Algebraic Structure of Lepton and Quark Flavor Invariants

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Introduction

- Observation of neutrino oscillations $\Rightarrow$ neutrino *flavor* eigenstates ν_e, ν_μ, ν_τ are not *mass* eigenstates ν_1, ν_2, ν_3. Constitutes first evidence for new physics beyond the Standard Model.
- Leading theory for massive light neutrinos is non-renormalizable theory = SM + $d = 5$ Weinberg operator $+ \ldots$ in which weak-doublet neutrinos acquire Majorana masses upon EWSB from unique $d = 5$ operator which respects gauge symmetry. This theory is the low-energy EFT obtained from the renormalizable seesaw theory by integrating out the gauge-singlet neutrinos with Majorana masses $M \gg M_W, M_Z, m_t, m_H$!
- Flavor structure of these theories is of interest. Useful to discuss flavor structure in terms of flavor invariants, which are basis independent.

Introduction

There is extensive literature on flavor invariants, both

quark invariants Jarlskog, Greenberg, Kusenko & Shrock,$\cdots$ and

lepton invariants Branco & Rebelo, Branco, Rebelo & Silva-Marcos, Kusenko & Shrock, Dreiner, Kim, Lebedev & Thormeier, $\cdots$

It is interesting to address the classification of flavor invariants using invariant theory. Mathematics of invariant theory describes the algebraic structure of invariants. The number of invariants of a given degree in the flavor-symmetry breaking mass matrices is encoded in Hilbert series. Flavor invariants with usual operations of addition and multiplication form a ring, which is finitely generated. It is interesting to determine the generators of the ring and the non-trivial relations (syzygies) among invariants.

Introduction

This talk is based on the references

- E. E. Jenkins and A. V. Manohar, “Algebraic Structure of Lepton and Quark Flavor Invariants and CP Violation,” JHEP10 (2009) 094.
- A. Hanany, E. E. Jenkins, A. V. Manohar and G. Torri, “Hilbert Series for Flavor Invariants of the Standard Model," JHEP03 (2011) 096.
- E. Jenkins and A. V. Manohar, “Rephasing Invariants of Quark and Lepton Mixing Matrices," Nucl. Phys. B792 (2008) 187.

Lepton and Quark Flavor Invariants

- Flavor structure of our leading theories (SM + $d = 5$ operator, seesaw) is encoded by flavor invariants constructed from the quark and lepton mass matrices.
- There are a finite number of basic invariants, and a general invariant can be written as a polynomial in the basic invariants.
- Number of basic invariant generators is equal to number of independent physical parameters: quark and lepton masses, mixing angles and phases
- The basic invariants and all non-trivial relations (syzygies) between these invariants determines the flavor structure of a given theory.

Primer: Invariant Theory

Before addressing the physical problem of interest, it is useful to consider some simple examples which illustrate the mathematics of invariant theory in a very simple context.

Model I

- Two complex couplings m_1 and m_2 which transform under $G = U(1) \times U(1)$

$$m_1 \to e^{i\phi_1} m_1, \qquad m_2 \to e^{i\phi_2} m_2 \ . \tag{1}$$

- Ring of invariant polynomials generated by two basic invariants $I_1 = m_1 m_1^*$ and $I_2 = m_2 m_2^*$ with no non-trivial relations (syzygies) between I_1 and I_2
- General invariant is of the form

$$(m_1 m_1^*)^{r_1} (m_2 m_2^*)^{r_2} \tag{2}$$

- Hilbert series

Definition

$$H(q) = \sum_{r=0}^{\infty} c_r q^r = 1 + \sum_{r=1}^{\infty} c_r q^r \tag{3}$$

$c_r =$ the number of invariants of degree r, $\qquad c_r \geq 0$

Model I

General invariant is of the form

$$(m_1 m_1^*)^{r_1} (m_2 m_2^*)^{r_2}$$

- Hilbert series of Model I

$$\begin{aligned} H(q) &= 1 + 2q^2 + 3q^4 + 4q^6 + 5q^8 + \ldots \\ &= \sum_{n=0}^{\infty} (n+1) q^{2n} \\ &= \frac{1}{(1-q^2)^2} \\ &= \left(1 + q^2 + q^4 + q^6 + \ldots\right)^2 \end{aligned} \tag{5}$$

- Hilbert series

$$H(q) = \left(1 + q^2 + q^4 + q^6 + \dots\right)^2 = \frac{1}{(1-q^2)^2} \quad (6)$$

Theorem

$$H(q) = \frac{N(q)}{D(q)}$$

$$N(q) = 1 + c_1 q + c_2 q^2 + \cdots + c_{d_N - 2} q^{d_N - 2} + c_{d_N - 1} q^{d_N - 1} + q^{d_N}$$

$$c_r \geq 0, \qquad c_r = c_{d_N - r}$$

$$D(q) = \prod_{r=1}^{p} \left(1 - q^{d_r}\right)$$

$$d_D = \sum_r d_r$$

Model I

- Ring $\mathbb{C}[m_1, m_1^*, m_2, m_2^*]^{U(1)\times U(1)}$ of all polynomials which are invariant under $G = U(1) \times U(1)$

$$p = \mathrm{dim} V - \mathrm{dim} G, \qquad \mathrm{dim} V = 4, \quad \mathrm{dim} G = 2$$

- Knop's Theorem

Theorem

$$\mathrm{dim} V \geq d_D - d_N \geq p$$

$$\mathrm{dim} V = 4,\ d_D = 4,\ d_N = 0,\ p = 2$$

$$4 \geq 4 \geq 2$$

Model II

- Two couplings m_1 and m_2 which transform under $G = U(1)$

$$m_1 \to e^{i\phi} m_1, \qquad m_2 \to e^{2i\phi} m_2 \,.$$

- Invariants generated by four basic invariants $I_1 = m_1 m_1^*$, $I_2 = m_2 m_2^*$, $I_3 = m_2 m_1^{*2}$ and $I_4 = m_2^* m_1^2$, but the four basic invariants are not all independent since $I_3 I_4 = I_1^2 I_2$
- Hilbert series

$$\begin{aligned} H(q) &= 1 + 2q^2 + 2q^3 + 3q^4 + 6q^6 + \ldots \\ &= \frac{1+q^3}{(1-q^2)^2(1-q^3)} \end{aligned} \qquad (7)$$

I_1, I_2, I_3, I_4 not all independent is encoded in Hilbert series

$$\begin{aligned} H(q) &\neq \frac{1}{(1-q^2)^2(1-q^3)^2} \\ &= 1 + 2q^2 + 2q^3 + 3q^4 + 7q^6 + \ldots \end{aligned} \qquad (8)$$

Model II

$(I_3 - I_4)$ cannot be written in terms of I_1, I_2, $(I_3 + I_4)$

Syzygy

$$I_3 I_4 = I_1^2 I_2$$

$$(I_3 - I_4)^2 = (I_3 + I_4)^2 - 4 I_3 I_4 = (I_3 + I_4)^2 - 4 {I_1}^2 I_2$$

General polynomial in basic invariants

$$P_1(I_1, I_2, I_3 + I_4) + (I_3 - I_4) P_2(I_1, I_2, I_3 + I_4)$$

Model II

- Ring $\mathbb{C}[m_1, m_1^*, m_2, m_2^*]^{U(1)}$ of all polynomials which are invariant under $G = U(1)$

$$p = \dim V - \dim G, \qquad \dim V = 4, \quad \dim G = 1$$

- Knop's Theorem

Theorem

$$\dim V \geq d_D - d_N \geq p$$

$$\dim V = 4,\ d_D = 7,\ d_N = 3,\ p = 3$$

$$4 \geq 4 \geq 3$$

Model III

- Three couplings m_1, m_2 and m_3 which transform under $G = U(1)$

$$m_1 \to e^{i\phi} m_1, \; m_2 \to e^{2i\phi} m_2, \; m_3 \to e^{3i\phi} m_3 \; .$$

- 13 basic invariants

$$\begin{aligned} I_1 &= m_1 m_1^*, \\ I_2 &= m_2 m_2^*, \\ I_3 &= m_3 m_3^*, \\ I_4 &= m_1^2 m_2^*, \\ I_5 &= m_1^{*2} m_2, \\ I_6 &= m_1^3 m_3^*, \\ I_7 &= m_1^{*3} m_3, \end{aligned}$$

$$\begin{aligned} I_8 &= m_2^3 m_3^{*2}, \\ I_9 &= m_2^{*3} m_3^2, \\ I_{10} &= m_1 m_2 m_3^*, \\ I_{11} &= m_1^* m_2^* m_3, \\ I_{12} &= m_1 m_3 m_2^{*2}, \\ I_{13} &= m_1^* m_3^* m_2^2. \end{aligned}$$

Model III

- 35 relations between products $I_i I_j$, but now there are relations among relations (syzygies)

Example

$$I_4 I_5 I_6 I_7 = I_1^4 I_{10} I_{11}$$

obtained by multiplying relations $I_4 I_7 = I_1^2 I_{11}$ and $I_5 I_6 = I_1^2 I_{10}$ OR by multiplying $I_4 I_5 = I_1^2 I_2$, $I_6 I_7 = I_1^3 I_3$ and using $I_{10} I_{11} = I_1 I_2 I_3$, so $I_4 I_5 I_6 I_7 = I_1^5 I_2 I_3 = I_1^4 I_{10} I_{11}$

- Hilbert series

$$H(q) = \frac{1 + q^2 + 3q^3 + 4q^4 + 4q^5 + 4q^6 + 3q^7 + q^8 + q^{10}}{(1-q^2)^2(1-q^3)(1-q^4)(1-q^5)}.$$

Model III

- Ring $\mathbb{C}[m_1, m_1^*, m_2, m_2^*, m_3, m_3^*]^{U(1)}$ of all polynomials which are invariant under $G = U(1)$

$$p = \mathrm{dim}V - \mathrm{dim}G, \qquad \mathrm{dim}V = 6, \quad \mathrm{dim}G = 1$$

- Knop's Theorem

Theorem

$$\mathrm{dim}V \geq d_D - d_N \geq p$$

$$\mathrm{dim}V = 6,\ d_D = 16,\ d_N = 10,\ p = 5$$

$$6 \geq 6 \geq 5$$

Lepton and Quark Flavor Invariants

- Use invariant theory to solve classification of quark and lepton mass matrix invariants in the (i) seesaw model and (ii) SM + dim-5 operator (giving Majorana masses to weakly interacting neutrinos)
- Invariant structure in lepton sector is highly non-trivial with many non-linear relations (syzygies) among the basic invariants. Invariant structure depends on number of generations n_g of SM quarks and leptons and n_g' of neutrino singlets
- Able to solve problem for low-energy EFT with $n_g = 2, 3$ and for high-energy seesaw theory with $n_g = n_g' = 2, 3$.
- Hilbert series obtained in cases of physical interest. Number of independent invariants and syzygy structure encoded by Hilbert series.
- Algebraic structure of lepton invariants is much more complicated than for quark invariants.

What is new? for What is ν?

- Hilbert series of flavor invariants for Lagrangians (i) SM+ $d = 5$ operator and (ii) seesaw model determined. Syzygy relations follow from Hilbert series.
- Solution dependent on number of families. Cases of physical interest: $n_g = 3$ families of SM fermions and $n'_g = 2, 3$ right-handed neutrinos now solved.
- Algebraic structure of lepton invariants is very complicated.

For the purposes of this talk:

Definition

Standard Model $\equiv$ nonrenormalizable EFT containing only SM fields with gauge symmetry $SU(3) \times SU(2) \times U(1)$ truncated after unique $d = 5$ operator (higher dimensional operators $d = 6, \cdots$ neglected)

Definition

Seesaw Model $\equiv$ renormalizable $SU(3) \times SU(2) \times U(1)$ theory with additional gauge-singlet neutrinos N

Flavor Matrices

- High-Energy Seesaw Model

$$\mathcal{L} = -U_i^c\,(Y_U)_{ij}\,Q_j H - D_i^c\,(Y_D)_{ij}\,Q_j H^\dagger - E_i^c\,(Y_E)_{ij}\,L_j H^\dagger$$
$$-N_I^c\,(Y_\nu)_{Ij}\,L_j H - \frac{1}{2} N_I^c M_{IJ} N_J^c + \text{h.c.}$$

Mass matrices: m_U, m_D, m_E, m_ν, M

- Low-Energy Effective Theory $\equiv$ SM + $d = 5$ operator

$$\mathcal{L}^{\text{EFT}} = -U_i^c\,(Y_U)_{ij}\,Q_j H - D_i^c\,(Y_D)_{ij}\,Q_j H^\dagger - E_i^c\,(Y_E)_{ij}\,L_j H^\dagger$$
$$+\frac{1}{2}(L_i H)\,(C_5)_{ij}\,(L_j H) + \text{h.c.}$$

Mass matrices: m_U, m_D, m_E, m_5

Quark Flavor Invariants

$$\mathbb{C}\left[m_U, m_U^\dagger, m_D, m_D^\dagger\right]^{SU(n_g)_Q \times SU(n_g)_{U^c} \times SU(n_g)_{D^c} \times U(1)^2}$$

$$\begin{aligned} m_U &\rightarrow {\mathcal{U}_{U^c}}^T \, m_U \, \mathcal{U}_Q \\ m_D &\rightarrow {\mathcal{U}_{D^c}}^T \, m_D \, \mathcal{U}_Q \end{aligned}$$

$$\begin{aligned} X_U &\equiv m_U^\dagger m_U \\ X_D &\equiv {m_D}^\dagger m_D \end{aligned}$$

$$X_{U,D} \rightarrow \mathcal{U}_Q^\dagger \, X_{U,D} \, \mathcal{U}_Q$$

Quark Flavor Invariants $n_g = 2$

$$
\begin{aligned}
I_{2,0} &= \langle X_U \rangle = \langle {m_U}^\dagger m_U \rangle \\
I_{0,2} &= \langle X_D \rangle = \langle {m_D}^\dagger m_D \rangle \\
I_{4,0} &= \langle {X_U}^2 \rangle = \langle \left({m_U}^\dagger m_U \right)^2 \rangle \\
I_{2,2} &= \langle X_U X_D \rangle = \langle {m_U}^\dagger m_U {m_D}^\dagger m_D \rangle \\
I_{0,4} &= \langle {X_D}^2 \rangle = \langle \left({m_D}^\dagger m_D \right)^2 \rangle
\end{aligned}
$$

$$
\begin{aligned}
I_{2,0} &= m_u^2 + m_c^2 \\
I_{0,2} &= m_d^2 + m_s^2 \\
I_{4,0} &= m_u^4 + m_c^4 \\
I_{2,2} &= m_u^2 m_s^2 + m_c^2 m_d^2 + (m_s^2 - m_d^2)(m_c^2 - m_u^2) \cos^2 \theta \\
I_{0,4} &= m_d^4 + m_s^4
\end{aligned}
$$

Quark Flavor Invariants $n_g = 2$

$$H(q) = \frac{1}{(1-q^2)^2(1-q^4)^3}$$

$p = 5$: 4 masses, 1 mixing angles θ_{C}

$$\dim V = 16, \qquad \dim G = 11$$

$$d_N = 0, \qquad d_D = 16$$

Knop's Theorem

$$16 \geq 16 \geq 5$$

Quark Flavor Invariants $n_g = 3$

$$
\begin{aligned}
I_{2,0} &= \langle X_U \rangle \\
I_{0,2} &= \langle X_D \rangle \\
I_{4,0} &= \langle {X_U}^2 \rangle \\
I_{2,2} &= \langle X_U X_D \rangle \\
I_{0,4} &= \langle {X_D}^2 \rangle \\
I_{6,0} &= \langle {X_U}^3 \rangle \\
I_{4,2} &= \langle {X_U}^2 X_D \rangle \\
I_{2,4} &= \langle X_U {X_D}^2 \rangle \\
I_{0,6} &= \langle {X_D}^3 \rangle \\
I_{4,4} &= \langle {X_U}^2 {X_D}^2 \rangle
\end{aligned}
$$

$$
I_{6,6}^{(-)} = \langle {X_U}^2 {X_D}^2 X_U X_D \rangle - \langle {X_D}^2 {X_U}^2 X_D X_U \rangle \propto J
$$

Quark Flavor Invariants $n_g = 3$

$$H(q) = \frac{1+q^{12}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

General polynomial invariant

$$P_1 + I^-_{6,6} P_2$$

since there is a syzygy $\left(I^-_{6,6}\right)^2 = \cdots$

$p = 10$: 6 masses, 3 mixing angles, 1 phase $\delta_{\rm CKM}$

$$\dim V = 36, \qquad \dim G = 26$$

$$d_N = 12, \qquad d_D = 48$$

Knop's Theorem

$$36 \geq 36 \geq 10$$

Lepton Flavor Invariants: EFT

$$\mathbb{C}\left[m_E, m_E^\dagger, m_5, m_5^*\right]^{SU(n_g)_L \times SU(n_g)_{E^c} \times U(1)^2}$$

$$\begin{aligned}
m_E &\rightarrow \mathcal{U}_{E^c}{}^T \, m_E \, \mathcal{U}_L \\
m_E{}^\dagger &\rightarrow \mathcal{U}_L{}^\dagger \, m_E{}^\dagger \, \mathcal{U}_{E^c}{}^* \\
m_5 &\rightarrow \mathcal{U}_L{}^T \, m_5 \, \mathcal{U}_L \\
m_5{}^* &\rightarrow \mathcal{U}_L{}^\dagger \, m_5{}^* \, \mathcal{U}_L{}^*
\end{aligned}$$

$$\begin{aligned}
X_E &\equiv m_E{}^\dagger m_E \rightarrow \mathcal{U}_L^\dagger \, X_E \, \mathcal{U}_L \\
X_E{}^T &\equiv m_E{}^T m_E{}^* \rightarrow \mathcal{U}_L{}^T \, X_E{}^T \, \mathcal{U}_L{}^*
\end{aligned}$$

$$X_5 \equiv m_5{}^* m_5 \rightarrow \mathcal{U}_L{}^\dagger \, X_5 \, \mathcal{U}_L$$

$$\left(m_5{}^* \, (X_E{}^n)^T \, m_5\right) \rightarrow \mathcal{U}_L{}^\dagger \left(m_5{}^* \, (X_E{}^n)^T \, m_5\right) \mathcal{U}_L$$

Lepton Flavor Invariants: EFT $n_g = 2$

$$
\begin{aligned}
I_{2,0} &= \langle X_E \rangle = \langle m_E{}^\dagger m_E \rangle \\
I_{0,2} &= \langle X_5 \rangle = \langle m_5{}^* m_5 \rangle \\
I_{4,0} &= \langle X_E{}^2 \rangle = \langle \left(m_E{}^\dagger m_E \right)^2 \rangle \\
I_{2,2} &= \langle m_5{}^* \, X_E{}^T \, m_5 \rangle = \langle m_5 \, X_E \, m_5{}^* \rangle \\
&= \langle m_E{}^T \, m_E{}^* \, m_5 \, m_5{}^* \rangle = \langle m_E{}^\dagger \, m_E \, m_5{}^* \, m_5 \rangle \\
I_{0,4} &= \langle X_5{}^2 \rangle = \langle (m_5{}^* m_5)^2 \rangle \\
I_{4,2} &= \langle m_5{}^* \, X_E{}^T \, m_5 \, X_E \rangle \\
&= \langle m_5{}^* \, m_E{}^T m_E{}^* \, m_5 \, m_E{}^\dagger m_E \rangle \\
I_{4,4}^{(-)} &= \langle m_5{}^* \, X_E{}^T \, m_5 \, X_E \, m_5{}^* \, m_5 \rangle \\
&\quad - \langle m_5{}^* \, X_E{}^T \, m_5 \, m_5{}^* \, m_5 \, X_E \rangle \\
&= \langle m_5{}^* \, m_E{}^T m_E{}^* \, m_5 \, m_E{}^\dagger m_E \, m_5{}^* \, m_5 \rangle \\
&\quad - \langle m_5{}^* \, m_E{}^T m_E{}^* \, m_5 \, m_5{}^* \, m_5 \, m_E{}^\dagger m_E \rangle
\end{aligned}
$$

Lepton Flavor Invariants: EFT $n_g = 2$

$$H(q) = \frac{1+q^8}{(1-q^2)^2(1-q^4)^3(1-q^6)}$$

- Ring $\mathbb{C}[m_5, m_5^*, m_E, m_E^\dagger]^{G_{\rm Flavor}}$ of all polynomials which are invariant under

$$G_{\rm Flavor} = SU(2)_L \times SU(2)_{E^c} \times U(1)^2$$

$$p = \dim V - \dim G, \qquad \dim V = 14, \quad \dim G = 8$$

Lepton Flavor Invariants: EFT $n_g = 2$

- Knop's Theorem

Theorem

$$\dim V \geq d_D - d_N \geq p$$

$$\dim V = 14, \; d_D = 22, \; d_N = 8, \; p = 6$$

$p = 6$ consists of 4 masses, 1 angle and 1 phase

$$14 \geq 14 \geq 6$$

Lepton Flavor Invariants: EFT $n_g = 3$

$$
\begin{aligned}
I_{2,0} &= \langle X_E \rangle = \langle {m_E}^\dagger m_E \rangle \,, \\
I_{0,2} &= \langle X_5 \rangle = \langle {m_5}^* m_5 \rangle \,, \\
I_{4,0} &= \langle {X_E}^2 \rangle = \langle \left({m_E}^\dagger m_E \right)^2 \rangle \,, \\
I_{2,2} &= \langle X_E X_5 \rangle = \langle {m_E}^\dagger m_E {m_5}^* m_5 \rangle \,, \\
I_{0,4} &= \langle {X_5}^2 \rangle = \langle ({m_5}^* m_5)^2 \rangle \,, \\
I_{6,0} &= \langle {X_E}^3 \rangle = \langle \left({m_E}^\dagger m_E \right)^3 \rangle \,, \\
I'_{4,2} &= \langle {X_E}^2 X_5 \rangle = \langle \left({m_E}^\dagger m_E \right)^2 {m_5}^* m_5 \rangle \,, \\
I_{4,2} &= \langle {m_5}^* \, {X_E}^T \, m_5 \, X_E \rangle \\
&= \langle {m_5}^* \, {m_E}^T {m_E}^* \, m_5 \, {m_E}^\dagger m_E \rangle \,, \\
I_{2,4} &= \langle X_E {X_5}^2 \rangle = \langle {m_E}^\dagger m_E \, ({m_5}^* m_5)^2 \rangle \,, \\
I_{0,6} &= \langle {X_5}^3 \rangle = \langle ({m_5}^* m_5)^3 \rangle \,,
\end{aligned}
$$

Lepton Flavor Invariants: EFT $n_g = 3$

$$
\begin{aligned}
I_{6,2} &= \langle {m_5}^* \, {X_E}^T \, m_5 \, {X_E}^2 \rangle \\
&= \langle {m_5}^* \, {m_E}^T {m_E}^* \, m_5 \, \left({m_E}^\dagger m_E\right)^2 \rangle \,, \\
I_{4,4}^{(\pm)} &= \langle {m_5}^* \, {X_E}^T \, m_5 \, {m_5}^* \, m_5 \, X_E \rangle \\
&\quad \pm \langle {m_5}^* \, m_5 \, {m_5}^* \, {X_E}^T \, m_5 \, X_E \rangle \\
&= \langle {m_5}^* \, {m_E}^T {m_E}^* \, m_5 \, {m_5}^* \, m_5 \, {m_E}^\dagger m_E \rangle \\
&\quad \pm \langle {m_5}^* \, m_5 \, {m_5}^* \, {m_E}^T {m_E}^* \, m_5 \, {m_E}^\dagger m_E \rangle \,, \\
I_{8,2} &= \langle {m_5}^* \, ({X_E}^T)^2 \, m_5 \, {X_E}^2 \rangle \\
&= \langle {m_5}^* \, \left({m_E}^T {m_E}^*\right)^2 \, m_5 \, \left({m_E}^\dagger m_E\right)^2 \rangle \,,
\end{aligned}
$$

Lepton Flavor Invariants: EFT $n_g = 3$

$$
\begin{aligned}
I_{6,4}^{(\pm)} &= \langle m_5{}^* \, X_E{}^T \, m_5 \, m_5{}^* \, m_5 \, X_E{}^2 \rangle \\
&\quad \pm \langle m_5{}^* \, m_5 \, m_5{}^* \, X_E{}^T \, m_5 \, X_E{}^2 \rangle \\
&= \langle m_5{}^* \, m_E{}^T m_E{}^* \, m_5 \, m_5{}^* \, m_5 \, \left(m_E{}^\dagger m_E \right)^2 \rangle \\
&\quad \pm \langle m_5{}^* \, m_5 \, m_5{}^* \, m_E{}^T m_E{}^* \, m_5 \, \left(m_E{}^\dagger m_E \right)^2 \rangle , \\
I_{8,4}^{(\pm)} &= \langle m_5{}^* \, (X_E{}^T)^2 \, m_5 \, m_5{}^* \, m_5 \, X_E{}^2 \rangle \\
&\quad \pm \langle m_5{}^* \, m_5 \, m_5{}^* \, (X_E{}^T)^2 m_5 \, X_E{}^2 \rangle \\
&= \langle m_5{}^* \, \left(m_E{}^T m_E{}^* \right)^2 \, m_5 \, m_5{}^* \, m_5 \, \left(m_E{}^\dagger m_E \right)^2 \rangle \\
&\quad \pm \langle m_5{}^* \, m_5 \, m_5{}^* \, \left(m_E{}^T m_E{}^* \right)^2 m_5 \, \left(m_E{}^\dagger m_E \right)^2 \rangle .
\end{aligned}
$$

Lepton Flavor Invariants: EFT $n_g = 3$

$$H(q) = \frac{N(q)}{D(q)}$$

$$N(q) = 1 + q^6 + 2q^8 + 4q^{10} + 8q^{12} + 7q^{14} + 9q^{16} + 10q^{18} + 9q^{20} + 7q^{22} + 8q^{24} + 4q^{26} + 2q^{28} + q^{30} + q^{36}$$

$$D(q) = \left(1-q^2\right)^2 \left(1-q^4\right)^3 \left(1-q^6\right)^4 \left(1-q^8\right)^2 \left(1-q^{10}\right)$$

- Ring $\mathbb{C}[m_5, m_5^*, m_E, m_E^\dagger]^{G_{\text{Flavor}}}$ of all polynomials which are invariant under

$$G_{\text{Flavor}} = SU(3)_L \times SU(3)_{E^c} \times U(1)^2$$

$$p = \dim V - \dim G, \qquad \dim V = 30, \quad \dim G = 18$$

Lepton Flavor Invariants: EFT $n_g = 3$

- Knop's Theorem

Theorem

$$\dim V \geq d_D - d_N \geq p$$

$$\dim V = 30, \; d_D = 66, \; d_N = 36, \; p = 12$$

$p = 12$ consists of 6 masses, 3 angles and 3 phases

$$30 \geq 30 \geq 12$$

Lepton Flavor Invariants: Seesaw Theory

$$X_E = {m_E}^\dagger m_E, \qquad X_E \to \mathcal{U}_L^\dagger \, X_E \, \mathcal{U}_L$$

$$m_\nu \to {\mathcal{U}_{N^c}}^T \, m_\nu \, \mathcal{U}_L$$

$$M \to {\mathcal{U}_{N^c}}^T \, M \, \mathcal{U}_{N^c}$$

$$X_\nu = m_\nu^\dagger m_\nu, \qquad X_\nu \to \mathcal{U}_L^\dagger \, X_\nu \, \mathcal{U}_L$$

$$Z_\nu = m_\nu m_\nu^\dagger, \qquad Z_\nu \to {\mathcal{U}_{N^c}}^T \, Z_\nu \, {\mathcal{U}_{N^c}}^*$$

$${Z_\nu}^T = m_\nu^* {m_\nu}^T, \qquad {Z_\nu}^T \to {\mathcal{U}_{N^c}}^\dagger \, {Z_\nu}^T \, \mathcal{U}_{N^c}$$

$$X_N = M^* M, \qquad X_N \to \mathcal{U}_{N^c}^\dagger \, X_N \, \mathcal{U}_{N^c}$$

$$Z_N = M M^*, \qquad Z_N \to {\mathcal{U}_{N^c}}^T \, Z_N \, {\mathcal{U}_{N^c}}^*$$

$$Z_X = m_\nu \, X_E \, {m_\nu}^\dagger \qquad Z_X \to {\mathcal{U}_{N^c}}^T Z_X \, {\mathcal{U}_{N^c}}^*$$

Lepton Flavor Invariants: Seesaw Theory $n_g = 2$

$$
\begin{aligned}
I_{2,0,0} &= \langle X_E \rangle = \langle {m_E}^\dagger m_E \rangle \,, \\
I_{0,2,0} &= \langle X_\nu \rangle = \langle {m_\nu}^\dagger m_\nu \rangle \,, \\
I_{0,0,2} &= \langle X_N \rangle = \langle M^* M \rangle \,, \\
I_{4,0,0} &= \langle {X_E}^2 \rangle = \langle {m_E}^\dagger m_E {m_E}^\dagger m_E \rangle \,, \\
I_{2,2,0} &= \langle X_\nu X_E \rangle = \langle {m_\nu}^\dagger m_\nu {m_E}^\dagger m_E \rangle \,, \\
I_{0,4,0} &= \langle {X_\nu}^2 \rangle = \langle {m_\nu}^\dagger m_\nu {m_\nu}^\dagger m_\nu \rangle \,, \\
I_{0,2,2} &= \langle Z_\nu Z_N \rangle = \langle m_\nu {m_\nu}^\dagger M M^* \rangle \,, \\
I_{0,0,4} &= \langle {X_N}^2 \rangle = \langle M^* M M^* M \rangle \,, \\
I_{2,2,2} &= \langle Z_X Z_N \rangle = \langle m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M M^* \rangle \,, \\
I_{0,4,2} &= \langle M^* Z_\nu M {Z_\nu}^T \rangle = \langle M^* m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I_{2,4,2} &= \langle M^* Z_\nu M {Z_X}^T \rangle \\
&= \langle M^* m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_E}^T {m_E}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{2,4,2} &= \langle M^* Z_\nu Z_X M \rangle - \langle M^* Z_X Z_\nu M \rangle \\
&= \langle M^* m_\nu {m_\nu}^\dagger m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M \rangle \\
&\quad - \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger m_\nu {m_\nu}^\dagger M \rangle \,, \\
I^{(-)}_{0,4,4} &= \langle Z_N Z_\nu M {Z_\nu}^T M^* \rangle - \langle M^* Z_\nu Z_N M {Z_\nu}^T \rangle \\
&= \langle M M^* m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T M^* \rangle \\
&\quad - \langle M^* m_\nu {m_\nu}^\dagger M M^* M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I_{4,4,2} &= \langle M^* Z_X M {Z_X}^T \rangle \\
&= \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M {m_\nu}^* {m_E}^T {m_E}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{2,4,4} &= \langle Z_N Z_X M {Z_\nu}^T M^* \rangle - \langle M^* Z_X Z_N M {Z_\nu}^T \rangle \\
&= \langle M M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T M^* \rangle \\
&\quad - \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M M^\dagger M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{2,6,2} &= \langle M^* Z_\nu Z_X M {Z_\nu}^T \rangle - \langle M^* Z_X Z_\nu M {Z_\nu}^T \rangle \\
&= \langle M^* m_\nu {m_\nu}^\dagger m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T \rangle \\
&\quad - \langle M^* m_\nu {m_E}^\dagger m_E {m_\nu}^\dagger m_\nu {m_\nu}^\dagger M {m_\nu}^* {m_\nu}^T \rangle \,, \\
I^{(-)}_{4,4,4} &= \left\langle M^* Z_N Z_X M Z_X^T \right\rangle - \left\langle M^* Z_X^T Z_N M Z_X \right\rangle \,, \\
I^{(-)}_{4,6,2} &= \left\langle M^* Z_\nu Z_X M Z_X^T \right\rangle - \left\langle M^* Z_X Z_\nu M Z_X^T \right\rangle .
\end{aligned}
\tag{9}
$$

Lepton Flavor Invariants: Seesaw Theory $n_g = 2$

$$H(q) = \frac{1 + q^6 + 3q^8 + 2q^{10} + 3q^{12} + q^{14} + q^{20}}{(1-q^2)^3(1-q^4)^5(1-q^6)(1-q^{10})}$$

- Ring $\mathbb{C}[m_\nu, m_\nu^\dagger, m_E, m_E^\dagger, M, M^*]^{G_{\rm Flavor}}$ of all polynomials which are invariant under

$$G_{\rm Flavor} = SU(2)_L \times SU(2)_{E^c} \times U(2)_{N^c} \times U(1)^2$$

$$p = {\rm dim} V - {\rm dim} G, \qquad {\rm dim} V = 22, \quad {\rm dim} G = 12$$

Lepton Flavor Invariants: Seesaw Theory $n_g = 2$

- Knop's Theorem

Theorem

$$\dim V \geq d_D - d_N \geq p$$

$$\dim V = 22, \; d_D = 42, \; d_N = 20, \; p = 10$$

$p = 10$ consists of 6 masses, 2 angles and 2 phases

$$22 \geq 22 \geq 10$$

$$
\begin{aligned}
H(q) &= \frac{N(q)}{D(q)}, \\
N(q) &= 1 + q^4 + 5q^6 + 9q^8 + \cdots + 9q^{106} + 5q^{108} + q^{110} + q^{114}, \\
D(q) &= (1-q^2)^3(1-q^4)^4(1-q^6)^4(1-q^8)^2(1-q^{10})^2(1-q^{12})^3 \\
&\quad \times (1-q^{14})^2(1-q^{16})
\end{aligned}
$$

- Ring $\mathbb{C}[m_\nu, m_\nu^\dagger, m_E, m_E^\dagger, M, M^*]^{G_{\text{Flavor}}}$ of all polynomials which are invariant under

$$G_{\text{Flavor}} = SU(3)_L \times SU(3)_{E^c} \times U(3)_{N^c} \times U(1)^2$$

$$p = \dim V - \dim G, \qquad \dim V = 48, \quad \dim G = 27$$

Lepton Flavor Invariants: Seesaw Theory $n_g = 3$

- Knop's Theorem

Theorem

$$\dim V \geq d_D - d_N \geq p$$

$$\dim V = 48,\ d_D = 162,\ d_N = 114,\ p = 21$$

$p = 21$: 9 masses, 6 angles and 6 phases

$$48 \geq 48 \geq 21$$

Summary

- Lepton and quark mass matrix invariants studied in low-energy SM effective theory and seesaw model.
- Hilbert series found for cases of physical interest, namely SM EFT with $d = 5$ operator Majorana neutrino masses for $n_g = 2, 3$ and seesaw theory for $n_g = 2, 3$ generations of SM fermions and $n'_g = 2, 3$ gauge-singlet neutrinos in seesaw model.
- Non-trivial relations (syzygies) between lepton invariants is encoded in Hilbert series.
- Structure of lepton flavor invariants is extremely non-trivial.