Von Dyck symmetries and lepton mixing

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In colab. with A. Yu. Smirnov; 1204.0445

It all begins with large mixing angles in the leptonic sector

$$|U_{l\nu}|^2 = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} \sim 0.7 & \sim 0.3 & \sim 0 \\ \sim 0.1 & \sim 0.3 & \sim 0.5 \\ \sim 0.1 & \sim 0.3 & \sim 0.5 \end{pmatrix} \end{array}$$

$$(|U_{l\nu}|^2) = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array} \begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{pmatrix} \end{array}$$

TriBimaximal Mixing

Harrison, Perkins, Scott, 2002

Even before and for different reasons, Bimaximal mixing had been proposed

Bimaximal Mixing

Vissani, 1997

$$|\mathcal{U}| \approx \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \\ \frac{\cos\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \sin\theta \end{pmatrix}$$

Do these mixing patterns have something to do with the actual neutrino mass matrix??

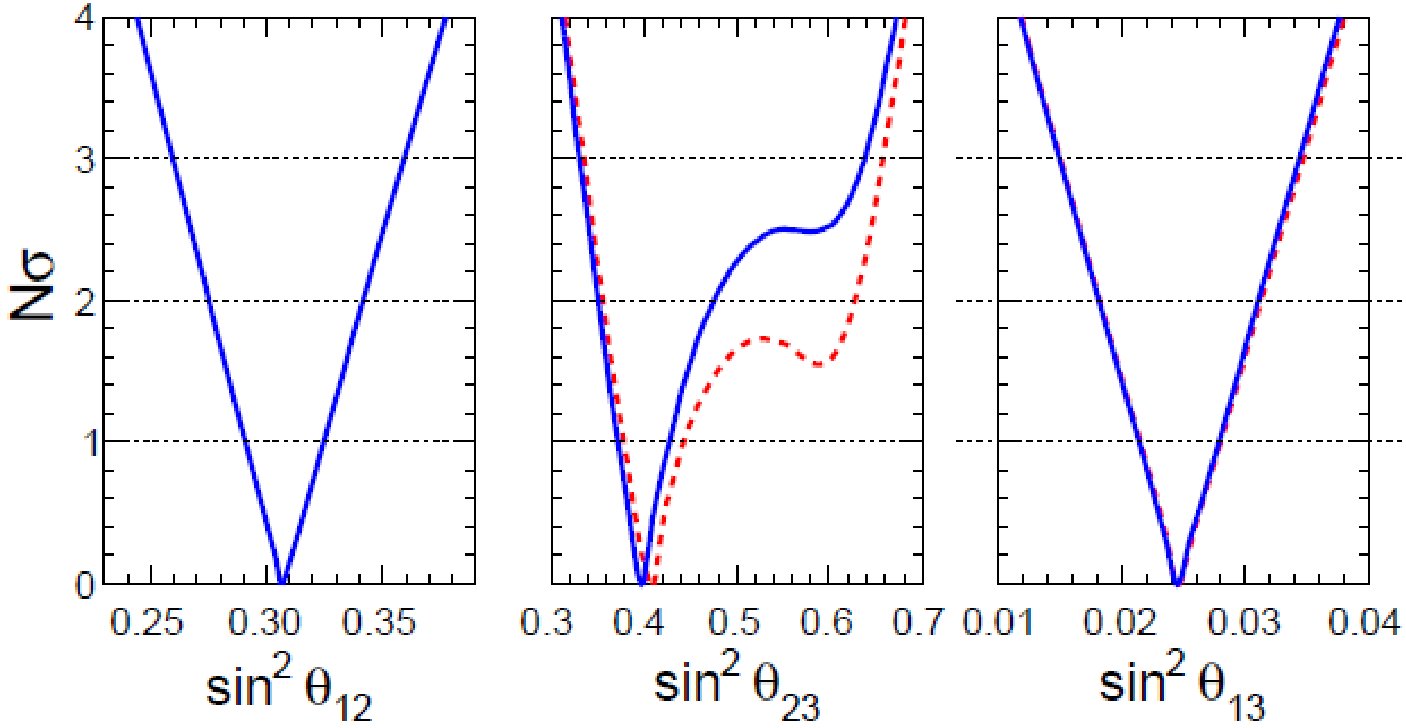
1. **Is is possible to reproduce this special mixing pattern from a fundamental Lagrangian??**

   Yes! Using discrete symmetries

2. **Is such Lagrangian believable?**

   Well… at least for many the answer is NO.

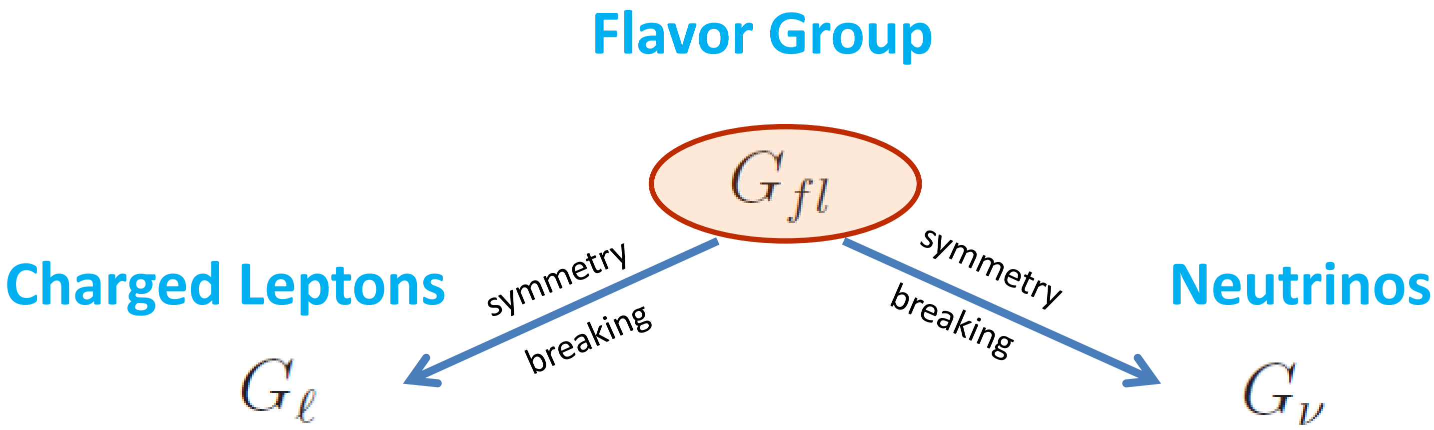
TBM now disfavoured



Fogli et al., 1205.5254

Can we make model-independent statements about the use of discrete symmetries in flavor??

General framework



Bottom-up approach: Identify G_ℓ and G_ν with accidental symmetries of the mass terms. Use them to define the flavor group G_{fl}

Identifying the accidental symmetries

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L U_{PMNS}\gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L + \ldots + \text{h.c.}$$

Focus on the mass terms

Charged Leptons

$\bar{E}_R m_\ell \ell_L$ is invariant under $U(1)^3$ **accidental**

$$E_R \to T E_R\,, \quad \ell_L \to T\ell_L \qquad\qquad T = \text{diag}\{e^{i\alpha},\, e^{i\beta},\, e^{i\gamma}\}$$

Identifying the flavor symmetry

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L U_{PMNS}\gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L + \ldots + \text{h.c.}$$

Focus on the mass terms

Neutrinos

$\frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L$ invariant under $Z_2 \otimes Z_2$ accidental

$$S_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad S_3 = S_1 S_2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

Enter mixing matrix

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L U_{PMNS}\gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L m_\nu \nu_L + \ldots + \text{h.c.}$$

Change of basis

$$\mathscr{L} = \frac{g}{\sqrt{2}}\bar{\ell}_L \gamma^\mu \nu_L W_\mu^+ + \bar{E}_R m_\ell \ell_L + \frac{1}{2}\bar{\nu^c}_L M_\nu \nu_L + \cdots + \text{h.c.} \qquad M_\nu = U^* m_\nu U^\dagger$$

Invariance of M_ν under $Z_2 \otimes Z_2$ accidental

$$S_{iU}^\dagger M_\nu S_{iU} = M_\nu \quad \text{with} \quad S_{iU} = U S_i U^\dagger$$

Still $S_{iU}^2 = 1$

Identified $U(1)^3$ for the charged leptons and Z2xZ2 for the neutrinos

For charged leptons, use a discrete subgroup as part of the group of flavor

Impose $T^m = 1$

Define T_α

$$T_e = \begin{pmatrix} 1 & & \\ & e^{2\pi ik/m} & \\ & & e^{-2\pi ik/m} \end{pmatrix}, \qquad T_\mu = \begin{pmatrix} e^{2\pi ik/m} & & \\ & 1 & \\ & & e^{-2\pi ik/m} \end{pmatrix},$$

$$T_\tau = \begin{pmatrix} e^{2\pi ik/m} & & \\ & e^{-2\pi ik/m} & \\ & & 1 \end{pmatrix}$$

Defining the flavor group

Symmetry group of neutrino mass matrix: Already discrete.

Choose at least one of the S_{iU} and T_α .

Define a relation between S_{iU} and T_α

$$T_\alpha^m = 1 \qquad\qquad S_{iU}^2 = 1$$

$$(S_{iU} T_\alpha)^p = (U S_i U^\dagger T_\alpha)^p = \mathbb{I}$$

The relations

$$S_{iU}^{n} = T^{m} = (S_{iU}T)^{p} = \mathbb{I}$$

define the **von Dyck group** $D(n, m, p)$

$D(2, 2, p)$ is the dihedral group $\mathbf{D}_p$

$$D(2,2,3) = \mathbf{S}_3$$
$$D(2,3,3) = \mathbf{A}_4$$
$$D(2,3,4) = \mathbf{S}_4$$
$$D(2,3,5) = \mathbf{A}_5$$

Notice that if

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{p} \leq 1$$

The von Dyck group is infinite

Now you know the flavor group and the symmetry breaking pattern, go and construct a model

Constraints on the mixing matrix

$$W_{i\alpha} = S_{iU} T_\alpha = U S_i U^\dagger T_\alpha \,, \quad W_{i\alpha}^p = 1$$

$$\mathrm{Det}[W_{i\alpha} - \lambda \mathbb{I}] = 0$$ cubic equation with $\lambda_i^p = 1$

Take one of the eigenvalues of $W_{i\alpha}$ equal to 1

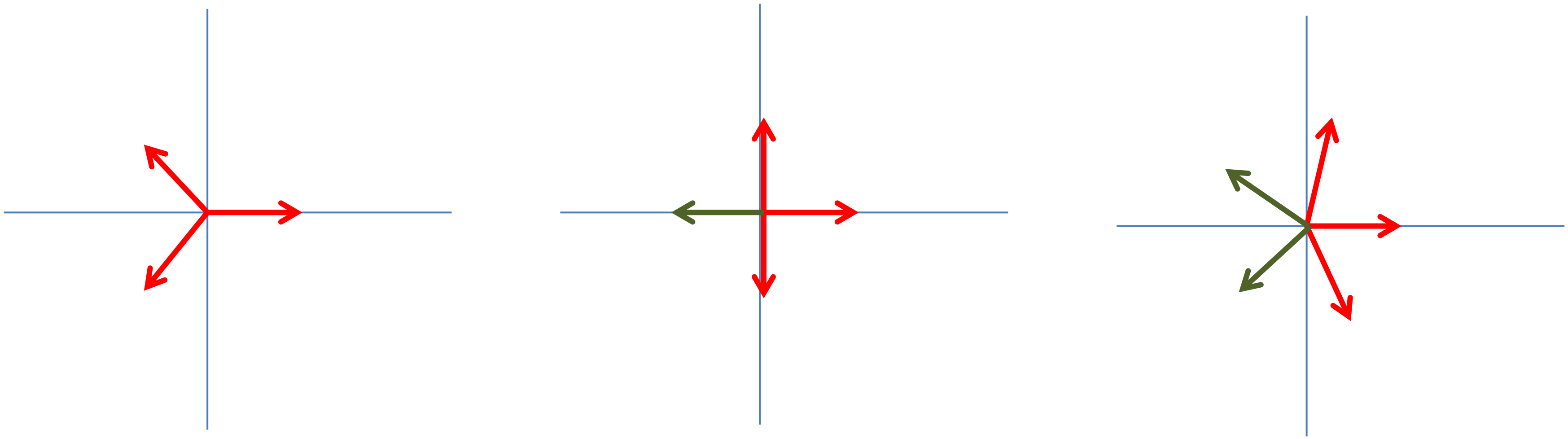
$$\lambda^3 + a\lambda^2 - a\lambda - 1 = 0$$ with $a = -\mathrm{Tr}[W_{i\alpha}]$

This is a real number!

Constraints on the mixing matrix

$$W_{i\alpha} = S_{iU} T_\alpha = U S_i U^\dagger T_\alpha \,, \quad W_{i\alpha}^p = 1$$

p=3 *p=4* *p=5*



For instance, for $p = 3 \longrightarrow (\lambda - 1)(\lambda - \omega)(\lambda - \omega^2) = \lambda^3 - 1 \longrightarrow a = 0$

or $p = 4 \longrightarrow (\lambda - 1)(\lambda + i)(\lambda - i) = \lambda^3 - \lambda^2 + \lambda - 1 \longrightarrow a = -1$

Implies two conditions on the mixing matrix

$$W_{i\alpha} = U S_i U T_\alpha \,, \quad W_{i\alpha}^p = 1$$

$$|U_{\beta i}|^2 = |U_{\gamma i}|^2$$
$$|U_{\alpha i}|^2 = \eta, \qquad \beta, \gamma \neq \alpha$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

First equation is general and depends only on the choice of S_{iU} and T_α

In the second equation η depends on the eigenvalues of $W_{i\alpha}$ through a, on the eigenvalues of T_α and on the choice of S_{iU}

Two equations lead to two constraints on the mixing angles.

The problem is reduced to a case study

$$W_{i\alpha} = US_iUT_\alpha\,, \quad W_{i\alpha}^p = 1$$

$$a = -\mathrm{Tr}[W_{i\alpha}]$$

$$|U_{\beta i}|^2 = |U_{\gamma i}|^2$$
$$|U_{\alpha i}|^2 = \eta, \qquad \beta, \gamma \neq \alpha$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

Remember $$|U_{l\nu}|^2 = \begin{matrix} \\ e \\ \mu \\ \tau \end{matrix} \begin{matrix} \nu_1 & \nu_2 & \nu_3 \\ \left(\begin{matrix} \sim 0.7 \\ \sim 0.1 \\ \sim 0.1 \end{matrix} \right. & \begin{matrix} \sim 0.3 \\ \sim 0.3 \\ \sim 0.3 \end{matrix} & \left. \begin{matrix} \sim 0 \\ \sim 0.5 \\ \sim 0.5 \end{matrix} \right) \end{matrix}$$

Hence, either $i = 2$ or $\alpha = e$

Recapitulating: What we assume

First and foremost

- The general framework for building a model with discrete symmetries

Other 'minor' assumptions

1. Neutrinos are Majorana.
2. The flavor symmetry is a subgroup of SU(3).
3. The remaining symmetry in each sector is a one-generator group
4. There is one charged lepton that doesn't transform under T
5. There is one neutrino that doesn't transform under S
6. ST has an eigenvalue that is equal to 1

Open to discussion!!

Recapitulating: What I have shown (under said assumptions)

After a number of choices have been made

1. T-charge of one charged lepton (*k* value)
2. The order of T (m value)
3. The eigenvalues of ST. *(a* value*)* .

A two-dimensional surface is cut in the parameter space of the mixing matrix.

Is is possible to fit the measured values of the PMNS matrix??

Hence, either $i = 2$ or $\alpha = e$

Choose $\alpha = e$

$$|U_{\mu i}|^2 = |U_{\tau i}|^2$$
$$|U_{ei}|^2 = \eta$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

Substituting the standard parameterization for $i = 1$

$$\tan 2\theta_{23} = -\frac{\sin^2\theta_{12} - \cos^2\theta_{12}\sin^2\theta_{13}}{\sin 2\theta_{12}\sin\theta_{13}\cos\delta}$$

S.F. Ge et al

$$\cos^2\theta_{12} = \frac{\eta}{\cos^2\theta_{13}} .$$

And for $i = 2$

$$\tan 2\theta_{23} = \frac{\cos^2\theta_{12} - \sin^2\theta_{12}\sin^2\theta_{13}}{\sin 2\theta_{12}\sin\theta_{13}\cos\delta}$$

$$\sin^2\theta_{12} = \frac{\eta}{\cos^2\theta_{13}} .$$

Choose $\alpha = e$

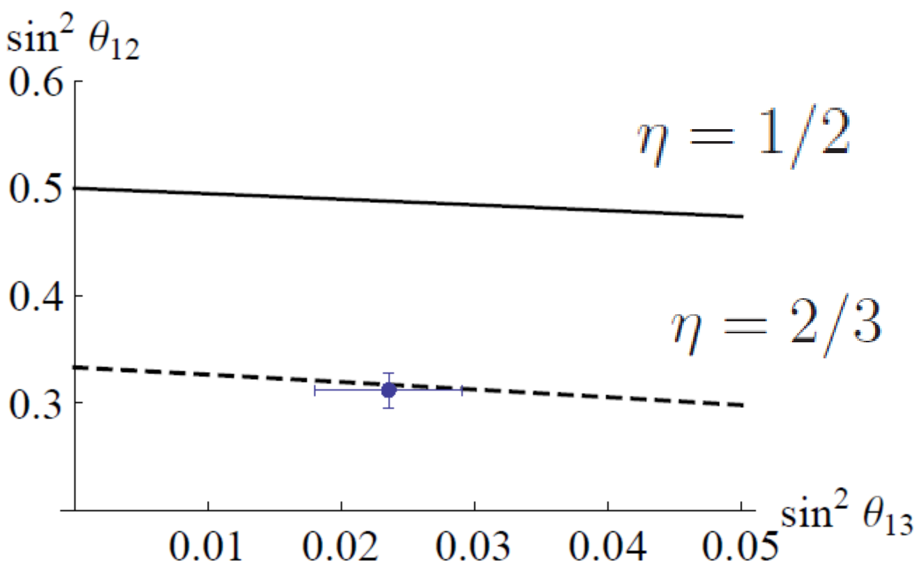
Taking $i = 1$

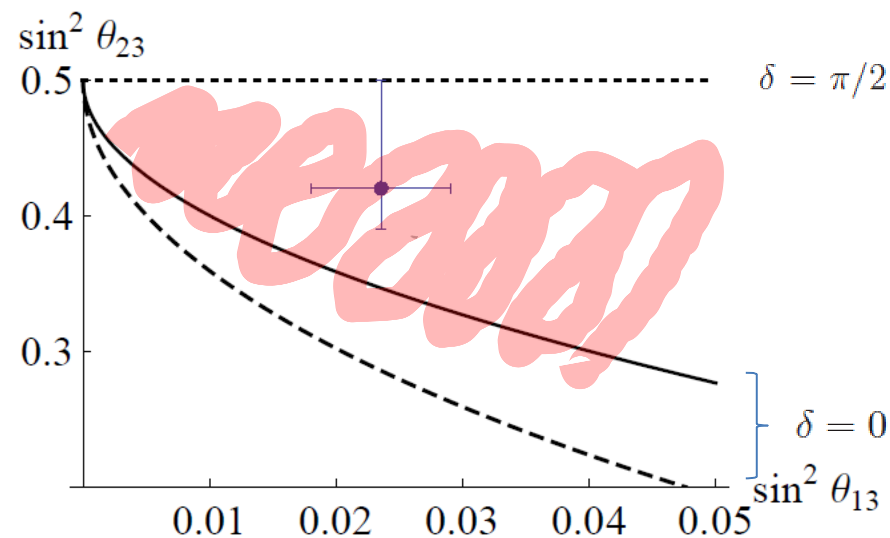
$$S_{iU}^n = T^m = (S_{iU}T)^p = \mathbb{I}$$

$$\lambda^3 + a\lambda^2 - a^*\lambda - 1 = 0$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

- Solid: *m = 4, p = 3. k=1* and from $(\lambda-1)(\lambda-\omega)(\lambda-\omega^2) = \lambda^3 - 1$, *a=0* . Group is $\mathbf{S}_4$
- Dashed: *m = 3, p = 4. k=1, a=-1.* Group is $\mathbf{S}_4$





Altarelli, Feruglio, Hagedorn, Merlo,…

Choose $\alpha = e$

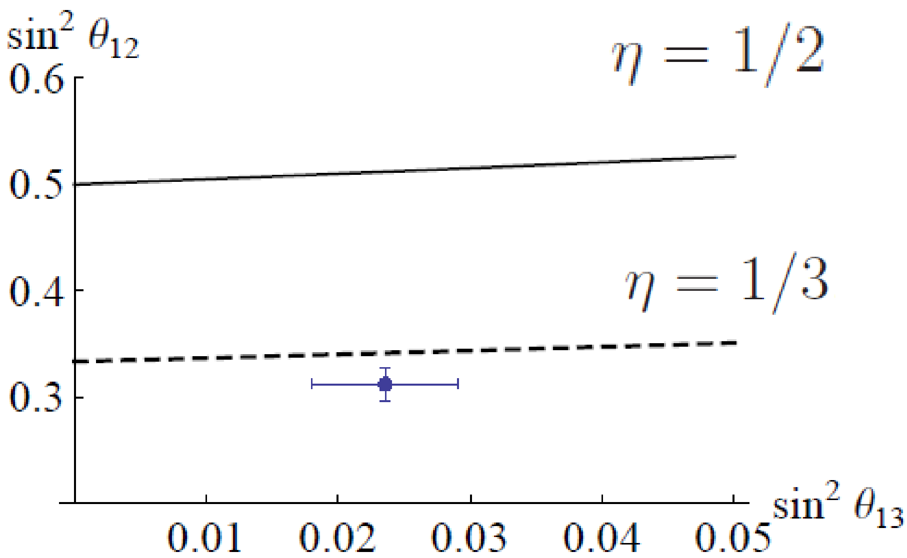
$$S_{iU}^n = T^m = (S_{iU}T)^p = \mathbb{I}$$

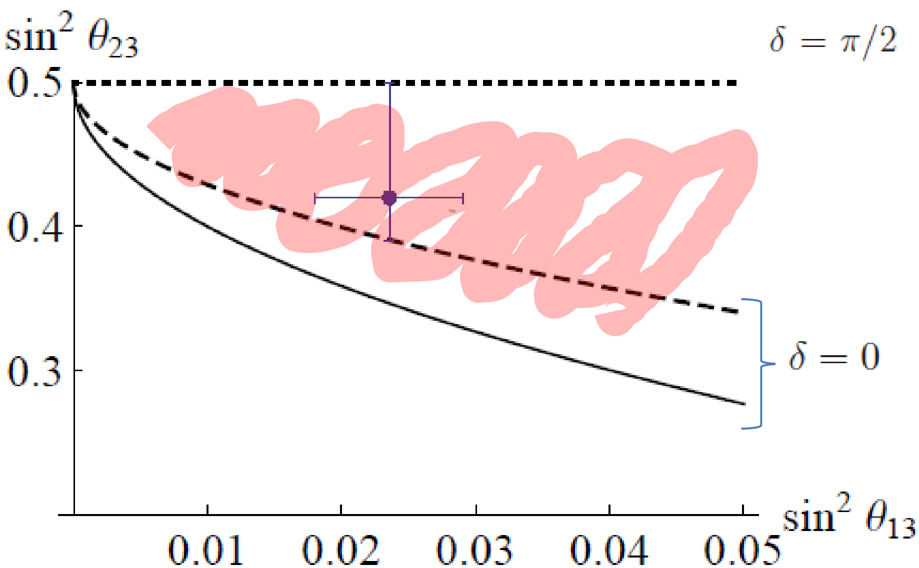
$$\lambda^3 + a\lambda^2 - a^*\lambda - 1 = 0$$

$$\eta \equiv \frac{1-a}{4\sin^2\left(\frac{\pi k}{m}\right)}$$

Taking $i = 2$

- Dashed: *m = 3, p = 3. k=1* and *a=0* . Group is $\mathbf{A}_4$
- Solid: *m = 4, p = 3. k=1, a=-1.* Group is $\mathbf{S}_4$





Ma, Babu, Valle, Altarelli, Feruglio, Merlo,…

Hence, either $i = 2$ or $\alpha = e$

Choose $i = 2$

$$|U_{e2}|^2 = |U_{\mu(\tau)2}|^2 , \qquad |U_{\tau(\mu)2}|^2 = \eta$$

$$\sin^2\theta_{12} = \frac{1-\eta}{2\cos^2\theta_{13}}$$

For the case of T_μ

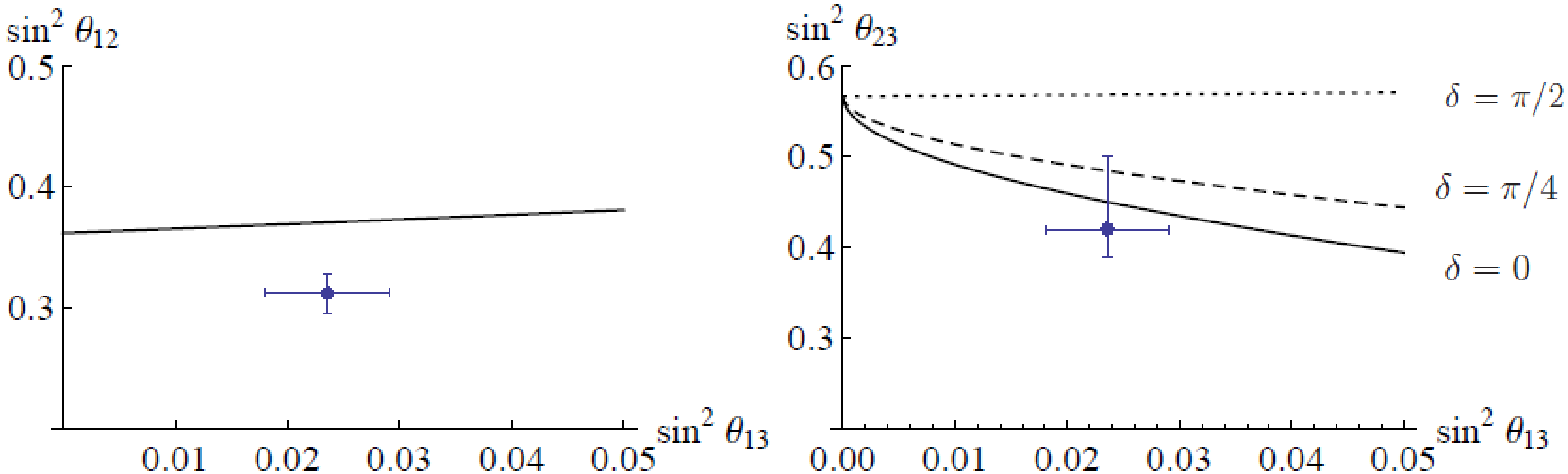
$$\cos\delta = -2\frac{\sin^2\theta_{12}(\cos^2\theta_{23}\sin^2\theta_{13} - \cos^2\theta_{13}) + \cos^2\theta_{12}\sin^2\theta_{23}}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}}$$

For the case of T_τ

$$\cos\delta = 2\frac{\sin^2\theta_{12}(\sin^2\theta_{23}\sin^2\theta_{13} - \cos^2\theta_{13}) + \cos^2\theta_{12}\cos^2\theta_{23}}{\sin 2\theta_{12}\sin 2\theta_{23}\sin\theta_{13}}$$

Unexplored

$$T_\mu : \quad a = 0 \quad k = 2 \quad m = 5$$



A few words about TBM

If one imposes that the two Z2 symmetries of the neutrino mass matrix should belong to the flavor group, then 4 relations appear between the entries of the mixing matrix

If they are compatible, they will fix all parameters of the mixing matrix.

TBM is indeed one solution for the case of $\mathbf{S_4}$.

This could be an argument pro TBM.

Lam

Conclusions

- Recipe for model building: upgrade the accidental symmetries of the mass terms by making them part of agroup.
- The minimal choice of generators (one Z_2 for neutrinos and one Z_N for charged leptons) leads to non-abelian discrete groups of the von Dyck type.
- In this scheme, two relations are imposed on the leptonic mixing matrix.
- One case with $\mathbf{S_4}$ shows a very good agreement with the measured values.

arXiv:1205.0075v2

New Simple A_4 Neutrino Model for Nonzero θ_{13} and Large δ_{CP}

Hajime Ishimori[1] and Ernest Ma[2,3]

Eq. (6) can be diagonalized exactly. Assuming that a, d are real and c complex, we find

$$\tan^2\theta_{12} = \frac{1 - 3\sin^2\theta_{13}}{2}, \quad (23)$$

arXiv:1205.5133v1

Tri-Bimaximal Neutrino Mixing and Discrete Flavour Symmetries

Guido Altarelli[1,2]*, Ferruccio Feruglio[3]**, and Luca Merlo[4,5]***

It is interesting to note that if we neglect the corrections proportional to ξ, we have an exact relation between the solar and the reactor angle:

$$\sin^2\theta_{12} = \frac{1}{3(1 - \sin^2\theta_{13})}, \quad (52)$$