV</th>
<th>orig. Shore</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\pi$</td>
<td>137.3</td>
<td>135.0</td>
<td>140.0</td>
<td></td>
<td>(138.0)$^{isospin\ average}$</td>
</tr>
<tr>
<td>$M_K$</td>
<td>495.7</td>
<td>494.9</td>
<td>495.0</td>
<td></td>
<td>(495.7)$^{isospin\ average}$</td>
</tr>
<tr>
<td>$M_{s\bar{s}}$</td>
<td>700.7</td>
<td>722.1</td>
<td>684.8</td>
<td></td>
<td>92.4 ± 0.3</td>
</tr>
<tr>
<td>$f_\pi$</td>
<td>93.1</td>
<td>92.9</td>
<td>92.0</td>
<td></td>
<td>113.0 ± 1.0</td>
</tr>
<tr>
<td>$f_K$</td>
<td>113.4</td>
<td>111.5</td>
<td>110.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_{s\bar{s}}$</td>
<td>135.0</td>
<td>132.9</td>
<td>119.1</td>
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<td></td>
</tr>
<tr>
<td>$M_\eta$</td>
<td>568.2</td>
<td>577.1</td>
<td>542.3</td>
<td></td>
<td>547.75 ± 0.12</td>
</tr>
<tr>
<td>$M_{\eta'}$</td>
<td>920.4</td>
<td>932.0</td>
<td>932.6</td>
<td></td>
<td>957.78 ± 0.14</td>
</tr>
<tr>
<td>$\phi$</td>
<td>41.42$^o$</td>
<td>39.56$^o$</td>
<td>40.75$^o$</td>
<td>38.24$^o$</td>
<td></td>
</tr>
<tr>
<td>$\theta$</td>
<td>−13.32$^o$</td>
<td>−15.18$^o$</td>
<td>−13.98$^o$</td>
<td>−16.5$^o$</td>
<td></td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>−2.86$^o$</td>
<td>−5.12$^o$</td>
<td>−6.80$^o$</td>
<td>−12.3$^o$</td>
<td></td>
</tr>
<tr>
<td>$\theta_8$</td>
<td>−22.59$^o$</td>
<td>−24.14$^o$</td>
<td>−20.58$^o$</td>
<td>−20.1$^o$</td>
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</tr>
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<td>$f_0$</td>
<td>108.8</td>
<td>107.9</td>
<td>101.8</td>
<td>106.6</td>
<td></td>
</tr>
<tr>
<td>$f_8$</td>
<td>122.6</td>
<td>121.1</td>
<td>110.7</td>
<td>104.8</td>
<td></td>
</tr>
<tr>
<td>$f_0^\eta$</td>
<td>5.4</td>
<td>9.6</td>
<td>12.1</td>
<td>22.8</td>
<td></td>
</tr>
<tr>
<td>$f_0^{\eta'}$</td>
<td>108.7</td>
<td>107.5</td>
<td>101.1</td>
<td>104.2</td>
<td></td>
</tr>
<tr>
<td>$f_8^\eta$</td>
<td>113.2</td>
<td>110.5</td>
<td>103.7</td>
<td>98.4</td>
<td></td>
</tr>
<tr>
<td>$f_8^{\eta'}$</td>
<td>−47.1</td>
<td>−49.5</td>
<td>−38.9</td>
<td>−60.1</td>
<td></td>
</tr>
</tbody>
</table>

$\eta'$ multipole and Witten-Veneziano relation at finite temperature

---

$a$ – p. 31/49
\( \chi \), topological susceptibility of QCD vacuum, at \( T > 0 \)

\[
\chi = \int d^4x \langle q(x)q(0) \rangle, \quad q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)
\]

- \( q(x) \) = topological charge density operator
- In WV rel., \( \chi \) is the pure-glue, YM one, \( \chi_{YM} \leftrightarrow \chi_{\text{quench}} \).

η′ multiplicity and Witten-Veneziano relation at finite temperature α – p. 32/49
Relative temperature \((T/T_{\chi})\) dependence of meson masses

\[
\chi = \chi_{YM} \Rightarrow \text{The required drop (} \sim 200 \text{ MeV) of } M_{\eta'} \text{ only for very unrealistically low } T_{\chi}/T_{\text{Ch}} \text{ ratio, where } \chi_{YM}(T) \text{ melts before } f_\pi(T) \text{ diminishes much.}
\]

![Graph showing the dependence of meson masses on relative temperature](image)

**Extremely low** \(T_{\chi} = 2/3 T_{\text{Ch}}\) \((T_{\text{Ch}} = 1.50 T_{\chi})\)
Relative temperature \((T/T_\chi)\) dependence of meson masses

\(\chi_{YM}/f_\pi^2\) implies \(\eta'\) mass increase → suppression of the \(\eta'\) multiplicity already for still unrealistically low topological susceptibility-melting temperature \(T_\chi \gtrsim 0.8 T_{Ch}\). Failure of WV relation at \(T \gtrsim T_{Ch}\).

Still unrealistically low \(T_\chi = 0.836 T_{Ch}\) (\(T_{Ch} = 1.20 T_\chi\))
Obviously WV relation fails as $T$ approaches $T_\chi \sim T_{\text{Ch}}$:

$\chi_{\text{YM}}/f_\pi^2$ implies a huge $\eta'$ mass increase, to 5 GeV for $T_\chi = T_{\text{Ch}}$,

⇒ total suppression of the $\eta'$ multiplicity, instead of enhancement.
Solution: another connection of different theories

Early work by Di Vecchia & Veneziano ... Leutwyler & Smilga [Phys. Rev. D46 (1992) 5607] derived, up to $O\left(\frac{1}{N_c}\right)$,

\[
\chi_{YM} = \frac{\chi}{1 + \chi \frac{N_f}{m\langle \bar{q}q \rangle_0}} \equiv \tilde{\chi}
\]

⇒ relates $\chi_{YM}$ to the full-QCD topological susceptibility $\chi$, chiral condensate $\langle \bar{q}q \rangle_0$ and $m \equiv N_f \times$ the reduced mass. Presently $N_f = 3$, i.e., $N_f/m = \sum_{q=u,d,s}(1/m_q)$.

- in the limit of very heavy quarks, $m_q, m \rightarrow \infty$, it confirms expectations that $\chi_{YM} = \text{value of topolog. susceptibility in quenched QCD}$, $\chi_{YM} = \chi(m_q = \infty)$

- It shows $\chi \leq \min\left(-m \langle \bar{q}q \rangle_0/N_f, \chi_{YM}\right)$

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature $^\alpha$ – p. 36/49
LS relation also holds in the opposite limit!

This (presently pertinent!) limit of light quarks = still a problem to get the full-QCD topol. susceptibility $\chi$ on lattice. Fortunately (Di Vecchia, Veneziano), there is the analytic result for small $m_q$:

$$\chi = -\frac{m \langle \bar{q}q \rangle_0}{N_f} + C(m),$$

- $C(m) = \text{small corrections of higher orders in small } m_q$, ...
  but $C(m)$ should not be neglected, since $C(m) = 0$ would imply that $\chi_{YM} = \infty$.
- LS relation fixes the value of the correction at $T = 0$:

$$\frac{1}{C(m)} = \frac{N_f}{m \langle \bar{q}q \rangle_0} - \chi_{YM}(0) \left(\frac{N_f}{m \langle \bar{q}q \rangle_0}\right)^2.$$
$T$-dependence of $\tilde{\chi}$

- LS relation must break down as $T$ rises towards the (pseudo)critical temperatures of full QCD ($\sim T_{\text{Ch}}$) since the YM quantity, $\chi_{\text{YM}}$, is much more $T$-resistant than RHS.

- RHS $\equiv \tilde{\chi}$ consists of the full-QCD quantities $\chi$ and $\langle \bar{q}q \rangle_0$, the quantities of full QCD with quarks, characterized by $T_{\text{Ch}}$, just as $f_\pi(T)$.

- Thus, the troublesome mismatch in $T$-dependences of $f_\pi(T)$ and the pure-gauge $\chi_{\text{YM}}(T)$ is expected to disappear if $\chi_{\text{YM}}(T)$ is replaced by $\tilde{\chi}(T)$, the $T$-extended RHS of LS relation.

- The usual, successful zero-$T$ WV relation is thereby retained, since $\chi_{\text{YM}} = \tilde{\chi}$ at $T = 0$. 

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature \textsuperscript{a} – p. 38/49
Extending the light-quark full-QCD topol. susceptibility $\chi$ is somewhat uncertain, as there is no guidance from lattice [unlike for $\chi_{Y\!M}(T)$].

The leading term in Di Vecchia-Veneziano relation $\propto \langle \bar{q}q \rangle_0(T)$ very plausibly, but for the correction term we have to explore a range of Ansätze, i.e.,

$$\chi(T) = -\frac{m \langle \bar{q}q \rangle_0(T)}{N_f} + C(m) \left[ \frac{\langle \bar{q}q \rangle_0(T)}{\langle \bar{q}q \rangle_0(T = 0)} \right]^{\delta}, \quad (0 \leq \delta < 2).$$

Then,

$$\tilde{\chi}(T) = \left\{ \frac{\langle \bar{q}q \rangle_0(T)}{\sum_{q=u,d,s} \left( \frac{1}{m_q} \right)} \left[ \frac{\langle \bar{q}q \rangle_0(T)}{\sum_{q=u,d,s} \left( \frac{1}{m_q} \right)} \right] \frac{1}{C(m)} \left[ \frac{\langle \bar{q}q \rangle_0(T = 0)}{\langle \bar{q}q \rangle_0(T) \langle \bar{q}q \rangle_0(T = 0)} \right]^{\delta} \right\}.$$
$T$-dependence of $\tilde{\chi}$ follows the chiral condensate $\langle \bar{q}q \rangle_0(T)$:

$-\langle \bar{q}q \rangle_0^{1/3}$

$\tilde{\chi}^{1/4}$

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature
Case 1: $T$-independent correction term in $\chi$

$\chi_{YM}=(0.1757 \text{ GeV})^4$, $\delta=0$

$M_p$ [GeV]

$\eta'$
$\eta_0$
$\eta_S$
$\eta_{NS}$
$\pi$
$\eta_8$
$

$T/T_{ch}$

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature — p. 41/49
Case 2: Strongly $T$-dependent correction term $\propto \langle \bar{q}q \rangle_0(T)$

$\chi_{YM}=(0.1757 \text{ GeV})^4$, $\delta=1$

$\eta'$ multiplicity and Witten-Veneziano relation at finite temperature$^\alpha$ – p. 42/49
Summary

- $\eta'$ enhanced multiplicity shows that WV relation cannot be straightforwardly extended to $T'$s close to $T_{Ch}$, because the $T$-dependence of the ratio $\chi_{YM}(T)/f_\pi(T)^2$ starts failing as $T \to T_{Ch}$. Rescalings of $T_{YM}$ useless!

- Leutwyler-Smilga and Di Vecchia-Veneziano relations enable one to retain unchanged WV relation, with $\chi_{YM}$, for $T = 0$ (in fact, any $T$ sufficiently below $T_{Ch}$) and 2.) to replace the $T$-dependence of $\chi_{YM}$ by that of the chiral condensate. This achieves consistency of the WV relation with the data on $\eta'$ multiplicities, and indicates how chiral restoration may be linked with the $U_A(1)$ one.

- Further work: – Extension to finite density - for $\eta'$ experiments, e.g., at NICA, as $\langle \bar{q}q \rangle_0(\mu) \to 0$ for $\mu \to \mu_{crit}$.
  – What happens in Shore’s generalization of WV relation?
Shore’s generalization of WV valid to all orders in $1/N_c$

- Inclusion of gluon anomaly in DGMOR relations $\rightarrow$

\[
(f^0\eta')^2 m_{\eta'} + (f^0\eta)^2 m_\eta = \frac{1}{3} \left( f^2_\pi m^2_\pi + 2 f^2_K m^2_K \right) + 6A \quad (1)
\]

\[
f^0\eta' f^8\eta' m^2_{\eta'} + f^0\eta f^8\eta m^2_\eta = \frac{2\sqrt{2}}{3} \left( f^2_\pi m^2_\pi - f^2_K m^2_K \right) \quad (2)
\]

\[
(f^8\eta')^2 m^2_{\eta'} + (f^8\eta)^2 m^2_\eta = -\frac{1}{3} \left( f^2_\pi m^2_\pi - 4 f^2_K m^2_K \right) \quad (3)
\]

\[A = \chi_{YM} + \mathcal{O}\left(\frac{1}{N_c}\right) = \text{full QCD topological charge.} \quad (1)+(3) \rightarrow\]

\[
(f^0\eta')^2 m^2_{\eta'} + (f^0\eta)^2 m^2_\eta + (f^8\eta)^2 m^2_\eta + (f^8\eta')^2 m^2_{\eta'} - 2 f^2_K m^2_K = 6A
\]

- Then, large $N_c$ limit and $f^0\eta, f^8\eta' \rightarrow 0$ as well as $f^0\eta', f^8\eta, f_K \rightarrow f_\pi$ recovers the standard WV.
\( \eta' \) and \( \eta \) have 4 independent decay constants

\[ f^0_{\eta'}, f^8_{\eta'}, f^0_{\eta}, f^8_{\eta} : \quad \langle 0 | A^\alpha \mu (x) | P(p) \rangle = i f^\alpha_P p^\mu e^{-ip \cdot x}, \quad a = 8, 0; \quad P = \eta, \eta' . \]

Equivalently, one has 4 related but different constants \( f^\alpha_{NS} \eta' \), \( f^\alpha_{NS} \eta \), \( f^\alpha_S \eta \), \( f^\alpha_S \eta' \) if instead of octet and singlet axial currents \( (a = 8, 0) \) one takes this matrix element of the nonstrange-strange axial currents \( (a = NS, S) \)

\[ A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^8 \mu (x) + \sqrt{\frac{2}{3}} A^0 \mu (x) = \frac{1}{2} (\bar{u}(x) \gamma^\mu \gamma_5 u(x) + \bar{d}(x) \gamma^\mu \gamma_5 d(x)) , \]

\[ A^\mu_{S}(x) = -\sqrt{\frac{2}{3}} A^8 \mu (x) + \frac{1}{\sqrt{3}} A^0 \mu (x) = \frac{1}{\sqrt{2}} \bar{s}(x) \gamma^\mu \gamma_5 s(x) , \]

\[ \begin{bmatrix} f^NS_{\eta} & f^S_{\eta} \\ f^NS_{\eta'} & f^S_{\eta'} \end{bmatrix} = \begin{bmatrix} f^8_{\eta} & f^0_{\eta} \\ f^8_{\eta'} & f^0_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{3} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} , \]

\[ a, P = NS, S : \quad \langle 0 | A^\mu_{NS}(x) | \eta_{NS}(p) \rangle = i f^{\alpha}_{NS} p^\mu e^{-ip \cdot x} , \quad \langle 0 | A^\mu_{NS}(x) | \eta_{S}(p) \rangle = 0 , \]

\[ a, P = NS, S : \quad \langle 0 | A^\mu_{S}(x) | \eta_{S}(p) \rangle = i f^{\alpha}_{S} p^\mu e^{-ip \cdot x} , \quad \langle 0 | A^\mu_{S}(x) | \eta_{NS}(p) \rangle = 0 , \]

Note: in our approach, \( f_{NS} = f_{u \bar{u}} = f_{d \bar{d}} = f_\pi \), \( f_S = f_{s \bar{s}} \) are calculated quantities

\( \eta' \) multiplicity and Witten-Veneziano relation at finite temperature \( ^{\alpha} - p. 45/49 \)
Two Mixing Angles and FKS one-angle scheme

- Any 4 \( \eta-\eta' \) decay constants conveniently parametrized in terms of two decay constants and two angles:

\[
\begin{align*}
 f_\eta^8 &= \cos \theta_8 \, f_8 , \quad f_\eta^0 = -\sin \theta_0 \, f_0 , \\
 f_{\eta'}^8 &= \sin \theta_8 \, f_8 , \quad f_{\eta'}^0 = \cos \theta_0 \, f_0 , \\
 f_{\eta}^{NS} &= \cos \phi_{NS} \, f_{NS} , \quad f_{\eta}^S = -\sin \phi_S \, f_S , \\
 f_{\eta'}^{NS} &= \sin \phi_{NS} \, f_{NS} , \quad f_{\eta'}^S = \cos \phi_S \, f_S
\end{align*}
\]

- Big practical difference between 0-8 and \( NS-S \) schemes:

- while \( \theta_8 \) and \( \theta_0 \) differ a lot from each other and from \( \theta \approx (\theta_8 + \theta_0)/2 \), FKS showed that \( \phi_{NS} \approx \phi_S \approx \phi \).

\[
\begin{bmatrix}
 f_{\eta}^{NS} & f_{\eta}^S \\
 f_{\eta'}^{NS} & f_{\eta'}^S
\end{bmatrix}
= \begin{bmatrix}
 \cos \phi & -\sin \phi \\
 \sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
 f_{NS} & 0 \\
 0 & f_S
\end{bmatrix}.
\]
For four decay constants, can use FKS one-angle scheme!

we can relate \( \{ f^8, f^8', f^0, f^0' \} \) with \( \{ f_{NS}, f_S \} = \{ f_\pi, f_{s\bar{s}} \} \):

\[
\begin{bmatrix}
  f^8_{\eta} & f^0_{\eta} \\
  f^8_{\eta'} & f^0_{\eta'}
\end{bmatrix} =
\begin{bmatrix}
  \cos \phi & -\sin \phi \\
  \sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
  f_{NS} & 0 \\
  0 & f_S
\end{bmatrix}
\begin{bmatrix}
  \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\
  -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}}
\end{bmatrix}
\]

Some other useful relations between quantities of \( NS-S \) (FKS) and \( 0-8 \) schemes:

\[
f^8 = \sqrt{\frac{1}{3} f^2_{NS} + \frac{2}{3} f^2_S}, \quad \theta^8 = \phi - \arctan \left( \sqrt{\frac{2 f_S}{f_{NS}}} \right),
\]

\[
f^0 = \sqrt{\frac{2}{3} f^2_{NS} + \frac{1}{3} f^2_S}, \quad \theta^0 = \phi - \arctan \left( \sqrt{\frac{2 f_{NS}}{f_S}} \right).
\]
For 3 DS models: $T = 0$ results of Shore’s generalization


Jain-Munczek $\langle A^2 \rangle$-induced separable

<table>
<thead>
<tr>
<th>$\chi_{YM}$</th>
<th>$191^4$</th>
<th>$175.7^4$</th>
<th>$191^4$</th>
<th>$175.7^4$</th>
<th>$191^4$</th>
<th>$175.7^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_\eta$</td>
<td>499.8</td>
<td>485.7</td>
<td>496.7</td>
<td>482.8</td>
<td>526.2</td>
<td>507.0</td>
</tr>
<tr>
<td>$M_{\eta'}$</td>
<td>931.4</td>
<td>815.8</td>
<td>934.9</td>
<td>818.4</td>
<td>983.2</td>
<td>868.7</td>
</tr>
<tr>
<td>$\phi$</td>
<td>52.01°</td>
<td>46.11°</td>
<td>51.85°</td>
<td>46.07°</td>
<td>47.23°</td>
<td>40.86°</td>
</tr>
<tr>
<td>$\theta_0$</td>
<td>7.74°</td>
<td>1.84°</td>
<td>7.17°</td>
<td>1.39°</td>
<td>-0.33°</td>
<td>-6.69°</td>
</tr>
<tr>
<td>$\theta_8$</td>
<td>-12.00°</td>
<td>-17.90°</td>
<td>-11.85°</td>
<td>-17.6°</td>
<td>-14.12°</td>
<td>-20.47°</td>
</tr>
<tr>
<td>$f_0$</td>
<td>108.8</td>
<td>108.8</td>
<td>107.9</td>
<td>107.9</td>
<td>101.8</td>
<td>101.8</td>
</tr>
<tr>
<td>$f_8$</td>
<td>122.6</td>
<td>122.6</td>
<td>121.1</td>
<td>121.1</td>
<td>110.7</td>
<td>110.7</td>
</tr>
<tr>
<td>$f_0^\eta$</td>
<td>-14.7</td>
<td>-3.5</td>
<td>-13.5</td>
<td>-2.6</td>
<td>0.6</td>
<td>11.9</td>
</tr>
<tr>
<td>$f_0^{\eta'}$</td>
<td>107.9</td>
<td>108.8</td>
<td>107.1</td>
<td>107.9</td>
<td>101.8</td>
<td>101.1</td>
</tr>
<tr>
<td>$f_8^\eta$</td>
<td>119.9</td>
<td>116.7</td>
<td>118.5</td>
<td>115.4</td>
<td>107.4</td>
<td>103.7</td>
</tr>
<tr>
<td>$f_8^{\eta'}$</td>
<td>-25.5</td>
<td>-37.7</td>
<td>-2.49</td>
<td>-37.6</td>
<td>-27.0</td>
<td>-38.7</td>
</tr>
</tbody>
</table>

For $T > 0$, the substitution $A \to \tilde{\chi}(T)$ leads to similar successful results as in the WV case!
The results of the approach through Witten-Veneziano relation and Shore’s approach are (qualitatively) quite similar.

Data on $\eta'$ enhanced multiplicity in RHIC experiments ⇒ neither the original Witten-Veneziano relation nor Shore’s scheme can be straightforwardly extended to $T$ close to $T_{Ch}$, because the the ratio $6\chi_{YM}(T)/f_\pi(T)^2$ tends to blow up as $T \to T_{Ch}$.

We find that the Leutwyler-Smilga relation enables one
1.) to retain unchanged Witten-Veneziano relation, with $\chi_{YM}$, for $T = 0$ (in fact, for any $T$ sufficiently below $T_{Ch}$) but also
2.) to replace the $T$-dependence of $\chi_{YM}$ by that of the chiral condensate. This ties the $U_A(1)$ symmetry restoration with the chiral symmetry restoration, and achieves consistency of both Witten-Veneziano and Shore’s approach with the data on $\eta'$ multiplicity.