η' multiplicity and Witten-Veneziano relation at finite temperature[a]

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Introduction and motivation

RHIC experiments show that *hot QCD matter has very intricate properties* = *a big challenge to understand.*
But it is clear that:

- Hot QCD matter may be called "QGP", but this cannot be perturbatively interacting quark-gluon gas (as once expected) until much higher T
- Still no direct “smoking gun" signal of deconfinement, etc ...
  compelling signals of new form of matter sought:
- e.g., a change in symmetries obeyed by the strong interaction: the restoration of the $[SU_A(3)$ flavor] *chiral symmetry* and $U_A(1)$ symmetry $\Rightarrow$ a good understanding of the *light-quark pseudoscalar nonet* is needed - *especially* η, η'.

Hot hadrons = important for understanding hot QCD matter

- especially since lattice (& other, e.g., see Gossiaux's talk) show: J/Ψ and η_c stay bound till $\sim 2T_{cri}$, maybe higher ... + similar indications about light-quark mesons = motivation to:

- explore validity of meson relations, e.g., *WV relation*:

$$M_{\eta'}{}^2 + M_{\eta}{}^2 - 2\, M_K{}^2 = \frac{2N_f}{f_\pi^2}\,\chi_{\rm YM} \qquad \left(+O\left(\frac{1}{N_c}\right)\right)$$

  (... and test validity of various T-rescaling procedures).

- use, even at high T, ***bound-state equations*** ... here, Dyson-Schwinger approach (by Zagreb group: Horvatić *et al.* PRD76 (2007) 096009) for non-anomalous sector, but results of Benić *et al.*, Phys. Rev. D84 (2011) 016006, for $U_A(1)$-anomalous sector of η-η' complex at $T > 0$.

Introduction and motivation

$U_A(1)$ symmetry is broken by the nonabelian ("gluon") axial anomaly: even in the chiral limit (ChLim, where $m_q \to 0$),

$$\partial_\alpha \bar{\psi}(x)\gamma^\alpha\gamma_5 \frac{\lambda^0}{2}\psi(x) \propto F^a(x)\cdot\widetilde{F}^a(x) \equiv \epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x) \neq 0\,.$$

This breaks the $U_A(1)$ symmetry of QCD and precludes the 9^{th} Goldstone pseudoscalar meson $\Rightarrow$ very massive η': even in ChLim, where $m_\pi, m_K, m_\eta \to 0$, still ('ChLim WVR')

$$0 \neq \Delta M^2_{\eta_0} = \Delta M^2_{\eta'} = \frac{(A = \text{qty.dim.mass})^4}{(\text{``}f_{\eta'}\text{''})^2} = \frac{6\,\chi_{\text{YM}}}{f_\pi^2} + O(\frac{1}{N_c})$$

... but uncertain fate of $U_A(1)$ breaking as T grows towards T_{YM} and T_{Ch}, where $\chi_{\text{YM}}(T)$ and $f_\pi(T)$ **strongly drop/vanish**!

Experimental observation of in-medium η' mass reduction

- High E heavy ion collisions $\Rightarrow$ hot and dense medium
- $\sqrt{s} = 200$ GeV Au+Au collide at RHIC $\Rightarrow$ enhanced η' abundance = 1^{st} exp. signature of a partial $U_A(1)$ sym. restoration ... is it before chiral symmetry restoration?!?!
- Combined STAR & PHENIX data analyzed robustly through six popular models for multiplicities (ALCOR, FRITIOF, ...) $\Rightarrow$ at 99,9% confidence level, η' mass is reduced by at least 200 MeV inside fireball.
  $M^*_{\eta'} = 340^{+50}_{-60}(statist.)^{+280}_{-140}(model) \pm 42(system.)\mathrm{MeV}$
  (Csörgő, Vértesi & Sziklai, Phys. Rev. Lett. 105 (2010) 182301.

  = *"The return of the prodigal Goldstone boson!"*

What are implications for the WV relation at $T > 0$?

Dyson-Schwinger approach to quark-hadron physics

- = the bound state approach which is nopertubative, covariant and chirally well-behaved.
- a) direct contact with QCD through *ab initio* calculations
- b) phenomenological modeling of hadrons as quark bound states (used also here, for example)
- coupled system of integral equations for Green functions of QCD
- ... but ... equation for n-point function calls (n+1)-point function ... $\to$ cannot solve in full the growing tower of DS equations
- $\to$ various degrees of truncations, approximations and modeling is unavoidable (more so in phenomenological modeling of hadrons, as here)

Dyson-Schwinger approach to quark-hadron physics

For the present purposes, the most important advantage of DS approach is that it is chirally well-behaved: non-anomalous parts of the masses of the light pseudoscalar $q\bar{q}'$ mesons (i.e., all parts except ΔM_{η_0}) behave as $M^2_{q\bar{q}'} = \text{const}\,(m_q + m_{q'})\,, \quad (q, q' = u, d, s)\,.$

$\Rightarrow$ non-anomalous parts of the masses in WVR cancel:

$M_{\eta'}{}^2 + M_\eta{}^2 - 2\,M_K{}^2 \approx \Delta M_{\eta_0}$

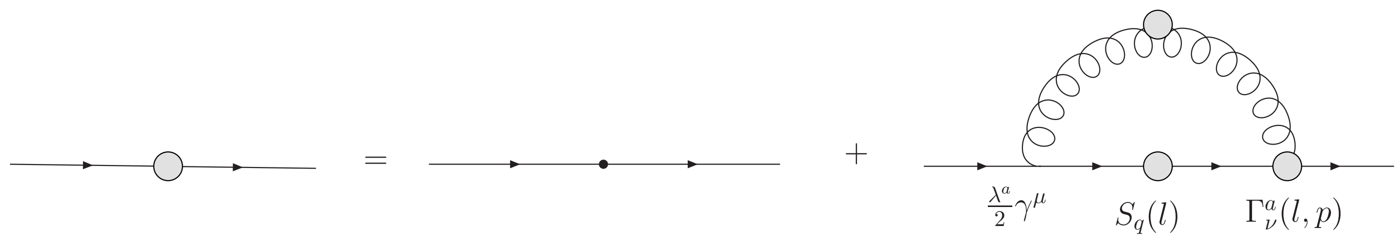
$\Rightarrow$ already ChLim WVR reveals the essence of the influence of the gluon anomaly on the masses in η-η' complex.

= IMPORTANT, since it shows almost model-independently that the WVR containing $\chi_{\rm YM}(T)$ implies $M_{\eta'}(T)$ in conflict with experiment

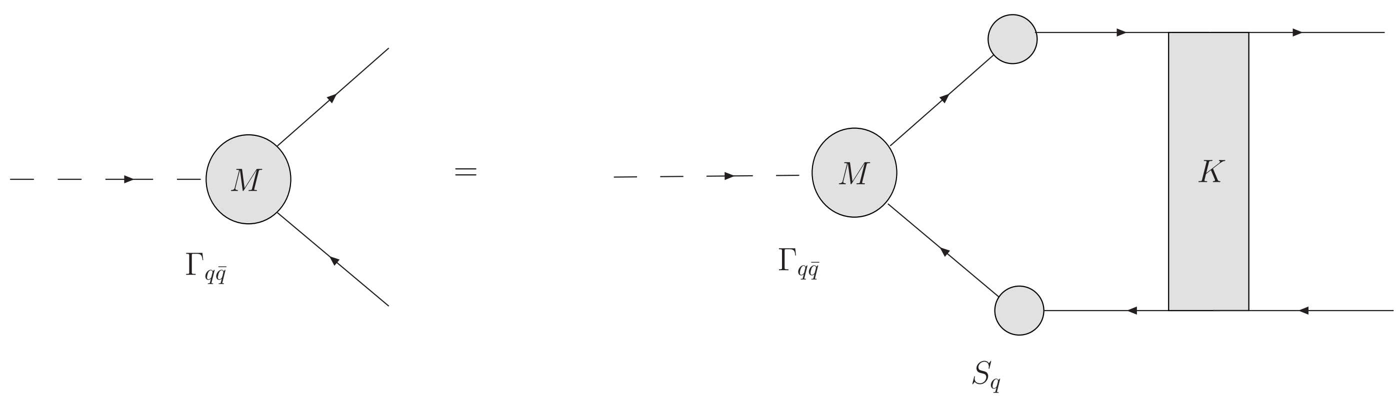
$\Rightarrow$ Model dependence of our discussion is minimal as everything boils down to the ratio $\chi_{\rm YM}(T)/f_\pi(T)^2$ ...

Dyson-Schwinger approach to quark-hadron physics

- Gap equation for propagator S_q of dressed quark q



- Homogeneous Bethe-Salpeter (BS) equation for a Meson $q\bar{q}$ bound state vertex $\Gamma_{q\bar{q}}$



Gap and BS equations in ladder truncation

$$S_q(p)^{-1} = i\gamma \cdot p + \widetilde{m}_q + \frac{4}{3}\int \frac{d^4\ell}{(2\pi)^4}\, g^2 G^{\rm eff}_{\mu\nu}(p-\ell)\gamma_\mu S_q(\ell)\gamma_\nu$$

$$\to S_q(p) = \frac{1}{i\not{p}\, A_q(p^2) + B_q(p^2)} = \frac{-i\not{p}\, A_q(p^2) + B_q(p^2)}{p^2 A_q(p^2)^2 + B_q(p^2)^2} = \frac{1}{A_q(p^2)}\,\frac{-i\not{p} + m_q(p^2)}{p^2 + m_q(p^2)^2}$$

$$\Gamma_{q\bar{q}'}(p,P) = -\frac{4}{3}\int \frac{d^4\ell}{(2\pi)^4} g^2 G^{\rm eff}_{\mu\nu}(p-\ell)\gamma_\mu S_q(\ell+\frac{P}{2})\Gamma_{q\bar{q}'}(\ell,P) S_q(\ell-\frac{P}{2})\gamma_\nu$$

- Euclidean space: $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$, $a\cdot b = \sum_{i=1}^4 a_i b_i$
- P is the total momentum, $M^2 = -P^2$ meson mass^2
- $G^{\rm eff}_{\mu\nu}(k)$ an “effective gluon propagator” - modeled !

From the gap and BS equations ...

- solutions of the gap equation $\rightarrow$ the dressed quark mass function

$$m_q(p^2) = \frac{B_q(p^2)}{A_q(p^2)}$$

- propagator solutions $A_q(p^2)$ and $B_q(p^2)$ pertain to confined quarks if

$$m_q^2(p^2) \neq -p^2 \quad \text{for} \;\; \text{real} \quad p^2$$

- The BS solutions $\Gamma_{q\bar{q}'}$ enable the calculation of the properties of $q\bar{q}$ bound states, such as the decay constants of pseudoscalar mesons:

$$\begin{aligned} f_{PS}\, P_\mu &= \langle 0|\bar{q}\, \frac{\lambda^{PS}}{2}\, \gamma_\mu\, \gamma_5\, q|\Phi_{PS}(P)\rangle \\ \longrightarrow \quad f_\pi P_\mu &= N_c\, \mathrm{tr_s} \int \frac{d^4\ell}{(2\pi)^4}\, \gamma_5\gamma_\mu\, S(\ell + P/2)\, \Gamma_\pi(\ell; P)\, S(\ell - P/2) \end{aligned}$$

Renormalization-group improved interactions

$$\text{Landau gauge gluon propagator}: \ g^2 G^{\text{eff}}_{\mu\nu}(k) = G(-k^2)(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}),$$

$$G(Q^2) \equiv 4\pi \frac{\alpha_s^{\text{eff}}(Q^2)}{Q^2} = G_{\mathsf{UV}}(Q^2) + G_{\mathsf{IR}}(Q^2), \qquad Q^2 \equiv -k^2 \ .$$

$$G_{\mathsf{UV}}(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \approx \frac{4\pi^2 d}{Q^2 \ln(x_0 + \frac{Q^2}{\Lambda^2_{\mathsf{QCD}}})} \{1 + b \, \frac{\ln[\ln(x_0 + \frac{Q^2}{\Lambda^2_{\mathsf{QCD}}})]}{\ln(x_0 + \frac{Q^2}{\Lambda^2_{\mathsf{QCD}}})}\} \ ,$$

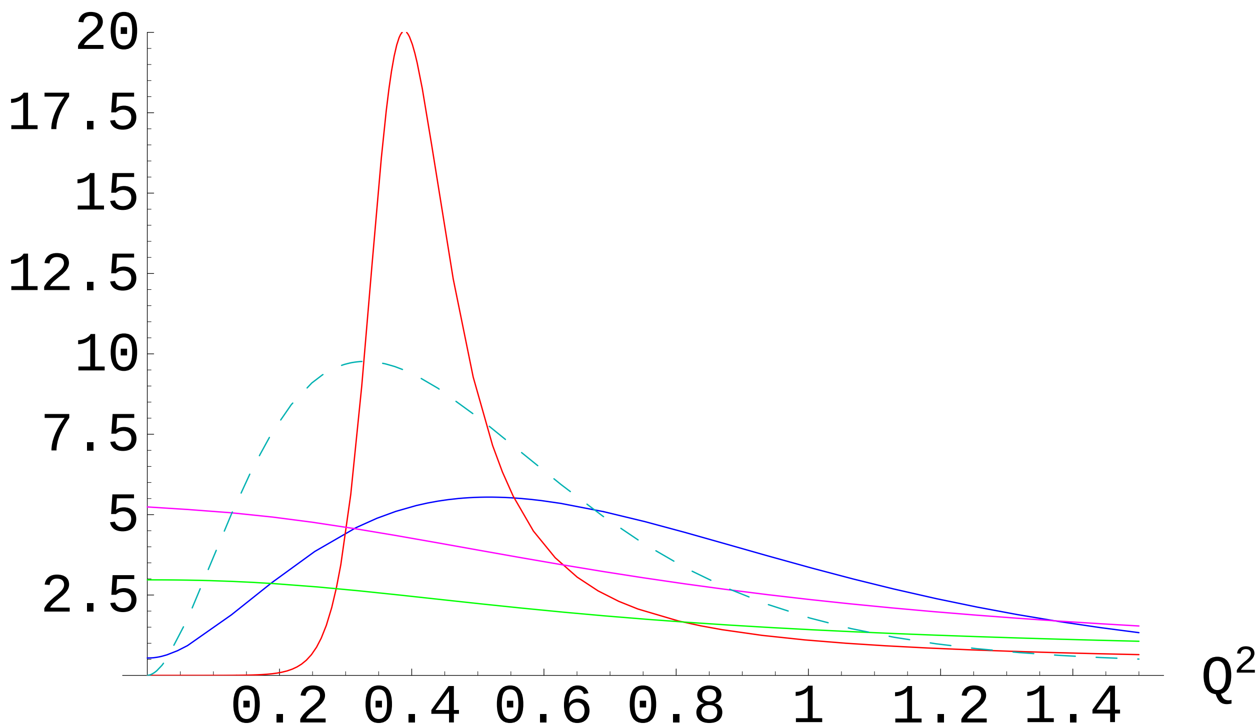
- but modeled non-perturbative part, e.g., Jain & Munczek:

$$G_{\mathsf{IR}}(Q^2) = G_{\mathsf{non\text{-}pert}}(Q^2) = 4\pi^2 \, a \, Q^2 \, \exp(-\mu Q^2) \quad (\text{similar}: \ \text{Maris, Roberts...})$$

- or, the dressed propagator with dim. 2 gluon condensate $\langle A^2 \rangle$-induced dynamical gluon mass (Kekez & Klabučar):

$$G(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \left(\frac{Q^2}{Q^2 - M^2_{\mathsf{gluon}} + \frac{c_{\mathsf{ghost}}}{Q^2}} \right)^2 \frac{Q^2}{Q^2 + M^2_{\mathsf{gluon}} + \frac{c_{\mathsf{gluon}}}{Q^2}} \ .$$

Some effective strong couplings $\alpha_s^{\rm eff}(Q^2) \equiv Q^2\, G(Q^2)/4\pi$



- Blue = Munczek & Jain model. Red = K & K propagator with $\langle A^2 \rangle$-induced dynamical gluon mass. Green = Alkofer. Magenta = Bloch. Turquoise dashed: Maris, Roberts & Tandy model.
  Important: integrated IR strength must be sufficient for **DChSB**!

Separable model = good, + easier at $T > 0$

- Calculations simplify with the separable Ansatz for $G^{\rm eff}_{\mu\nu}$:

$$G^{\rm eff}_{\mu\nu}(p-q) \to \delta_{\mu\nu}\, G(p^2, q^2, p\cdot q)$$

$$G(p^2, q^2, p\cdot q) = D_0\ f_0(p^2) f_0(q^2) + D_1\ f_1(p^2)(p\cdot q) f_1(q^2)$$

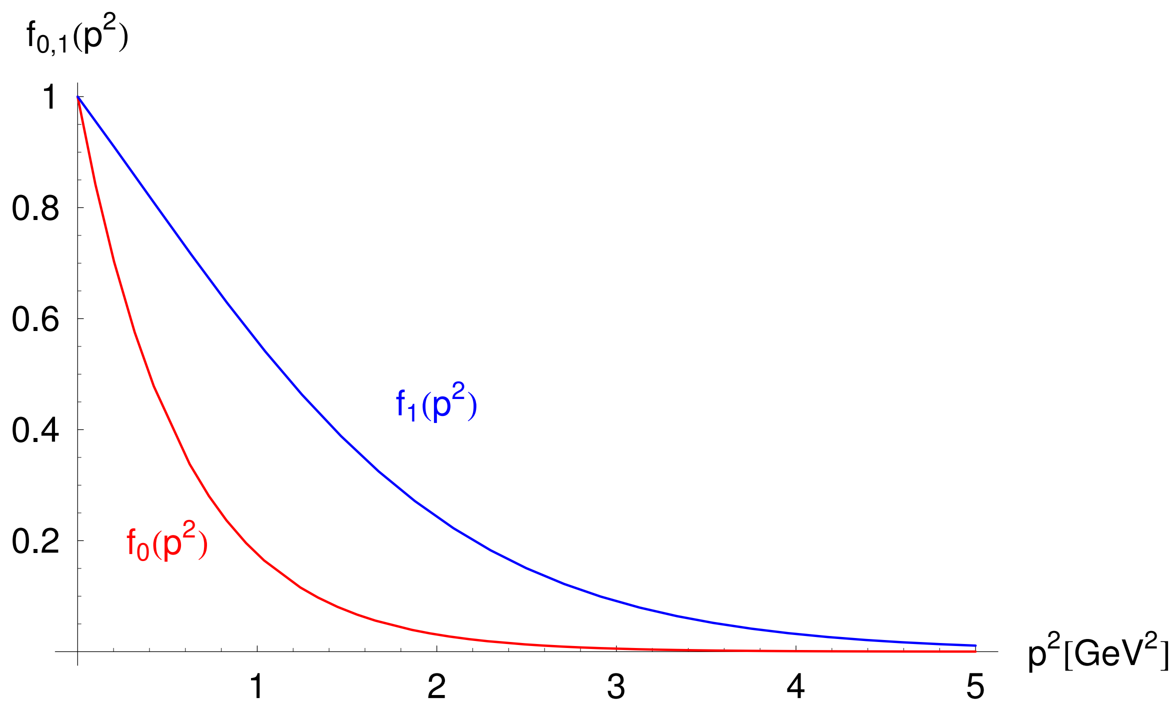
- two strength parameters D_0, D_1, and corresponding form factors $f_i(p^2)$. In the separable model, gap equation yields

$$\begin{aligned} B_f(p^2) &= \widetilde{m}_f + \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} G(p^2, q^2, p\cdot q) \frac{B_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)} \\ \left[A_f(p^2) - 1\right] p^2 &= \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} G(p^2, q^2, p\cdot q) \frac{(p\cdot q) A_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)} \,. \end{aligned}$$

- This gives $B_f(p^2) = \widetilde{m}_f + b_f\ f_0(p^2)$ and $A_f(p^2) = 1 + a_f\ f_1(p^2)$, reducing to nonlinear equations for constants b_f and a_f.

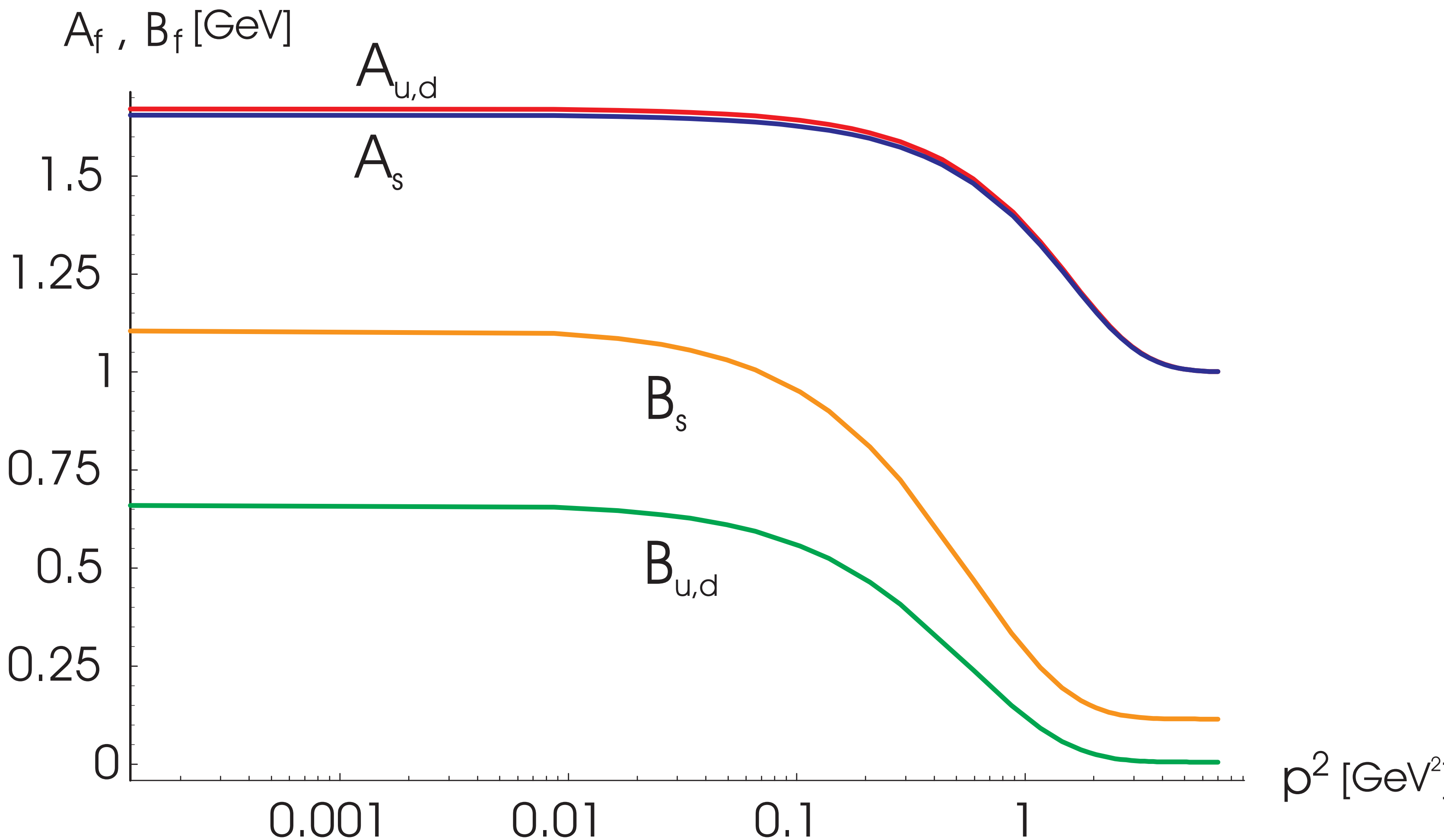
A simple choice for ‘interaction form factors’ of the separable model:

- $f_0(p^2) = \exp(-p^2/\Lambda_0^2)$
- $f_1(p^2) = [1 + \exp(-p_0^2/\Lambda_1^2)]/[1 + \exp((p^2 - p_0^2))/\Lambda_1^2]$
  gives good description of pseudoscalar properties if the interaction is strong enough for realistic DChSB, when $m_{u,d}(p^2 \sim small) \sim$ the typical constituent quark mass scale $\sim m_\rho/2 \sim m_N/3$.



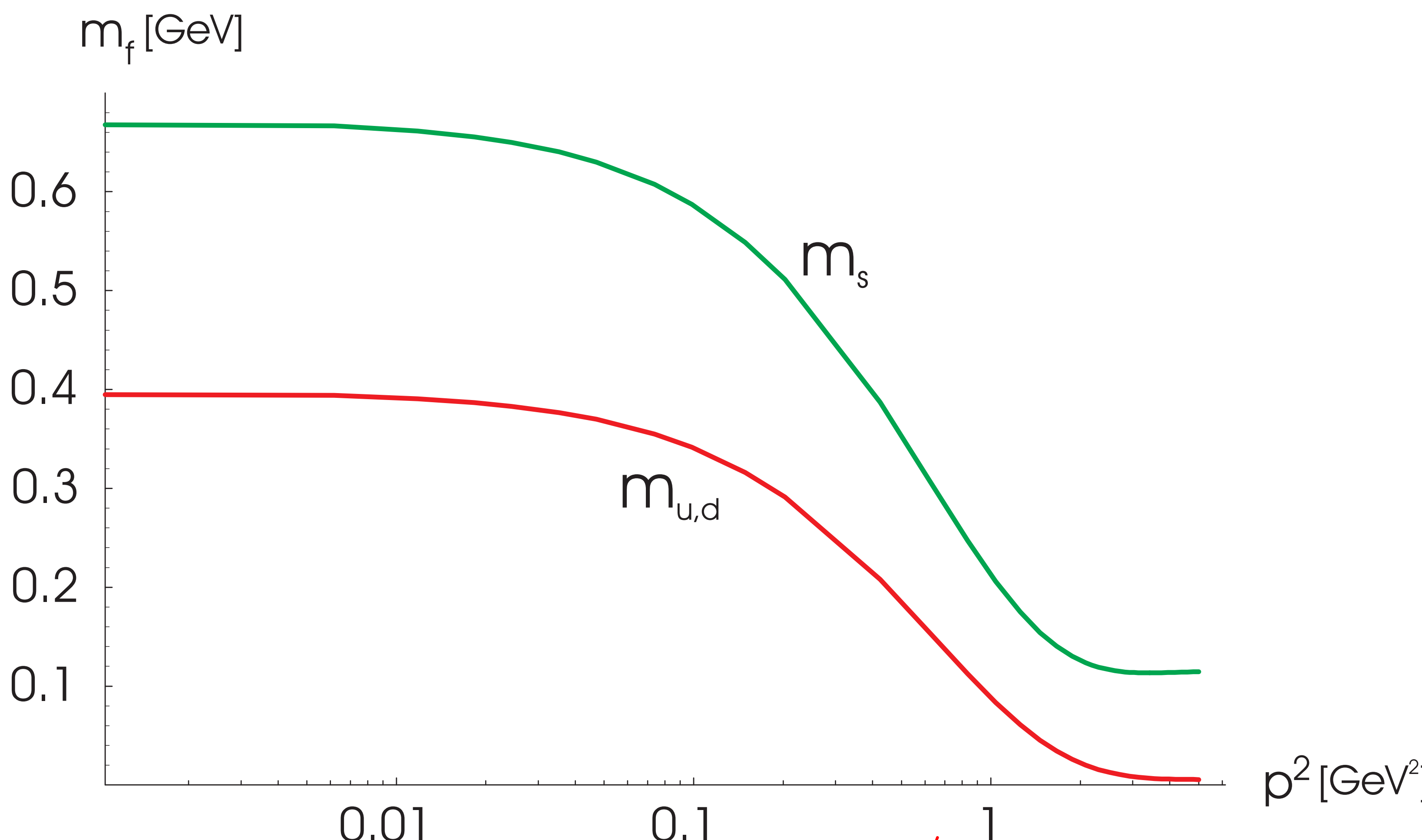
Nonperturbative dynamical propagator dressing

- $\longrightarrow$ Dynamical Chiral Symmetry Breaking (DChSB)



DChSB = nonperturb. generation of large quark masses ...

- ... even in the chiral limit ($\widetilde{m}_f \to 0$), where the octet pseudoscalar mesons are Goldstone bosons of DChSB!



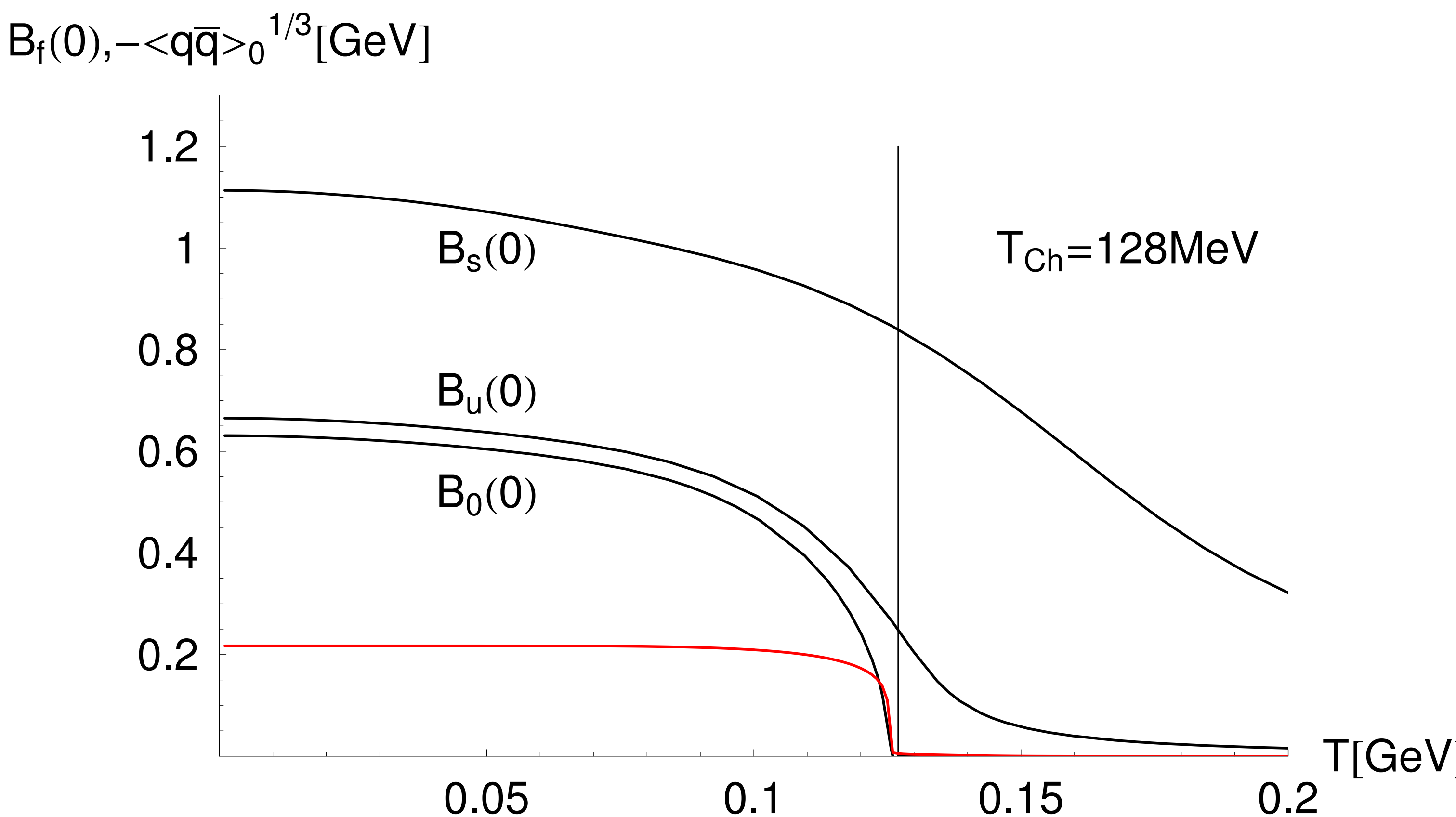
At $T = 0$, good DS results; e.g., "non-anomalous":

- Separable model parameter values reproducing experimental data:
- $\widetilde{m}_{u,d} = 5.5$ MeV, $\Lambda_0 = 758$ MeV, $\Lambda_1 = 961$ MeV, $p_0 = 600$ MeV, $D_0\Lambda_0^2 = 219$, $D_1\Lambda_1^4 = 40$ (fixed by fitting M_π, f_π, M_ρ, $g_{\rho\pi^+\pi^-}$, $g_{\rho e^+e^-}$ $\to$ pertinent predictions $a_{u,d} = 0.672$, $b_{u,d} = 660$ MeV, i.e., $m_{u,d}(p^2)$, $\langle\bar{u}u\rangle$)
- $\widetilde{m}_s = 115$ MeV (fixed by fitting $M_K \to$ predictions $a_s = 0.657$, $b_s = 998$ MeV, i.e., $m_s(p^2)$, $\langle\bar{s}s\rangle$, $M_{s\bar{s}}$, f_K, $f_{s\bar{s}}$)
- Summary of results (all in GeV) for $q = u, d, s$ and pseudoscalar mesons without the influence of gluon anomaly:

PS	M_{PS}	M_{PS}^{exp}	f_{PS}	f_{PS}^{exp}	$m_q(0)$	$-\langle q\bar{q}\rangle_0^{1/3}$
π	0.140	0.1396	0.092	0.0924 ± 0.0003	0.398	0.217
K	0.495	0.4937	0.110	0.1130 ± 0.0010		
$s\bar{s}$	0.685		0.119		0.672	

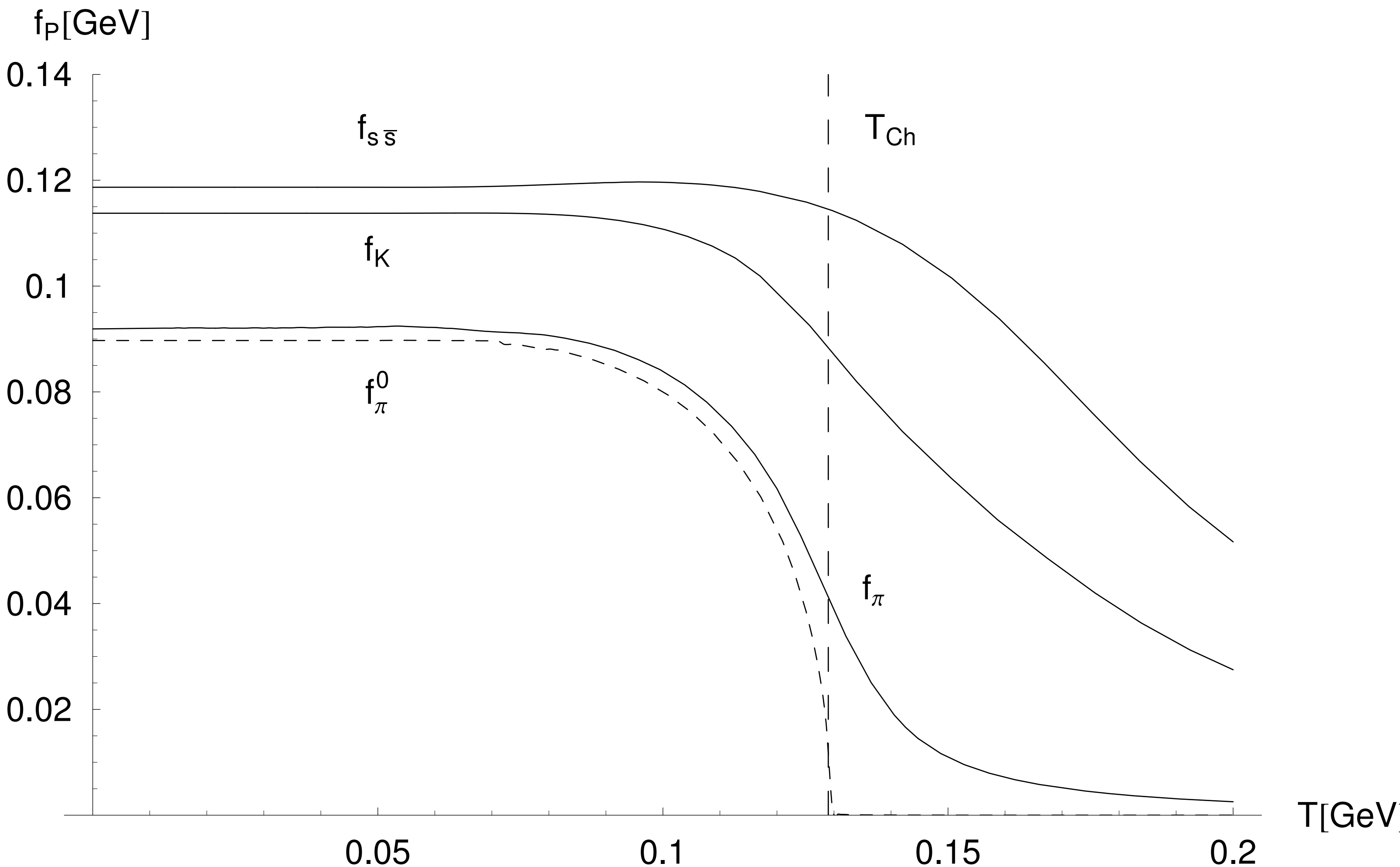
At $T > 0$, good and less good DS results

E.g., chiral symmetry restoration qualitatively good, but T_{Ch} lower than lattice (maybe up to 35%, and even more for 'more realistic' DS models unless they contain δ-function):



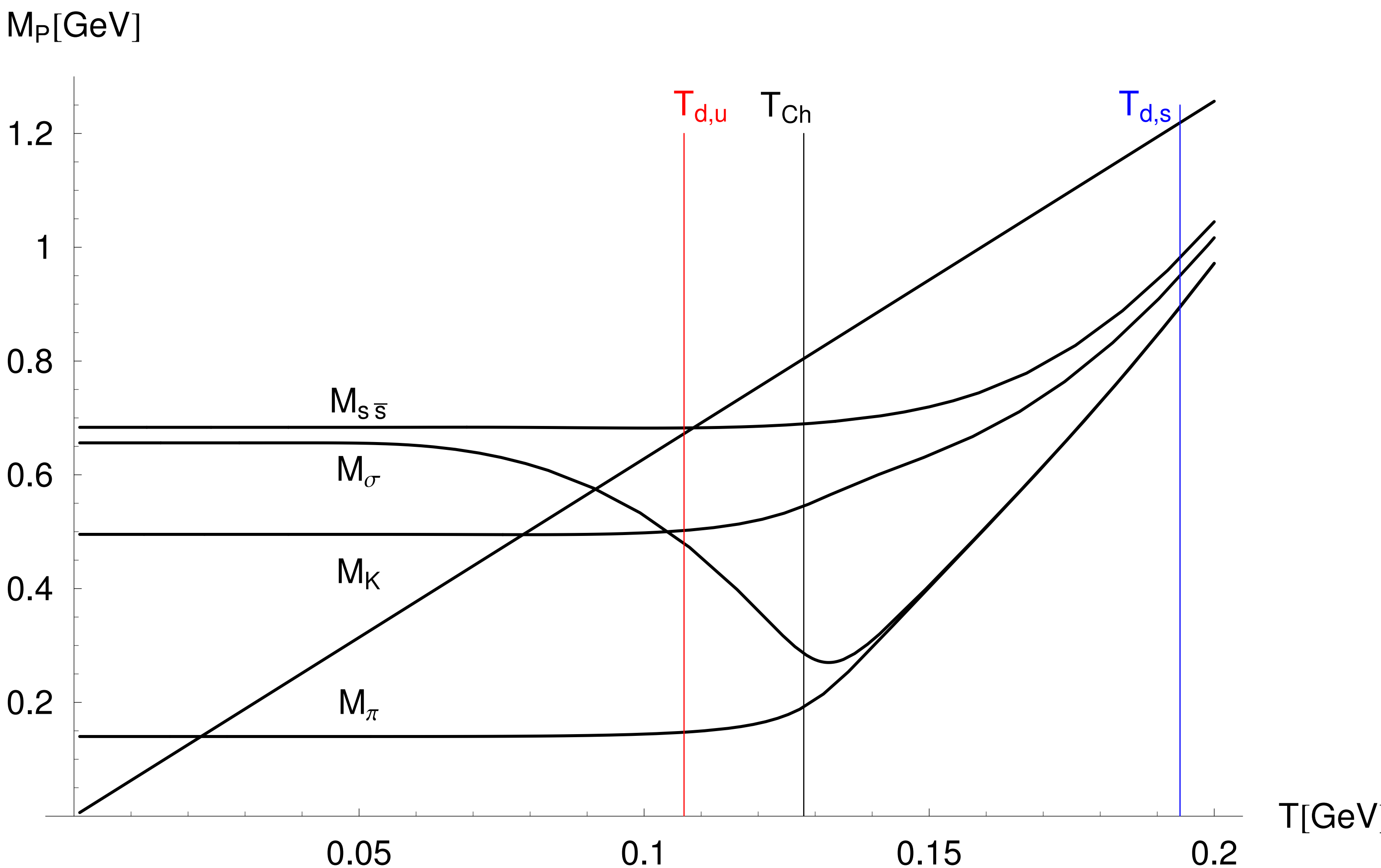
Same with pseudoscalar decay constants $f_P(T)$:

Both crossover and Ch-limit behavior OK, but $T_{\rm Ch} = 128$ MeV
... but this is cured by introducing Polyakov loop (PL)



Similarly with the T-dependence of $\pi, K, s\bar{s}, \sigma$ masses:

'Deconfinement' $T_{d,q}$ from S_q pole - very different $T_{d,u}$, $T_{d,s}$ ... also cured/synchronized with $T_{\rm Ch} (= T_{cri})$ by PL



Anomaly and mixing in η-η' complex

- present approach yields mass^2 eigenvalues $M^2_{u\bar{d}} = M^2_{\pi^+}, M^2_{u\bar{s}} = M^2_K, ..., \hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$
- $|u\bar{d}\rangle = |\pi^+\rangle, |u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles (at $T = 0$ at least!), although in the isospin limit (adopted from now on) $M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_\pi$. I is a good quantum number!
- $\Rightarrow$ recouple into "more physical" $I_3 = 0$ octet-singlet basis

$$
\begin{aligned}
I = 1 \qquad |\pi^0\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \,, \\
I = 0 \qquad |\eta_8\rangle &= \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) \,, \\
I = 0 \qquad |\eta_0\rangle &= \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \,.
\end{aligned}
$$

Anomaly and mixing in η-η' complex

- the "non-anomalous" (chiral-limit-vanishing!) part of the mass-squared matrix of π^0 and η's is (in π^0-η_8-η_0 basis)

$$\hat{M}^2_{NA} = \begin{pmatrix} M^2_\pi & 0 & 0 \\ 0 & M^2_{88} & M^2_{80} \\ 0 & M^2_{08} & M^2_{00} \end{pmatrix}$$

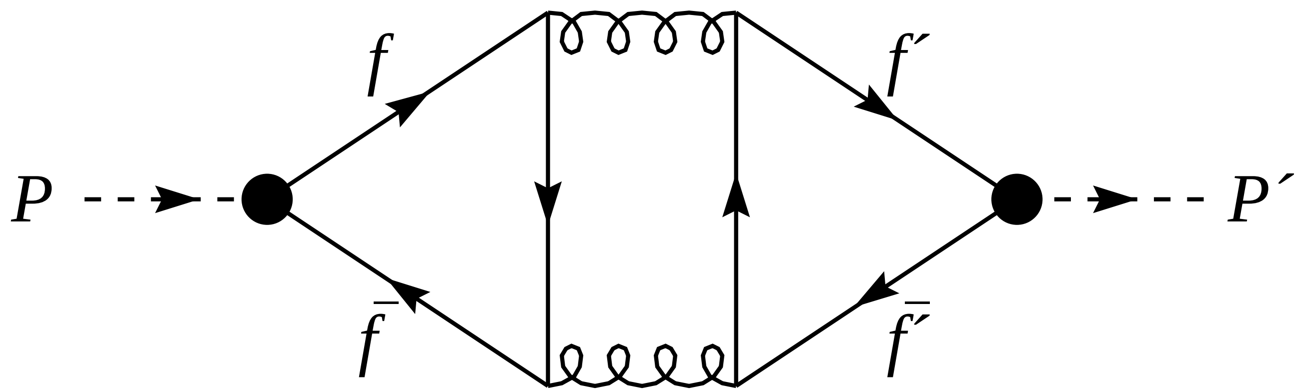
$$M^2_{88} \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_8 \rangle = \frac{2}{3}(M^2_{s\bar{s}} + \frac{1}{2}M^2_\pi),$$

$$M^2_{80} \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_0 \rangle = M^2_{08} = \frac{\sqrt{2}}{3}(M^2_\pi - M^2_{s\bar{s}})$$

$$M^2_{00} \equiv \langle \eta_0 | \hat{M}^2_{NA} | \eta_0 \rangle = \frac{2}{3}(\frac{1}{2}M^2_{s\bar{s}} + M^2_\pi),$$

- Not enough! In order to avoid the U$_A$(1) problem, one must break the U$_A$(1) symmetry (as it is destroyed by the gluon anomaly) at least at the level of the masses.

Gluon anomaly is not accessible to ladder approximation!



- **Diamond graph**: an example of a transition $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ ($q, q' = u, d, s[...]$), contributing to the anomalous masses in the η-η' complex, but not included in the interaction kernel in the ladder approximation.

Anomaly and mixing in η-η' complex

- All masses in $\hat{M}^2_{NA}$ are calculated in the ladder approx., which cannot include the gluon anomaly contributions.
- Large N_c: the gluon anomaly suppressed as $1/N_c$! $\rightarrow$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the η_0 (squared) mass by anomalous contribution 3β.
- complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$ where

$$\hat{M}^2_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{pmatrix} \qquad \text{does not vanish in the chiral limit.}$$

$3\beta = \Delta M^2_{\eta_0}$ = the anomalous mass^2 of η_0 [in SU(3) limit incl. ChLim] is related to the YM topological susceptibility. Fixed by phenomenology or (here) taken from the lattice.

Anomaly and mixing in η-η' complex

- we can also rewrite $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$

$$\hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{breaking}]{\text{flavor}} \hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}$$

- We introduced the effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \to |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$. $s\bar{s}$ transition suppression estimated by $X \approx f_\pi / f_{s\bar{s}}$.
- Then, $\hat{M}_A^2$ in the octet-singlet basis is modified to

$$\hat{M}_A^2 = \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2-X-X^2) \\ 0 & \frac{\sqrt{2}}{3}(2-X-X^2) & \frac{1}{3}(2+X)^2 \end{pmatrix}$$

- $\to$ In the isospin limit, one can always restrict to 2×2 submatrix of etas $(I=0)$, as π^0 $(I=1)$ is decoupled then.

Anomaly and mixing in η-η' complex

- nonstrange (*NS*) – strange (*S*) basis

$$
\begin{aligned}
|\eta_{NS}\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle \ , \\
|\eta_S\rangle &= |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle \ .
\end{aligned}
$$

- the η–η' matrix in this basis is

$$
\hat{M}^2 = \begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S \eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} = \begin{pmatrix} M^2_{u\bar{u}} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{pmatrix} \overset{\phi}{\rightarrow} \begin{pmatrix} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{pmatrix}
$$

- *NS*–*S* mixing relations

$$
|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle \ , \qquad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle \ .
$$

$$
\theta = \phi - \arctan\sqrt{2}
$$

Anomaly and mixing in η-η' complex

- Let lowercase m_M's denote the empirical mass of meson M. From our calculated, model mass matrix in *NS*–*S* basis, we form its empirical counterpart $\hat{m}^2_{\rm exp}$ by
- *i)* obvious substitutions $M_{u\bar{u}} \equiv M_\pi \to m_\pi$, $M_{s\bar{s}} \to m_{s\bar{s}}$
- *ii)* by noting that $m_{s\bar{s}}$, the “empirical" mass of the unphysical $s\bar{s}$ pseudoscalar bound state, is given in terms of masses of physical particles as $m^2_{s\bar{s}} \approx 2m^2_K - m^2_\pi$. Then,

$$\hat{m}^2_{\rm exp} = \left[\begin{array}{cc} m^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & 2m^2_K - m^2_\pi + \beta X^2 \end{array} \right] \xrightarrow[\phi_{\rm exp}]{} \left[\begin{array}{cc} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{array} \right].$$

Finally, fix anomalous contribution to η-η':

- the trace of the empirical $\hat{m}^2_{\rm exp}$ demands the 1^{st} equality in

$$\beta(2+X^2) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{2N_f}{f_\pi^2}\chi_{\rm YM} \quad (2^{\rm nd}\text{equality} = \text{WV relation})$$

- requiring that the experimental trace $(m_\eta^2 + m_{\eta'}^2)_{exp} \approx 1.22$ GeV^2 be reproduced by the theoretical $\hat{M}^2$, yields $\beta_{\rm fit} = \frac{1}{2+X^2}[(m_\eta^2 + m_{\eta'}^2)_{exp} - (M^2_{u\bar{u}} + M^2_{s\bar{s}})]$
- Or, get β from lattice $\chi_{\rm YM}$! Then no free parameters!
- then, can calculate the NS-S mixing angle ϕ

$$\tan 2\phi = \frac{2\,M^2_{\eta_S\eta_{NS}}}{M^2_{\eta_S} - M^2_{\eta_{NS}}} = \frac{2\sqrt{2}\beta X}{M^2_{\eta_S} - M^2_{\eta_{NS}}} \qquad \text{and}$$

$$M^2_{\eta_{NS}} = M^2_{u\bar{u}} + 2\beta = M^2_\pi + 2\beta, \quad M^2_{\eta_S} = M^2_{s\bar{s}} + \beta X^2 = M^2_{s\bar{s}} + \beta\frac{f_\pi^2}{f^2_{s\bar{s}}}$$

Anomaly and mixing in η-η' complex

- The diagonalization of the NS-S mass matrix then finally gives us the *calculated* η and η' masses:

$$
\begin{aligned}
M_\eta^2 &= \cos^2\phi\, M_{\eta_{NS}}^2 - \sqrt{2}\beta X \sin 2\phi + \sin^2\phi\, M_{\eta_S}^2 \\
M_{\eta'}^2 &= \sin^2\phi\, M_{\eta_{NS}}^2 + \sqrt{2}\beta X \sin 2\phi + \cos^2\phi\, M_{\eta_S}^2
\end{aligned}
$$

- Equivalently, from the secular determinant,

$$
\begin{aligned}
M_\eta^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 - \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 8\beta^2 X^2} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) - \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right] \\
M_{\eta'}^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 + \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 8\beta^2 X^2} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) + \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right]
\end{aligned}
$$

Separable model results on η and η' mesons (at $T = 0$)

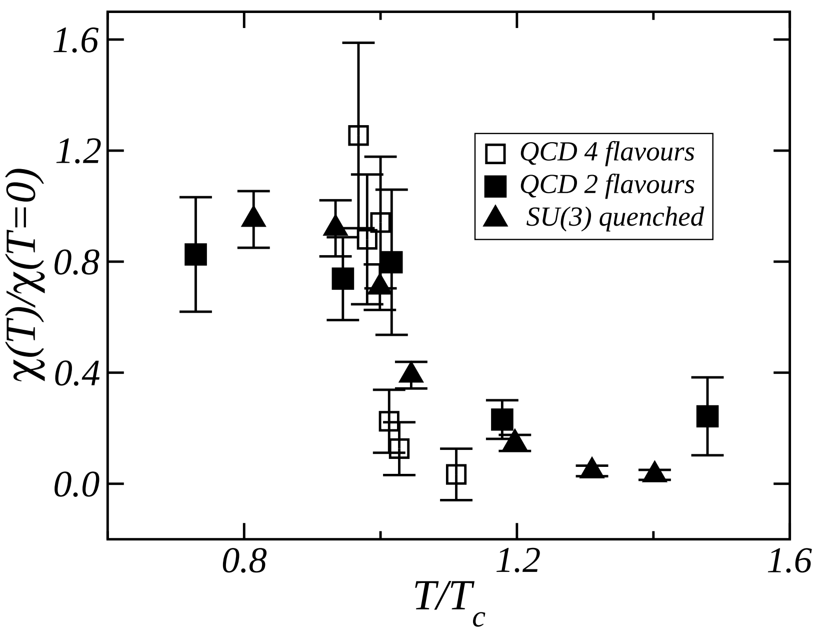
	β_{fit}	$\beta_{\text{latt.}}$	Exp.
θ	-12.22°	-13.92°	
M_η	548.9	543.1	547.75
$M_{\eta'}$	958.5	932.5	957.78
X	0.772	0.772	
3β	0.845	0.781	

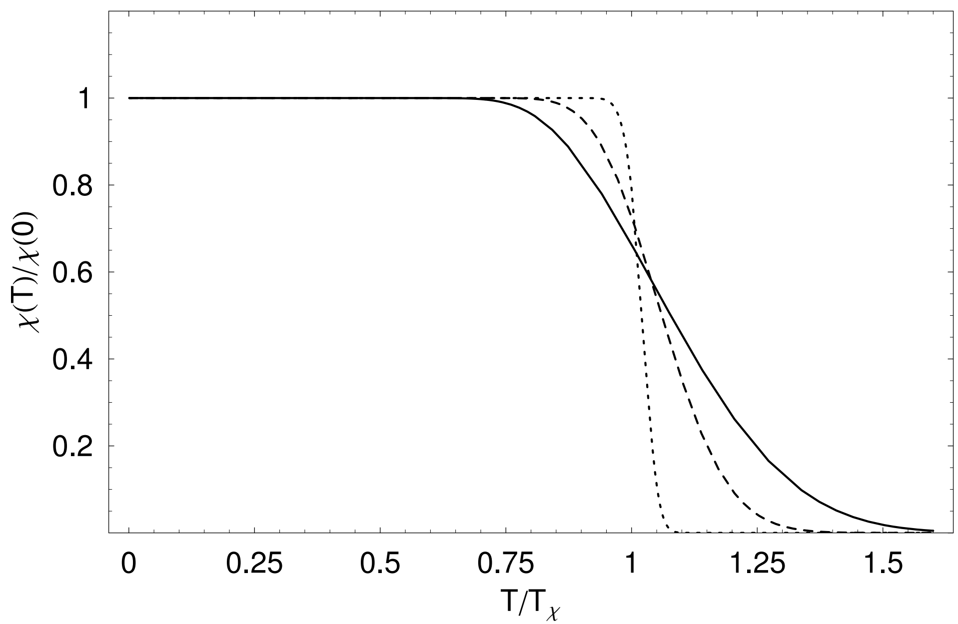
- masses are in units of MeV, 3β in units of GeV^2 and the mixing angles are dimensionless.
- $\beta_{\text{latt.}}$ was obtained from $\chi_{\text{YM}}(T = 0) = (175.7 \text{ MeV})^4$
- $X = f_\pi / f_{s\bar{s}}$ as well as the whole $\hat{M}^2_{NA}$ (consisting of M_π and $M_{s\bar{s}}$) are calculated model quantities.

For three DS models: summary of $T = 0$ results from WV

from Ref.	J-M&WV	A^2 &WV	separab&WV	orig. Shore	Experiment
M_π	137.3	135.0	140.0		$(138.0)^{isospin}_{average}$
M_K	495.7	494.9	495.0		$(495.7)^{isospin}_{average}$
$M_{s\bar{s}}$	700.7	722.1	684.8		
f_π	93.1	92.9	92.0		92.4 ± 0.3
f_K	113.4	111.5	110.1		113.0 ± 1.0
$f_{s\bar{s}}$	135.0	132.9	119.1		
M_η	568.2	577.1	542.3		547.75 ± 0.12
$M_{\eta'}$	920.4	932.0	932.6		957.78 ± 0.14
ϕ	41.42^o	39.56^o	40.75^o	38.24^o	
θ	-13.32^o	-15.18^o	-13.98^o	-16.5^o	
θ_0	-2.86^o	-5.12^o	-6.80^o	-12.3^o	
θ_8	-22.59^o	-24.14^o	-20.58^o	-20.1^o	
f_0	108.8	107.9	101.8	106.6	
f_8	122.6	121.1	110.7	104.8	
f_η^0	5.4	9.6	12.1	22.8	
$f_{\eta'}^0$	108.7	107.5	101.1	104.2	
f_η^8	113.2	110.5	103.7	98.4	
$f_{\eta'}^8$	-47.1	-49.5	-38.9	-36.1	

χ, topological susceptibility of QCD vacuum, at $T > 0$



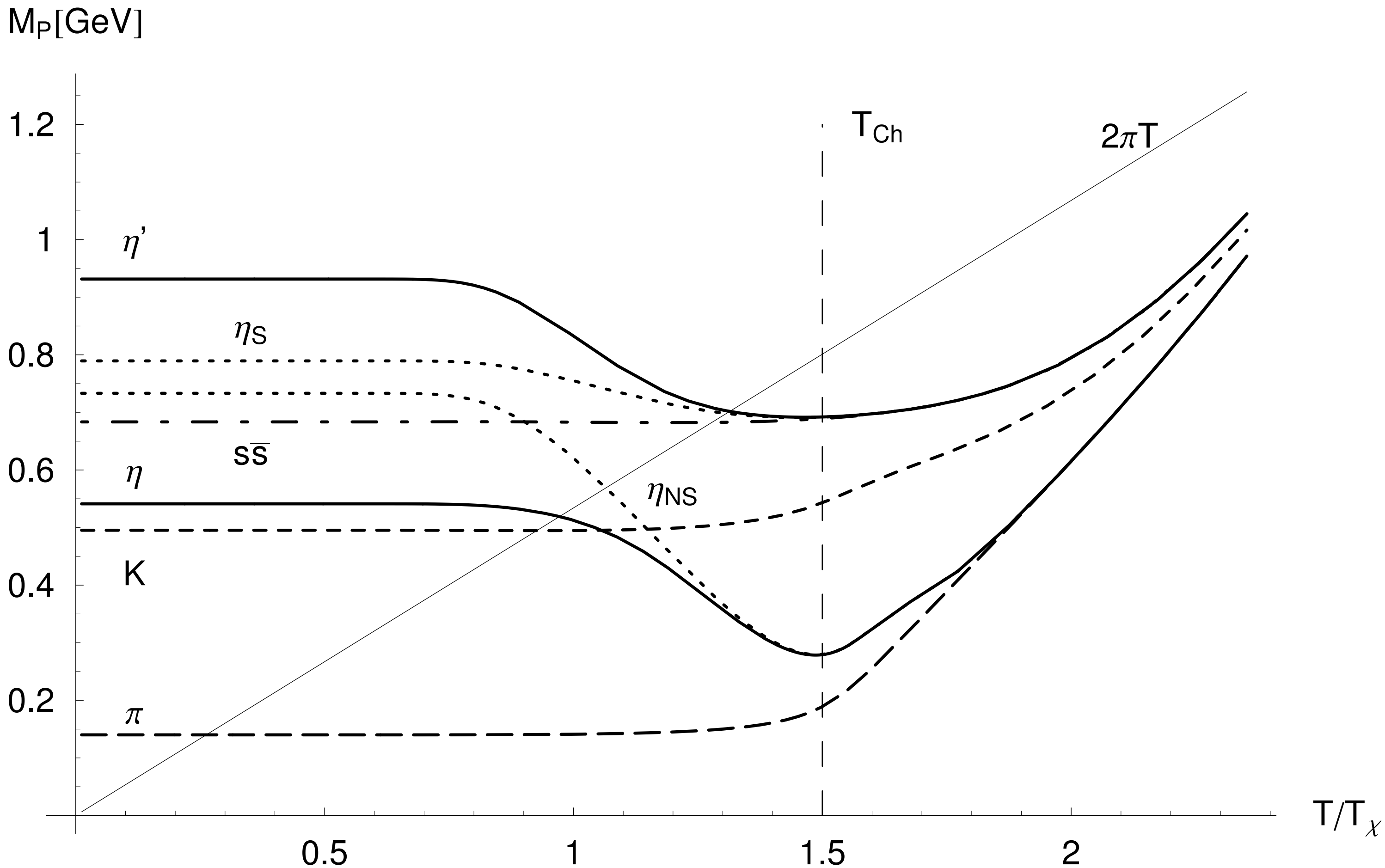


$$\chi = \int d^4x \; \langle q(x) q(0) \rangle \; , \qquad q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

- $q(x)$ = topological charge density operator
- In WV rel., χ is the pure-glue, YM one, $\chi_{\rm YM} \leftrightarrow \chi_{quench}$.

Relative temperature (T/T_χ) dependence of meson masses

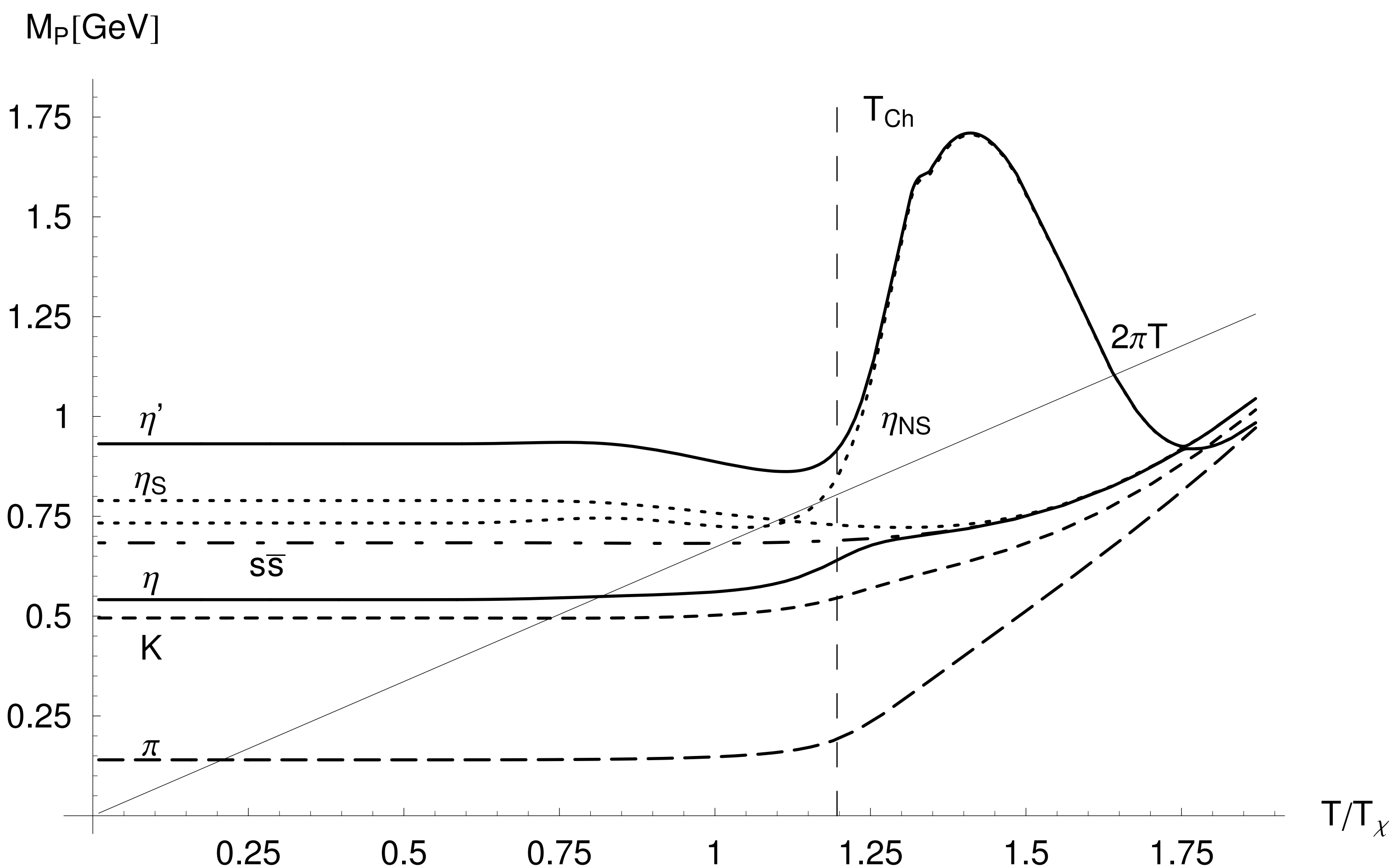
$\chi = \chi_{\rm YM} \Rightarrow$ The required drop (~ 200 MeV) of $M_{\eta'}$ only for very unrealistically low $T_\chi/T_{\rm Ch}$ ratio, where $\chi_{\rm YM}(T)$ melts before $f_\pi(T)$ diminishes much.



Extremely low $T_\chi = 2/3\, T_{\rm Ch}$ $\quad (T_{\rm Ch} = 1.50\, T_\chi)$

Relative temperature (T/T_χ) dependence of meson masses

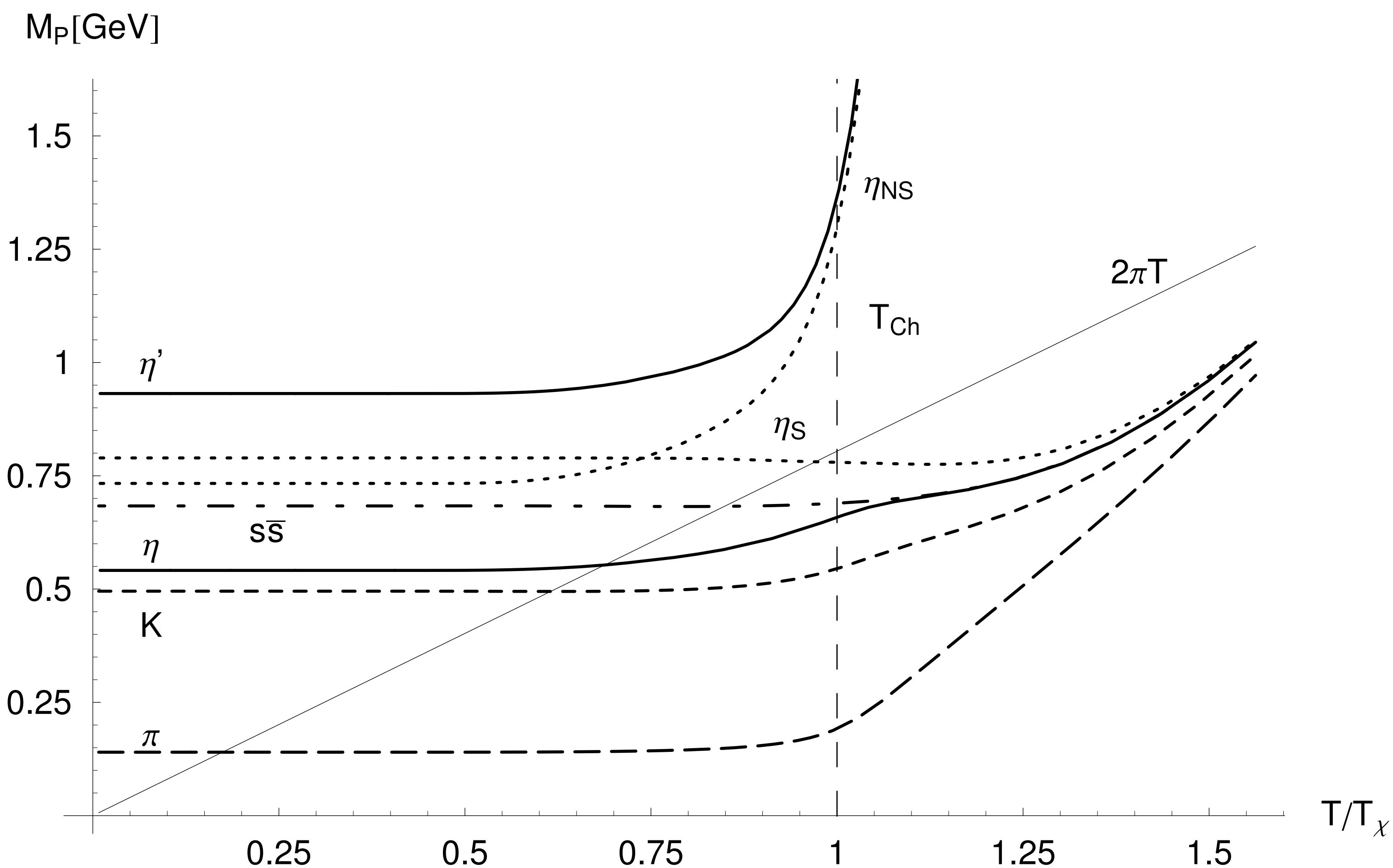
$\chi_{\rm YM}/f_\pi^2$ implies η' mass increase $\rightarrow$ suppression of the η' multiplicity already for still unrealistically low topological susceptibility-melting temperature $T_\chi \gtrsim 0.8\,T_{\rm Ch}$. Failure of WV relation at $T \gtrsim T_{\rm Ch}$.



Still unrealistically low $T_\chi = 0.836\,T_{\rm Ch}$ $\quad (T_{\rm Ch} = 1.20\,T_\chi)$

Obviously WV relation fails as T approaches $T_\chi \sim T_{\rm Ch}$:

$\chi_{\rm YM}/f_\pi^2$ implies a huge η' mass increase, to 5 GeV for $T_\chi = T_{\rm Ch}$,
$\Rightarrow$ total suppression of the η' multiplicity, instead of enhancement



$T_\chi = T_{\rm Ch}$

Solution: another connection of different theories

Early work by Di Vecchia & Veneziano ... Leutwyler & Smilga [Phys. Rev. D46 (1992) 5607] derived, up to $O(\frac{1}{N_c})$,

$$(\text{at}\, T = 0), \qquad \chi_{\rm YM} = \frac{\chi}{1 + \chi \, \frac{N_f}{m \, \langle \bar{q} q \rangle_0}} \equiv \widetilde{\chi}$$

$\Rightarrow$ relates $\chi_{\rm YM}$ to the full-QCD topological susceptibility χ, chiral condensate $\langle \bar{q} q \rangle_0$ and $m \equiv N_f \times$ the reduced mass. Presently $N_f = 3$, *i.e.*, $N_f/m = \sum_{q=u,d,s} (1/m_q)$.

- in the limit of very heavy quarks, $m_q, m \to \infty$, it confirms expectations that $\chi_{\rm YM} =$ value of topolog. susceptibility in *quenched* QCD, $\quad \chi_{\rm YM} = \chi(m_q = \infty)$
- It shows $\quad \chi \leq \min(-m \, \langle \bar{q} q \rangle_0 / N_f, \chi_{\rm YM})$

LS relation also holds in the oposite limit!

This (presently pertinent!) limit of light quarks = still a problem to get the full-QCD topol. susceptibility χ on lattice. Fortunately (Di Vecchia, Veneziano), there is the analytic result for small m_q:

$$\chi = -\frac{m\,\langle\bar{q}q\rangle_0}{N_f} + \mathcal{C}(m)\,,$$

- $\mathcal{C}(m)$ = small corrections of higher orders in small m_q, ... but $\mathcal{C}(m)$ should not be neglected, since $\mathcal{C}(m) = 0$ would imply that $\chi_{\rm YM} = \infty$.
- LS relation fixes the value of the correction at $T = 0$:

$$\frac{1}{\mathcal{C}(m)} = \frac{N_f}{m\,\langle\bar{q}q\rangle_0} - \chi_{\rm YM}(0)\left(\frac{N_f}{m\,\langle\bar{q}q\rangle_0}\right)^2 .$$

T-dependence of $\widetilde{\chi}$

- LS relation must break down as T rises towards the (pseudo)critical temperatures of full QCD ($\sim T_{\rm Ch}$) since the YM quantity, $\chi_{\rm YM}$, is much more T-resistant than RHS.
- RHS $\equiv \widetilde{\chi}$ consists of the full-QCD quantities χ and $\langle \bar{q} q \rangle_0$, the quantities of full QCD with quarks, characterized by $T_{\rm Ch}$, just as $f_\pi(T)$.
- Thus, the troublesome mismatch in T-dependences of $f_\pi(T)$ and the pure-gauge $\chi_{\rm YM}(T)$ is expected to disappear if $\chi_{\rm YM}(T)$ is replaced by $\widetilde{\chi}(T)$, the T-extended RHS of LS relation
- The usual, successful zero-T WV relation is thereby retained, since $$\chi_{\rm YM} = \widetilde{\chi} \quad \text{at } T = 0.$$

T-dependence of χ and $\widetilde{\chi}$

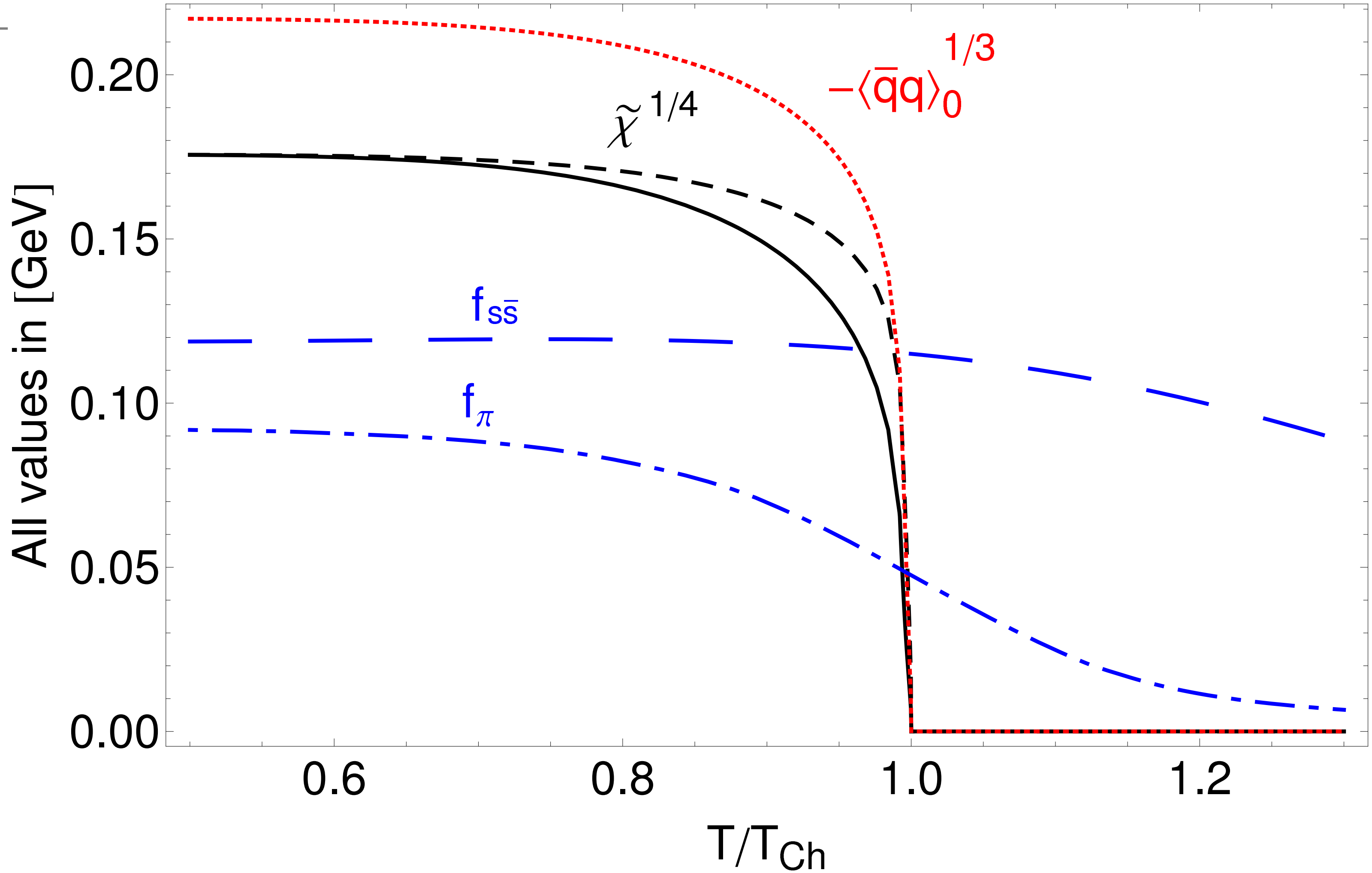
- Extending the light-quark full-QCD topol. susceptibility χ is somewhat uncertain, as there is no guidance from lattice [unlike for $\chi_{\rm YM}(T)$].
- The leading term in Di Vecchia-Veneziano relation $\propto \langle\bar{q}q\rangle_0(T)$ very plausibly, but for the correction term we have to explore a range of Ansätze, i.e.,

$$\chi(T) = -\frac{m\,\langle\bar{q}q\rangle_0(T)}{N_f} + \mathcal{C}(m)\,\left[\frac{\langle\bar{q}q\rangle_0(T)}{\langle\bar{q}q\rangle_0(T=0)}\right]^{\delta}, \quad (0 \leq \delta < 2).$$

Then, $\widetilde{\chi}(T) =$

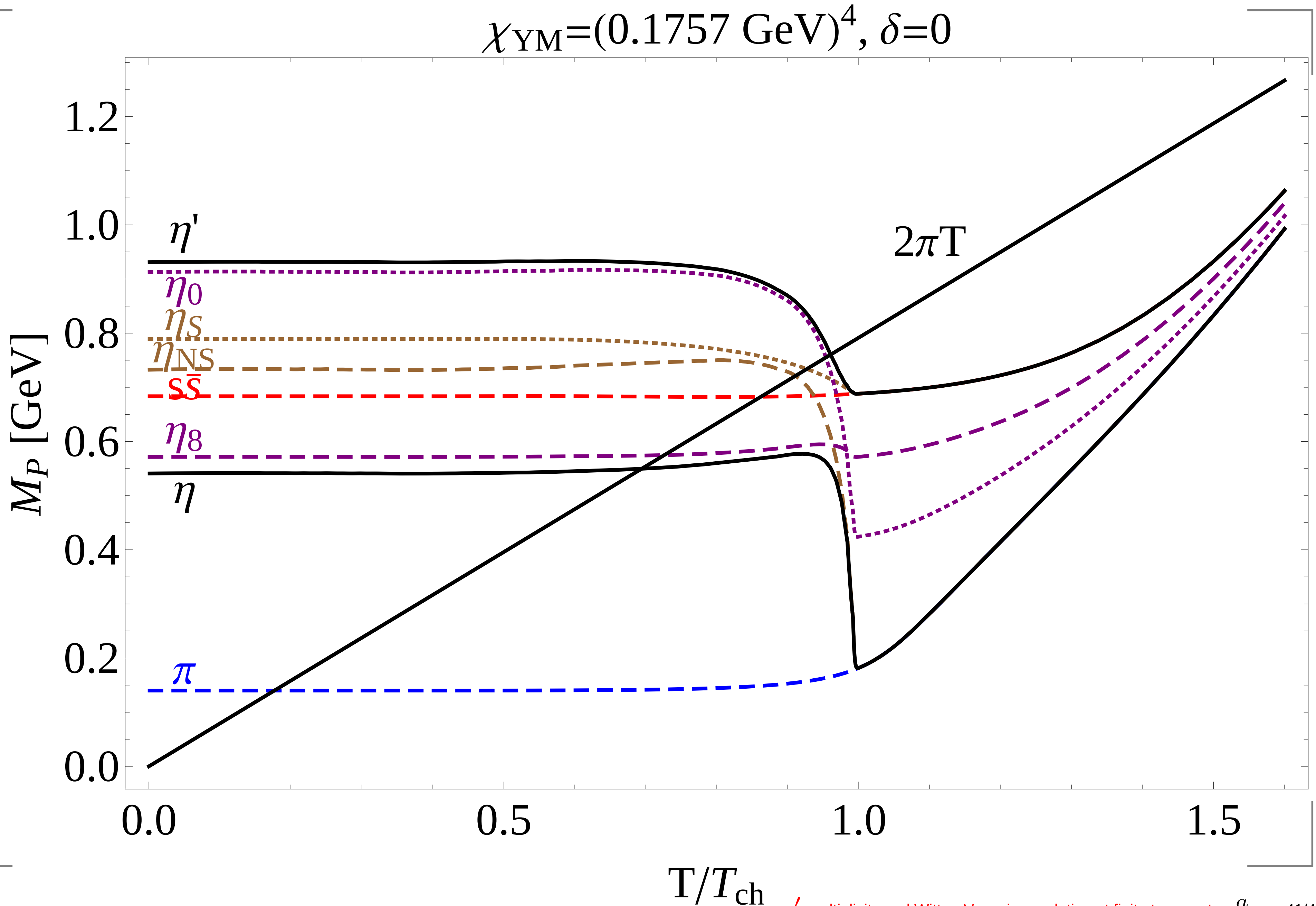
$$= \frac{\langle\bar{q}q\rangle_0(T)}{\sum_{q=u,d,s}\left(\frac{1}{m_q}\right)} \left\{ 1 - \frac{\langle\bar{q}q\rangle_0(T)}{\sum_{q=u,d,s}\left(\frac{1}{m_q}\right)} \frac{1}{\mathcal{C}(m)} \left[\frac{\langle\bar{q}q\rangle_0(T=0)}{\langle\bar{q}q\rangle_0(T)}\right]^{\delta} \right\}.$$

T-dependence of $\widetilde{\chi}$ follows the chiral condensate $\langle\bar{q}q\rangle_0(T)$:

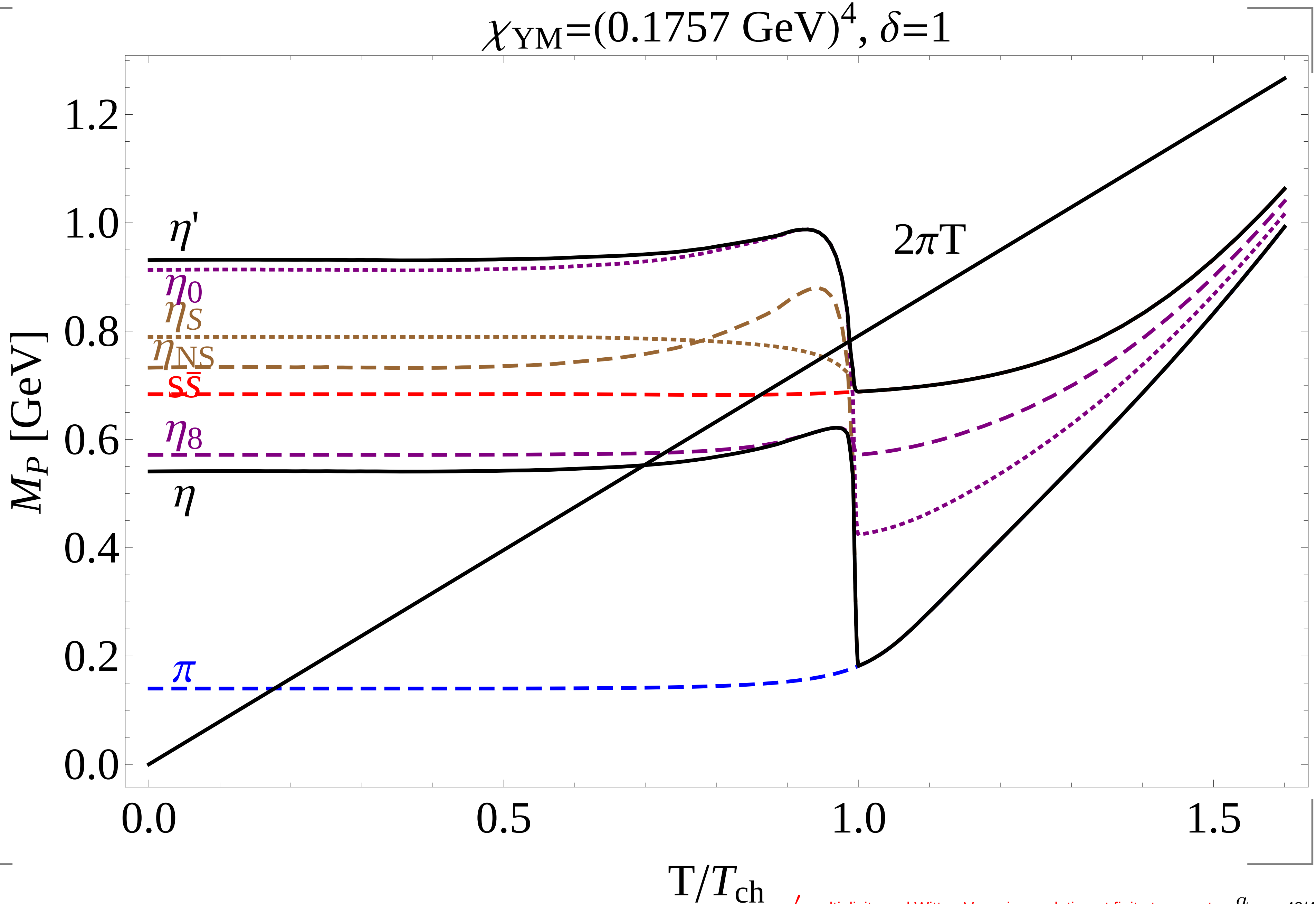


$\rightarrow$ T-dependence of the anomalous mass contribution will also follow the chiral condensate $\langle\bar{q}q\rangle_0(T)$!

Case 1: T-independent correction term in χ



Case 2: Strongly T-dependent correction term $\propto \langle \bar{q}q \rangle_0(T)$



Summary

- η' enhanced multiplicity shows that WV relation cannot be straightforwardly extended to T's close to $T_{\rm Ch}$, because the T-dependence of the ratio $\chi_{\rm YM}(T)/f_\pi(T)^2$ starts failing as $T \to T_{\rm Ch}$. Rescalings of $T_{\rm YM}$ useless!
- Leutwyler-Smilga and Di Vecchia-Veneziano relations 1.) enable one to retain unchanged WV relation, with $\chi_{\rm YM}$, for $T = 0$ (in fact, any T sufficiently below $T_{\rm Ch}$) and 2.) to replace the T-dependence of $\chi_{\rm YM}$ by that of the chiral condensate. This achieves consistency of the WV relation with the data on η' multiplicities, and indicates how chiral restoration may be linked with the $U_A(1)$ one.
- Further work: – Extension to finite density - for η' experiments, e.g., at NICA, as $\langle\bar{q}q\rangle_0(\mu) \to 0$ for $\mu \to \mu_{crit}$.
  – What happens in Shore's generalization of WV relation?

Shore's generalization of WV valid to all orders in $1/N_c$

- Inclusion of gluon anomaly in DGMOR relations $\rightarrow$

$$(f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_\eta^2 = \frac{1}{3}\left(f_\pi^2 m_\pi^2 + 2 f_K^2 m_K^2\right) + 6A \quad (1)$$

$$f^{0\eta'} f^{8\eta'} m_{\eta'}^2 + f^{0\eta} f^{8\eta} m_\eta^2 = \frac{2\sqrt{2}}{3}\left(f_\pi^2 m_\pi^2 - f_K^2 m_K^2\right) \quad (2)$$

$$(f^{8\eta'})^2 m_{\eta'}^2 + (f^{8\eta})^2 m_\eta^2 = -\frac{1}{3}\left(f_\pi^2 m_\pi^2 - 4 f_K^2 m_K^2\right) mber \quad (3)$$

$A = \chi_{\mathrm{YM}} + \mathcal{O}(\frac{1}{N_c})$ = full QCD topological charge. (1)+(3)$\rightarrow$

$$(f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_\eta^2 + (f^{8\eta})^2 m_\eta^2 + (f^{8\eta'})^2 m_{\eta'}^2 - 2 f_K^2 m_K^2 = 6A$$

- Then, large N_c limit and $f^{0\eta}, f^{8\eta'} \rightarrow 0$ as well as $f^{0\eta'}, f^{8\eta}, f_K \rightarrow f_\pi$ recovers the standard WV.

η' and η have 4 independent decay constants

$$f^0_{\eta'}, f^8_{\eta}, f^0_{\eta}, f^8_{\eta'}: \qquad \langle 0|A^{a\,\mu}(x)|P(p)\rangle = i f^a_P\, p^\mu e^{-ip\cdot x}, \;\; a=8,0; \;\; P=\eta,\eta' .$$

- Equivalently, one has 4 related but different constants $f^{NS}_{\eta'}, f^{NS}_{\eta}, f^{S}_{\eta}, f^{S}_{\eta'}$ if instead of octet and singlet axial currents ($a = 8, 0$) one takes this matrix element of the nonstrange-strange axial currents ($a = NS, S$)

$$A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^{8\,\mu}(x) + \sqrt{\frac{2}{3}} A^{0\,\mu}(x) = \frac{1}{2}\left(\bar{u}(x)\gamma^\mu\gamma_5 u(x) + \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right) \;,$$

$$A^\mu_{S}(x) = -\sqrt{\frac{2}{3}} A^{8\,\mu}(x) + \frac{1}{\sqrt{3}} A^{0\,\mu}(x) = \frac{1}{\sqrt{2}} \bar{s}(x)\gamma^\mu\gamma_5 s(x) \;,$$

$$\begin{bmatrix} f^{NS}_{\eta} & f^{S}_{\eta} \\ f^{NS}_{\eta'} & f^{S}_{\eta'} \end{bmatrix} = \begin{bmatrix} f^{8}_{\eta} & f^{0}_{\eta} \\ f^{8}_{\eta'} & f^{0}_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \;,$$

$$a, P = NS, S: \qquad \langle 0|A^\mu_{NS}(x)|\eta_{NS}(p)\rangle = i f_{NS}\, p^\mu e^{-ip\cdot x} \;, \qquad \langle 0|A^\mu_{NS}(x)|\eta_{S}(p)\rangle = 0 \;,$$

$$a, P = NS, S: \qquad \langle 0|A^\mu_{S}(x)|\eta_{S}(p)\rangle = i f_{S}\, p^\mu e^{-ip\cdot x} \;, \qquad \langle 0|A^\mu_{S}(x)|\eta_{NS}(p)\rangle = 0 \;,$$

- Note: in our approach, $f_{NS} = f_{u\bar{u}} = f_{d\bar{d}} = f_\pi$, $f_S = f_{s\bar{s}}$ are calculated quantities

Two Mixing Angles and FKS one-angle scheme

- Any 4 η-η' decay constants conveniently parametrized in terms of two decay constants and two angles:

$$f_\eta^8 = \cos\theta_8\, f_8\ , \qquad f_\eta^0 = -\sin\theta_0\, f_0\ , \qquad f_\eta^{NS} = \cos\phi_{NS}\, f_{NS}\ , \qquad f_\eta^S = -\sin\phi_S\, f_S\ ,$$

$$f_{\eta'}^8 = \sin\theta_8\, f_8\ , \qquad f_{\eta'}^0 = \cos\theta_0\, f_0\ , \qquad f_{\eta'}^{NS} = \sin\phi_{NS} f_{NS}\ , \qquad f_{\eta'}^S = \cos\phi_S f_S$$

- Big practical difference between 0-8 and NS-S schemes:

- while θ_8 and θ_0 differ a lot from each other and from $\theta \approx (\theta_8 + \theta_0)/2$, FKS showed that $\phi_{NS} \approx \phi_S \approx \phi$.

$$\begin{bmatrix} f_\eta^{NS} & f_\eta^{S} \\ f_{\eta'}^{NS} & f_{\eta'}^{S} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} .$$

For four decay constants, can use FKS one-angle scheme!

- we can relate $\{f_\eta^8, f_{\eta'}^8, f_\eta^0, f_{\eta'}^0\}$ with $\{f_{NS}, f_S\} = \{f_\pi, f_{s\bar{s}}\}$:

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

- Some other useful relations between quantities of NS-S (FKS) and 0-8 schemes:

$$f_8 = \sqrt{\frac{1}{3} f_{NS}^2 + \frac{2}{3} f_S^2}\,, \qquad \theta_8 = \phi - \text{arctan}\left(\frac{\sqrt{2} f_S}{f_{NS}}\right)\,,$$

$$f_0 = \sqrt{\frac{2}{3} f_{NS}^2 + \frac{1}{3} f_S^2}\,, \qquad \theta_0 = \phi - \text{arctan}\left(\frac{\sqrt{2} f_{NS}}{f_S}\right)\,.$$

For 3 DS models: $T = 0$ results of Shore's generalization

Published by D. Horvatić et al., Eur. Phys. J. A **38** (2008) 257.

	Jain-Munczek		$\langle A^2 \rangle$-induced		separable	
χ_{YM}	191^4	175.7^4	191^4	175.7^4	191^4	175.7^4
M_η	499.8	485.7	496.7	482.8	526.2	507.0
$M_{\eta'}$	931.4	815.8	934.9	818.4	983.2	868.7
ϕ	52.01^o	46.11^o	51.85^o	46.07^o	47.23^o	40.86^o
θ	-2.72^o	-8.62^o	-2.89^o	-8.67^o	-7.51^o	-13.87^o
θ_0	7.74^o	1.84^o	7.17^o	1.39^o	-0.33^o	-6.69^o
θ_8	-12.00^o	-17.90^o	-11.85^o	-17.6^o	-14.12^o	-20.47^o
f_0	108.8	108.8	107.9	107.9	101.8	101.8
f_8	122.6	122.6	121.1	121.1	110.7	110.7
f_η^0	-14.7	-3.5	-13.5	-2.6	0.6	11.9
$f_{\eta'}^0$	107.9	108.8	107.1	107.9	101.8	101.1
f_η^8	119.9	116.7	118.5	115.4	107.4	103.7
$f_{\eta'}^8$	-25.5	-37.7	-2.49	-37.6	-27.0	-38.7

For $T > 0$, the substitution $A \to \widetilde{\chi}(T)$ leads to similar successful results as in the WV case!

Summary 2

- The results of the approach through Witten-Veneziano relation and Shore's approach are (qualitatively) quite similar.
- Data on η' enhanced multiplicity in RHIC experiments $\Rightarrow$ neither the original Witten-Veneziano relation nor Shore's scheme can be straightforwardly extended to T close to $T_{\rm Ch}$, because the the ratio $6\chi_{\rm YM}(T)/f_\pi(T)^2$ tends to blow up as $T \to T_{\rm Ch}$.
- We find that the Leutwyler-Smilga relation enables one
  1.) to retain unchanged Witten-Veneziano relation, with $\chi_{\rm YM}$, for $T = 0$ (in fact, for any T sufficiently below $T_{\rm Ch}$) but also
  2.) to replace the T-dependence of $\chi_{\rm YM}$ by that of the chiral condensate. This ties the U$_A$(1) symmetry restoration with the chiral symmetry restoration, and achieves consistency of both Witten-Veneziano and Shore's approach with the data on η' multiplicity.