

Analytic LO Gluon Distributions from the  
proton structure function  $F_2(x, Q^2)$ ---  
---> New PDF's for the LHC

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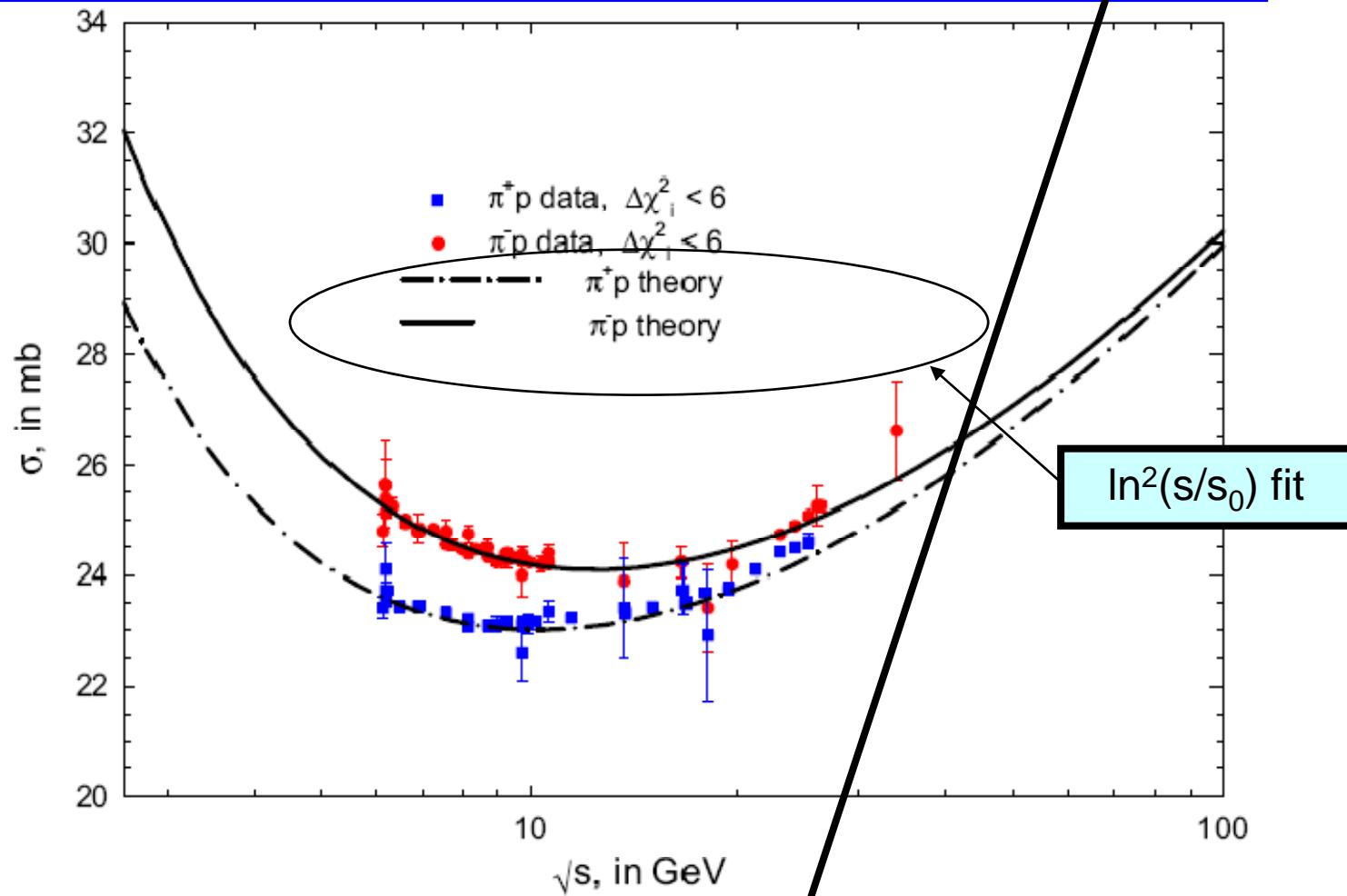
## OUTLINE: Background to talk

- 1) Fitting the accelerator data---“New evidence for the saturation of the Froissart bound”,  
M. Block and F. Halzen, Phys. Rev. D **72**, 036006 (2005).
- 2) The Proton Structure Function  $F_2^p(x, Q^2)$  : “Small-x behavior of parton distributions from the observed Froissart energy dependence of the deep-inelastic-scattering cross sections”,  
M. M. Block, Edmund L. Berger and Chung-I Tan,  
Phys. Rev. Lett. 308 (2006).  
“Analytic expression for the joint x and  $Q^2$  dependences of the deep-inelastic structure function”,  
Edmund L. Berger, M. M. Block and Chung-I Tan,  
Phys. Rev. Lett. 98 242001 (2007).

## OUTLINE: Talk

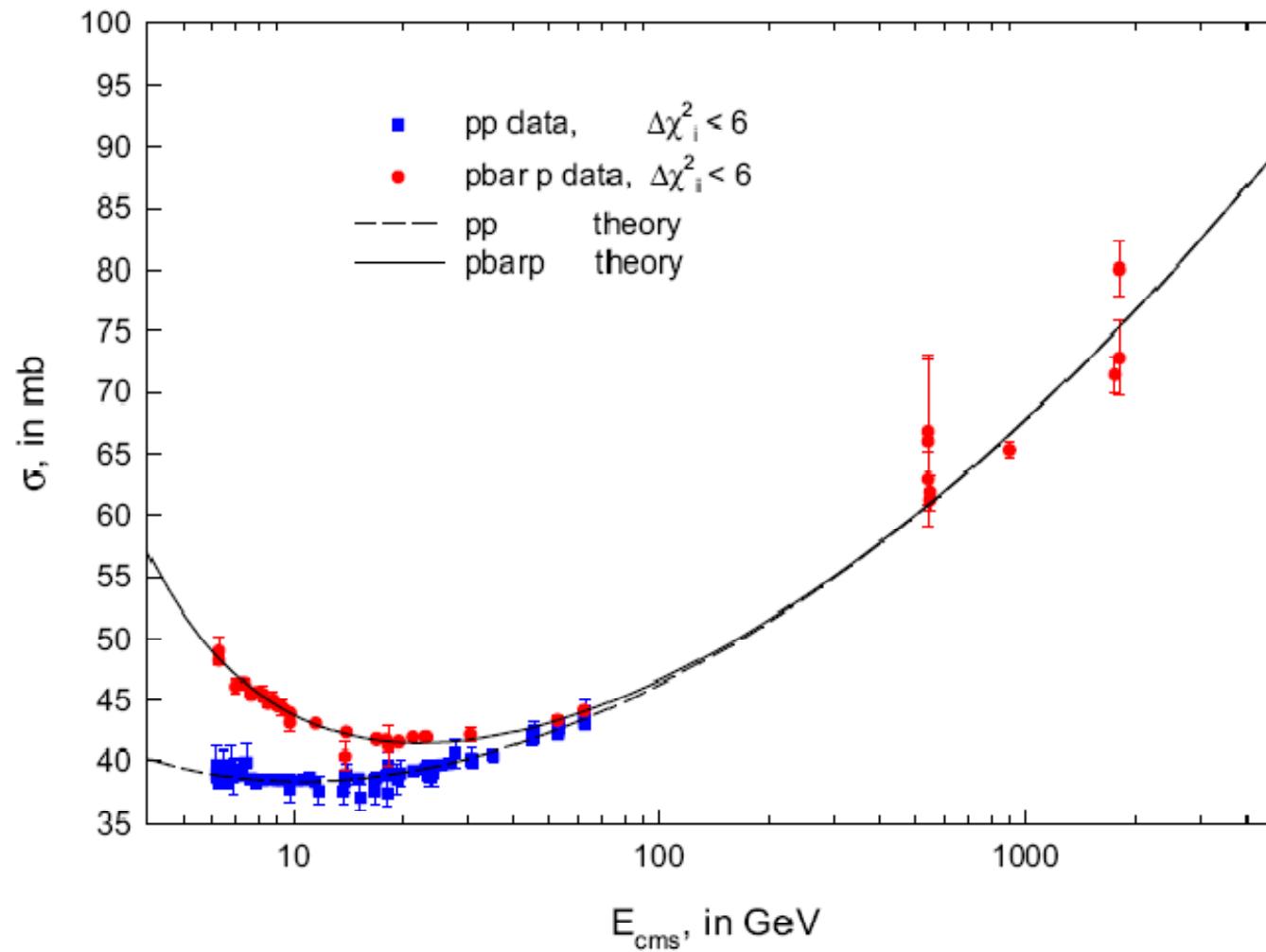
- New Analytic LO Gluon Distributions using the Froissart Bound and massless quarks:  
“Analytic derivation of the leading-order gluon distribution function  $G(x, Q^2) = xg(x, Q^2)$  from the proton structure function  $F_2^p(x, Q^2)$ ”, M. M. Block, L. Durand and D. McKay, arXiv hep/ph 0710.312 (2007).
- Analytic LO Gluon Distributions using the Froissart Bound, for 5 quarks: 3 massless quarks + massive c and b quarks:  
M. M. Block, L. Durand and D. McKay, Aspen Winter Physics Conference, Jan. 2008

Part I: Cross section fits for  $E_{\text{cms}} > 6 \text{ GeV}$ , anchored at 2.6 GeV,  
 $\pi^+p$  and  $\pi^-p$

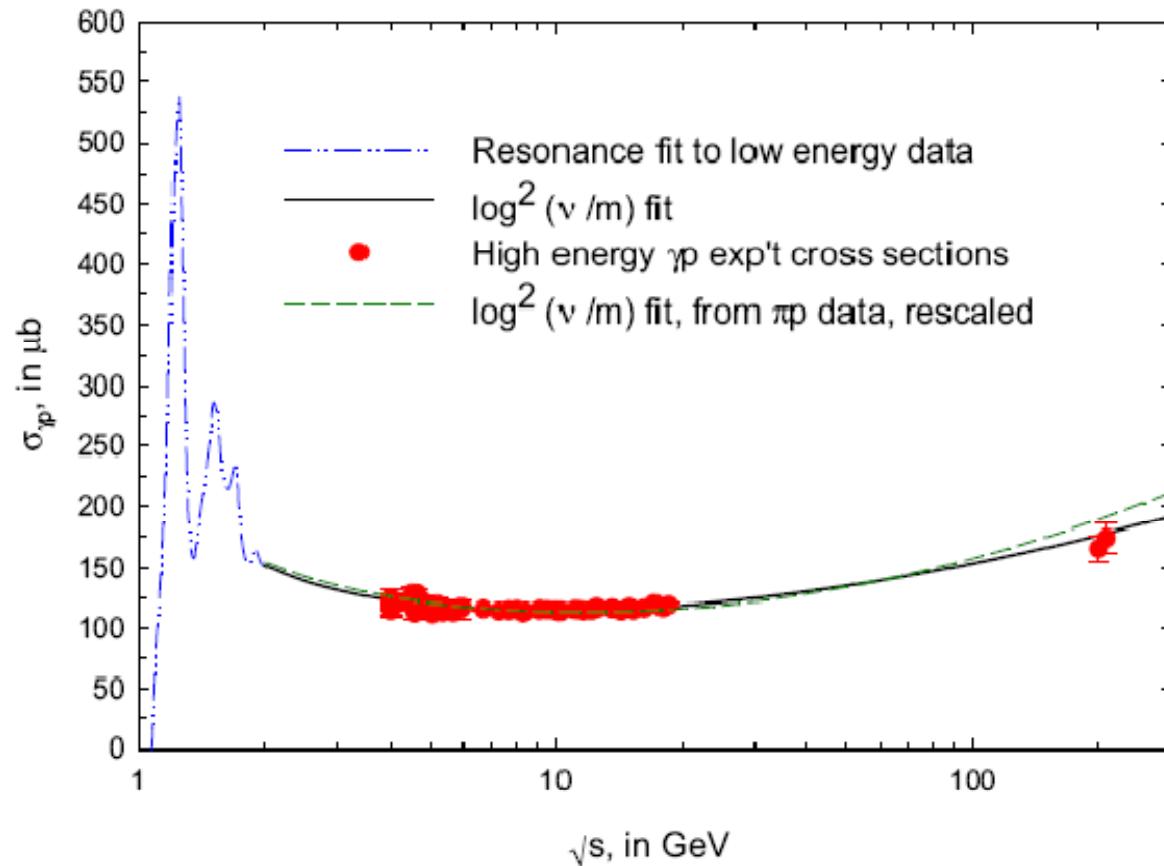


M. Block and F. Halzen, Phys. Rev. D **72**, 036006 (2005); K. Igi and M. Ishida, Phys. Lett. B **262**, 286 (2005). It gives 4 constraints! Only 2 parameters needed for cross sections.

Cross section fits for  $E_{\text{cms}} > 6 \text{ GeV}$ , anchored at 4 GeV,  
pp and pbar p



## $\gamma p$ $\log^2(v/m)$ fit, compared to the $\pi p$ even amplitude fit



M. Block and  
F. Halzen,  
Phys. Rev. D **70**,  
091901, (2004)

Part 2: Proton Structure Function  $F_2(x, Q^2)$ , from Deep Inelastic Scattering, Block, Berger & Tan, PRL 99, 88 (2006); Berger, Block and Tan, PRL 98 242001 (2007).

Kinematics in the inclusive process  $ep \rightarrow e'X$

The invariant  $s$ , the square of the center of mass (c.m.) energy  $W$  of the  $\gamma^*p$  system, is  $s \equiv W^2 = (q + p)^2 = 2m\nu - Q^2 + m^2$ . The Lorentz invariant variables  $x$  and  $y$  are defined as  $x \equiv \frac{Q^2}{2p \cdot q} = \frac{Q^2}{2m\nu}$  and  $y \equiv \frac{p \cdot q}{p \cdot k}$ , where  $k$  is the incoming electron's four-vector momentum. Thus,

$$s = W^2 = \frac{Q^2}{x}(1 - x) + m^2.$$

## Global Fit

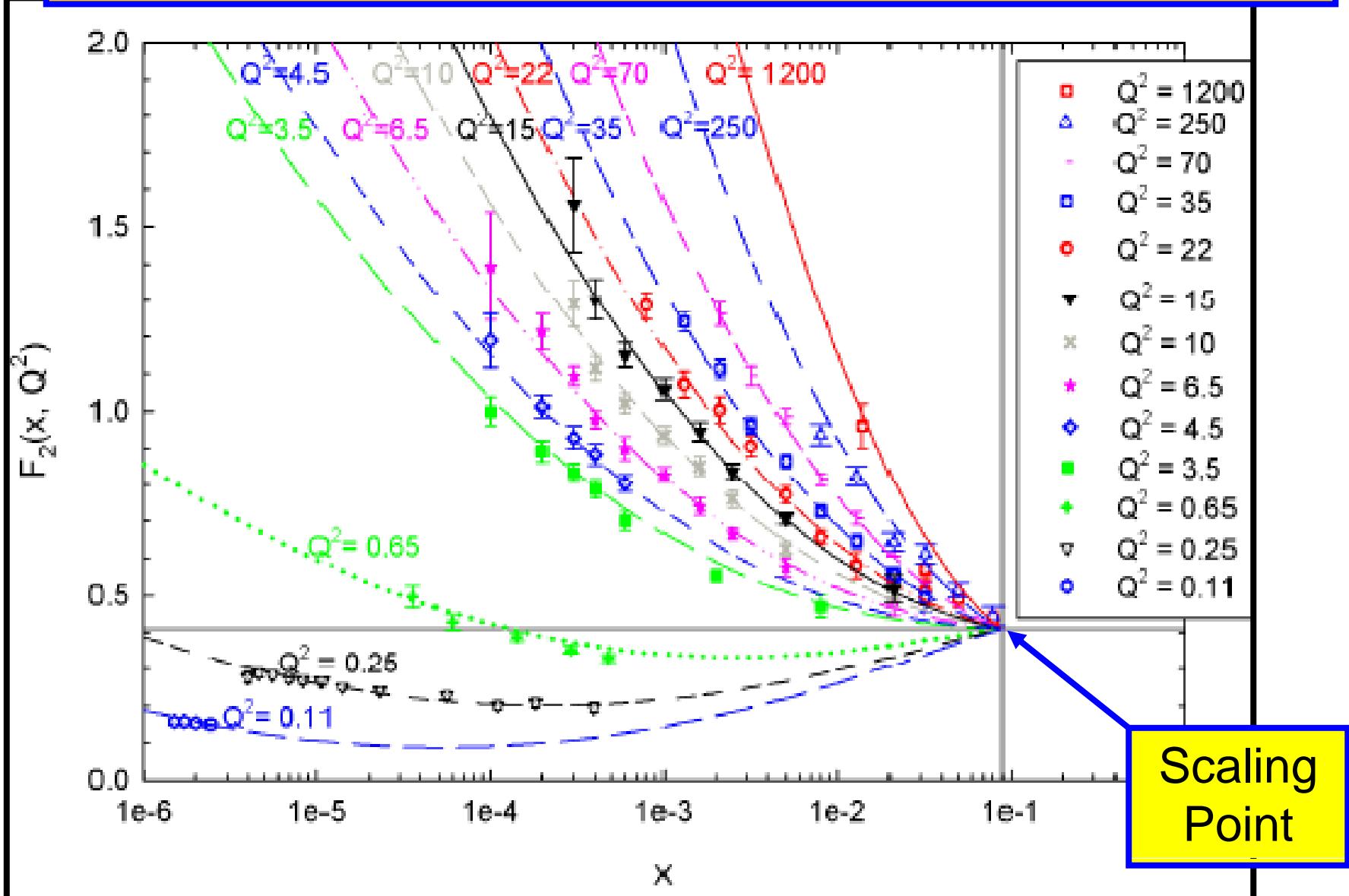
The H1 collaboration determined that, for *fixed*  $x$ , the  $Q^2$  dependence of  $F_2^p(x, Q^2)$  was reproduced by

$$F_2^p(x, Q^2) = \alpha_0(x) + \alpha_1(x) \ln(Q^2) + \alpha_2(x) \ln^2(Q^2),$$

so the ZEUS data were fit to:

$$\begin{aligned} F_2^p(x, Q^2) = & (1 - x) \left[ \frac{F_P}{1 - x_P} \right. \\ & + \left( a_0 + a_1 \ln(Q^2) + a_2 \ln^2(Q^2) \right) \ln \left[ \frac{x_P(1-x)}{x(1-x_P)} \right] \\ & \left. + \left( b_0 + b_1 \ln(Q^2) + b_2 \ln^2(Q^2) \right) \ln^2 \left[ \frac{x_P(1-x)}{x(1-x_P)} \right] \right], \end{aligned}$$

# Global (*Simultaneous*) Fit of ZEUS $F_2(x, Q^2)$ to $x$ and $Q^2$



Differentiate the  $F_2$  fit to obtain:

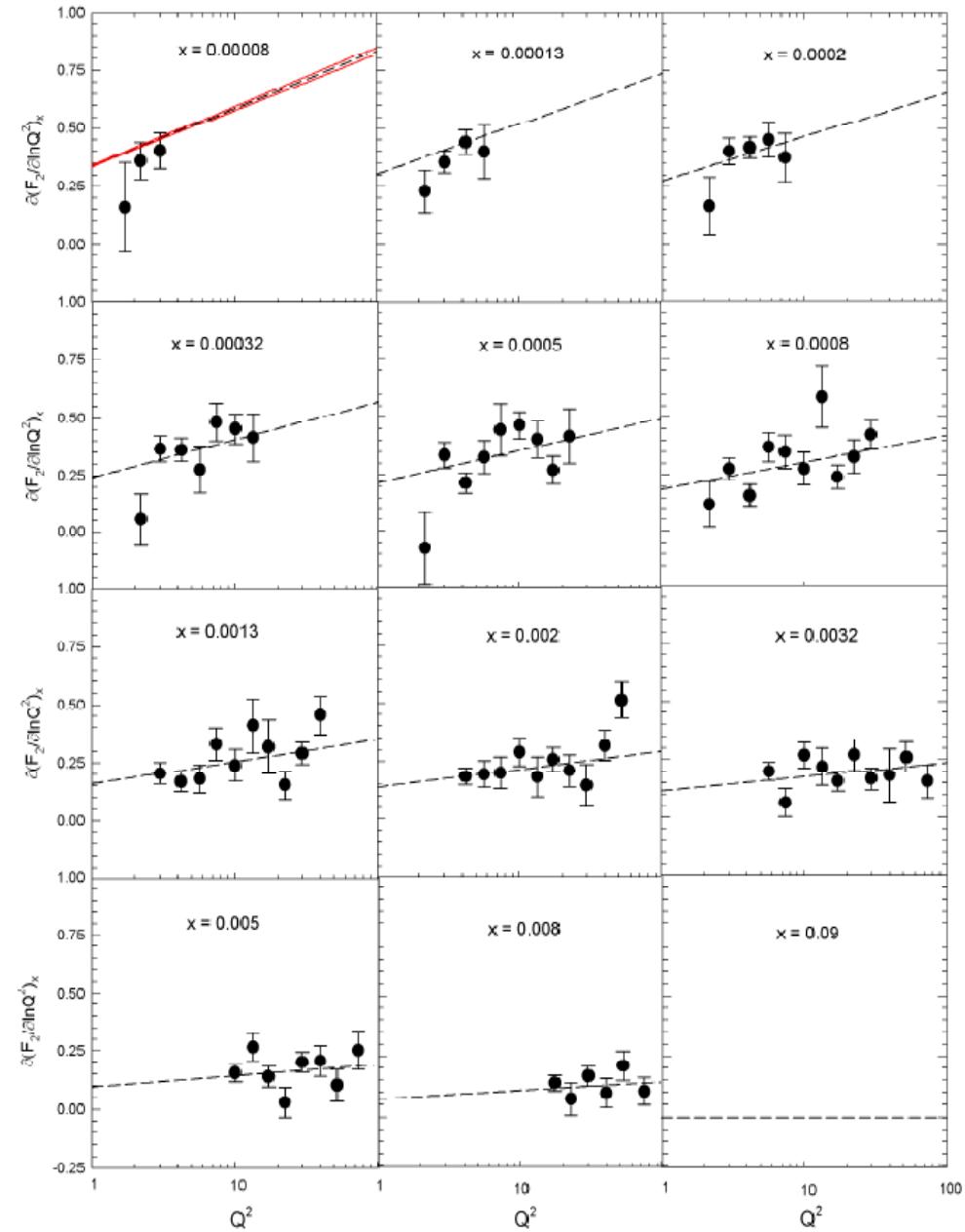
$$\frac{\partial_x F_2^p}{\partial \ln Q^2}(x, Q^2) \text{ vs. } Q^2$$

$$1 \leq Q^2 \leq 100 \text{ GeV}^2$$

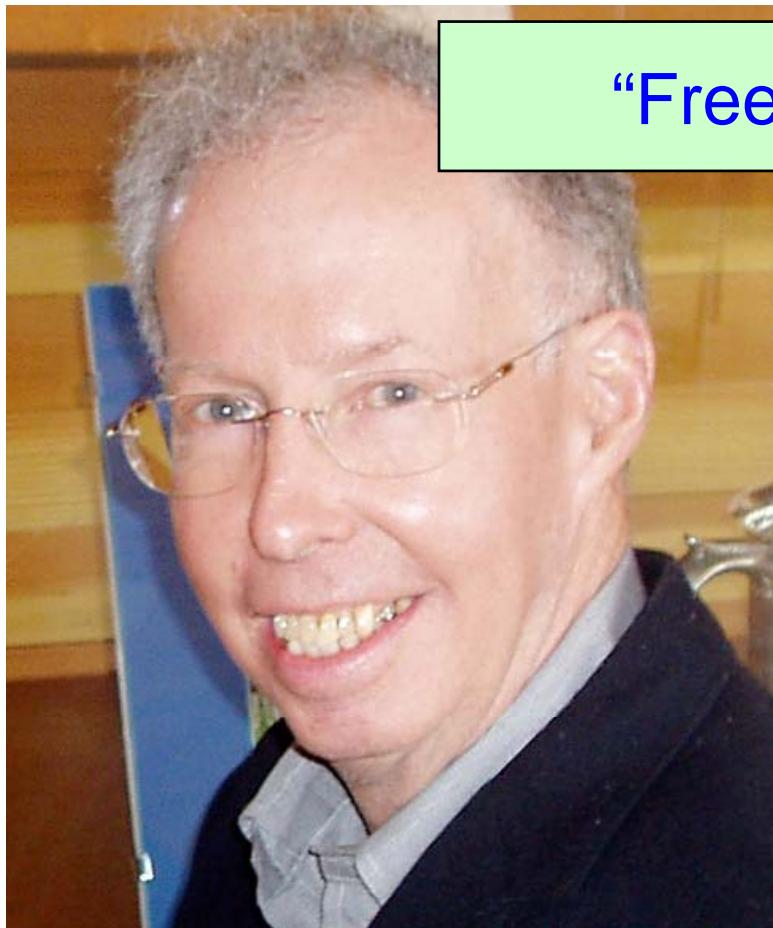
*Predictions* are made  
using ZEUS data in  
global fit

Experimental *data* are  
from H1 collaboration

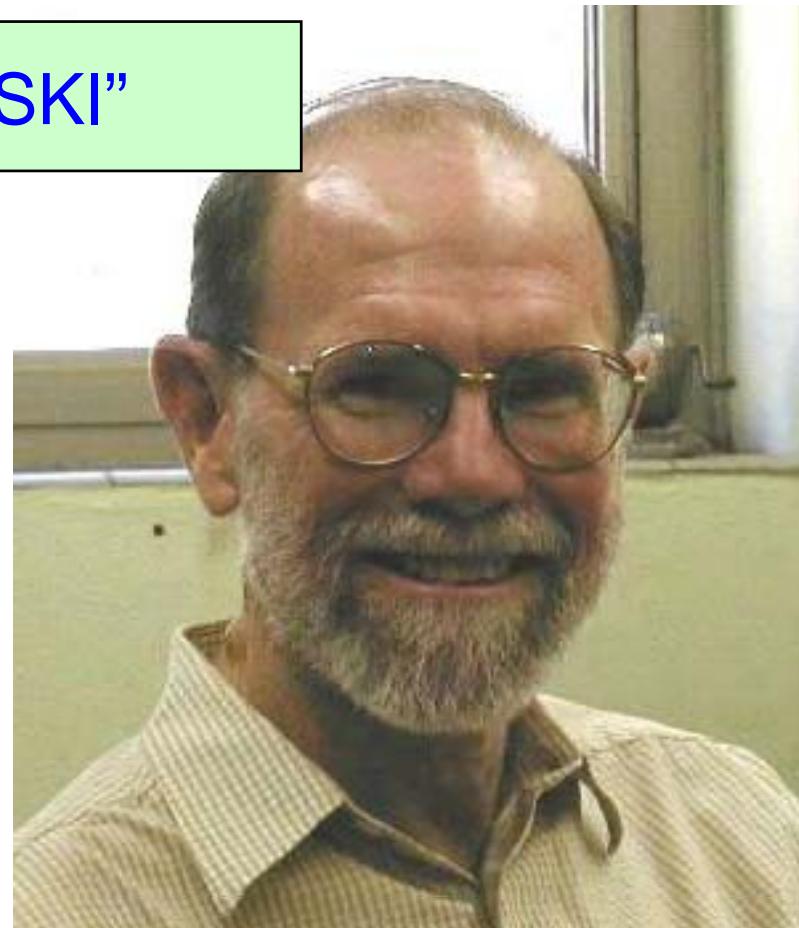
NO  
*RENORMALIZATION*  
made!



Part 3: Analytic gluon distributions from the proton structure function  $F_2(x, Q^2)$ : M. M. Block, L. Durand and D. McKay, arXiv hep/ph 0710.312 (2007).



“Free to SKI”



Jan, 2008

**Randy**

M. Block, Aspen Winter Physics Conference

**Doug**

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Traditionally, gluon and quark distribution functions have been determined *simultaneously* by starting with a virtuality  $Q_0^2$ , typically in the 1 to 2 GeV<sup>2</sup> range, and using the two coupled integral-differential (DGLAP) equations to evolve *individual* quark and gluon trial distributions to higher  $Q^2$ . The results are adjusted to fit the overall data (mainly the experimental data for proton structure function  $F_2^p(x, Q^2)$ ) by adjusting the parameters in the initial parton distributions, thus determining the evolved distributions.

## New Strategy for Determining Gluon Distributions

We propose a new and more accurate method for determining gluon distribution functions, illustrating it with a Froissart bound-type fit to the proton structure function  $F_2^p(x, Q^2)$

1. First, make a *global parametrization* of the experimental proton structure function  $F_2^p$ , simultaneously in  $x$  and  $Q^2$ .

2. Then, use only the DGLAP equation involving the evolution of the (now known) proton structure function  $F_2^p(x, Q^2)$  to solve explicitly for the gluon distribution function. We require no model-dependent assumptions about shapes of the gluon distribution or individual quark distributions. We find a global gluon distribution function  $G(x, Q^2)$  which is unique within uncertainties of the experimental data on  $F_2^p(x, Q^2)$ .

We now derive a *differential equation* for the LO gluon distribution  $G(x, Q^2) = xg(x, Q^2)$  from the integral A-P equation

The LO DGLAP integral equation for the evolution of the proton structure function  $F_2^p(x, Q^2)$ :

$$\begin{aligned} \frac{\partial F_2^p(x, Q^2)}{\partial \ln(Q^2)} = & x \left[ \int_x^1 \frac{F_2^p(z, Q^2) \times K_{qq}(x/z)}{z^2} dz \right. \\ & \left. + \int_x^1 \sum_f e_i^2(z, Q^2) \frac{G(z, Q^2) \times K_{qg}(x/z)}{z^2} dz \right]. \end{aligned}$$

Here  $G(x, Q^2) = xg(x, Q^2)$  and  $e_i^2(z, Q^2) = e_i^2 \theta(z - x_i)$  where  $e_i$  is the electric charge of the quark with flavor  $i$ ,  $x_i = x[1 + (4M_i^2/Q^2)]$ , and the step function enforces the parton level thresh-

old condition  $s \geq 4M^2$  for the production of a pair of quarks of mass  $M$ . The sum runs over all quarks.  $K_{qq}$  and  $K_{qg}$  are the LO splitting functions for quarks and gluons, respectively. Temporarily, we will ignore the threshold factors, which become irrelevant for  $Q^2 \gg 4M_i^2$ , and illustrate our method using the massless 4 quark families,  $u, d, c, s$ . Then  $\sum_{f,\bar{f}} e_i^2 = 20/9$ . Results will later be generalized to include the mass-dependent effects associated with both the charm quark  $c$  and the bottom quark  $b$ .

Introducing the LO Splitting functions, we write the Alterelli-Parisi evolution equation for the proton structure function as:

$$\begin{aligned} \frac{\partial F_2^p(x, Q^2)}{\partial \ln(Q^2)} = & \frac{\alpha_s}{4\pi} \left\{ 4F_2^p(x, Q^2) + \frac{16}{3} \left[ F_2^p(x, Q^2) \ln \frac{1-x}{x} \right. \right. \\ & + x \int_x^1 \left( \frac{F_2^p(z, Q^2)}{z} - \frac{F_2^p(x, Q^2)}{x} \right) \frac{dz}{z-x} \Big] \\ & - \frac{8}{3} x \int_x^1 F_2^p(z, Q^2) \left( 1 + \frac{x}{z} \right) \frac{dz}{z^2} \\ & \left. \left. + \frac{20}{9} x \int_x^1 G(z, Q^2) \left( \frac{x^2 + (z-x)^2}{z^2} \right) \frac{dz}{z^2} \right\} \right. \end{aligned}$$

After integrating by parts and some manipulation, we define  $\mathcal{F}_2^p(x, Q^2)$  by

$$\begin{aligned}\mathcal{F}_2^p(x, Q^2) \equiv & \frac{\partial F_2^p(x, Q^2)}{\partial \ln(Q^2)} \\ & - \frac{\alpha_s}{4\pi} \left\{ \frac{16}{3} \int_x^1 \frac{1}{z} \frac{\partial F_2^p}{\partial z}(z, Q^2) \ln \frac{z}{z-x} dz \right. \\ & \left. - \frac{4}{3} \int_x^1 \frac{\partial F_2^p}{\partial z}(z, Q^2) \left( \frac{x^2}{z^2} + \frac{2x}{z} \right) dz \right\}.\end{aligned}$$

After multiplying both sides by  $\left(\frac{\alpha_s}{4\pi}\right)^{-1}$ , we write

$$\begin{aligned}\left(\frac{\alpha_s}{4\pi}\right)^{-1} \mathcal{F}_2^p(x, Q^2)/x = & \\ & \frac{20}{9} \int_x^1 G(z, Q^2) \left( \frac{x^2 + (z-x)^2}{z^2} \right) \frac{dz}{z^2}.\end{aligned}$$

## Analytic solution for $G(x, Q^2)$

To simplify the notation, we define  $\mathcal{G}(x, Q^2)$  by

$$\mathcal{G}(x, Q^2) \equiv -\left(\frac{\alpha_s}{4\pi} \frac{80}{9}\right)^{-1} x^4 \frac{\partial^3 (\mathcal{F}_2^p(x, Q^2)/x)}{\partial x^3}.$$

Inhomogeneous r.h.s. of the linear 2<sup>nd</sup> order differential equation is **completely** determined by the proton structure function,  $F_2(x, Q^2)$

Introducing the new variable  $v = \ln(1/x)$ , we now write the linear differential equation as

$$\left(\frac{\partial^2}{\partial v^2} + 3\frac{\partial}{\partial v} + 4\right) \hat{G}(v, Q^2) = 4\hat{\mathcal{G}}(v, Q^2).$$

Given a smooth analytic parametrization of  $F_2^p(x, Q^2)$  as a simultaneous function of  $x$  and  $Q^2$  in a domain  $x_{\min} \leq x \leq x_{\max}$ ,  $Q_{\min}^2 \leq Q^2 \leq Q_{\max}^2$ , we calculate  $\hat{G}(v, Q^2)$ , determining the gluon distribution function  $G(x, Q^2)$  to the accuracy of the parametrization.

Different smooth parametrizations that fit the data on  $F_2^p(x, Q^2)$  equally well in the specified domain should give equivalent gluon distributions within that domain.

## Analytic solution for $G(x, Q^2)$

We will illustrate the procedure using the Froissart bound fit of Berger, Block and Tan to  $F_2(x, Q^2)$ , although any good parametrization of the structure function data can be used.

The final analytic answer is

$\ln^2(1/x)$ , for small  $x$

$$G_4(x, Q^2) = \alpha_4(1-x)^2 + \beta_4(1-x)\ln\frac{1}{x} + \gamma_4\ln^2\frac{1}{x},$$
$$0 < x \leq 1, \text{ where}$$

$$\alpha_4 = -0.47 - 0.16\ln(Q^2) - 0.019\ln^2(Q^2)$$

$$\beta_4 = 0.23 + 0.012\ln Q^2 - 0.014\ln^2 Q^2$$

$$\gamma_4 = 0.084 + 0.063\ln Q^2 + 0.011\ln^2 Q^2.$$

We used

$$\alpha_s(Q^2) \equiv \frac{12\pi}{25\ln(Q^2/\Lambda^2)},$$

Quadratic in  
 $\ln Q^2$

for four active flavors, with  $\Lambda = 0.153$  GeV  
adjusted to give  $\alpha_s(M_Z^2) = 0.118$ .

We emphasize that *only experimental  $F_2^p$  data in the region  $x \geq 0.0001$* , were needed to obtain our numerical results.. The 6-parameter fit which we used was constructed using all of the available ZEUS data with  $x_P \geq x \gtrsim 10^{-3}$  to  $10^{-4}$  (depending in the viruality and  $Q^2$ )  $0.11 \leq Q^2 \leq 1200 \text{ GeV}^2$ . The fit was excellent, with a  $\chi^2/\text{d.o.f.} = 1.09$  for 169 degrees of freedom.

3 massless  $u, d, s$  and 2 massive  $c$  and  $b$  quarks

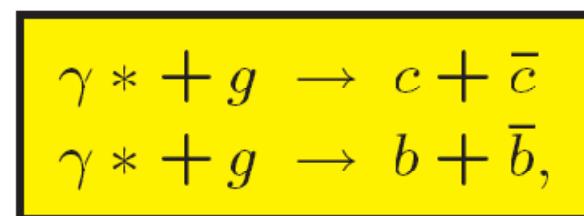
The sum over the 4 massless quark charges in the integral over  $G(z, Q^2)$  is replaced by

4 massless	3 massless	massive c	massive b
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$$2 \sum_{u,d,s,c} e_i^2 \rightarrow 2 \sum_{u,d,s} e_i^2 + 2e_c^2 \theta(z - x_c) + 2e_b^2 \theta(z - x_b).$$

$$x_c = x \left(1 + \frac{4M_c^2}{Q^2}\right), \quad x_b = x \left(1 + \frac{4M_b^2}{Q^2}\right), \quad 0 < z \leq 1.$$

The step functions enforce the threshold conditions for the reactions



W. K. Tung *et al*,  
JHeP 0702:053  
(2007)

## Integral Alterelli-Parisi Evolution for Heavy Quarks

$$\begin{aligned}
 \frac{\partial F_2^p(x, Q^2)}{\partial \ln(Q^2)} = & x \left[ \int_x^1 \frac{F_2^p(z, Q^2) \times K_{qg}(x/z)}{z^2} dz \right. \\
 & + 2 \int_x^1 \sum_{u,d,s} e_i^2 \int_x^1 \frac{G(z, Q^2) \times K_{qg}(x/z)}{z^2} dz \\
 & + 2e_c^2 \int_{x_c}^1 \frac{G(z, Q^2) \times K_{qg}(x/z)}{z^2} dz \\
 & \left. + 2e_b^2 \int_{x_b}^1 \frac{G(z, Q^2) \times K_{qg}(x/z)}{z^2} dz \right] \\
 x_c = x \left( 1 + \frac{4M_c^2}{Q^2} \right), \quad x_b = x \left( 1 + \frac{4M_b^2}{Q^2} \right), \quad 0 < z \leq 1.
 \end{aligned}$$

New lower limits due to  
 step functions

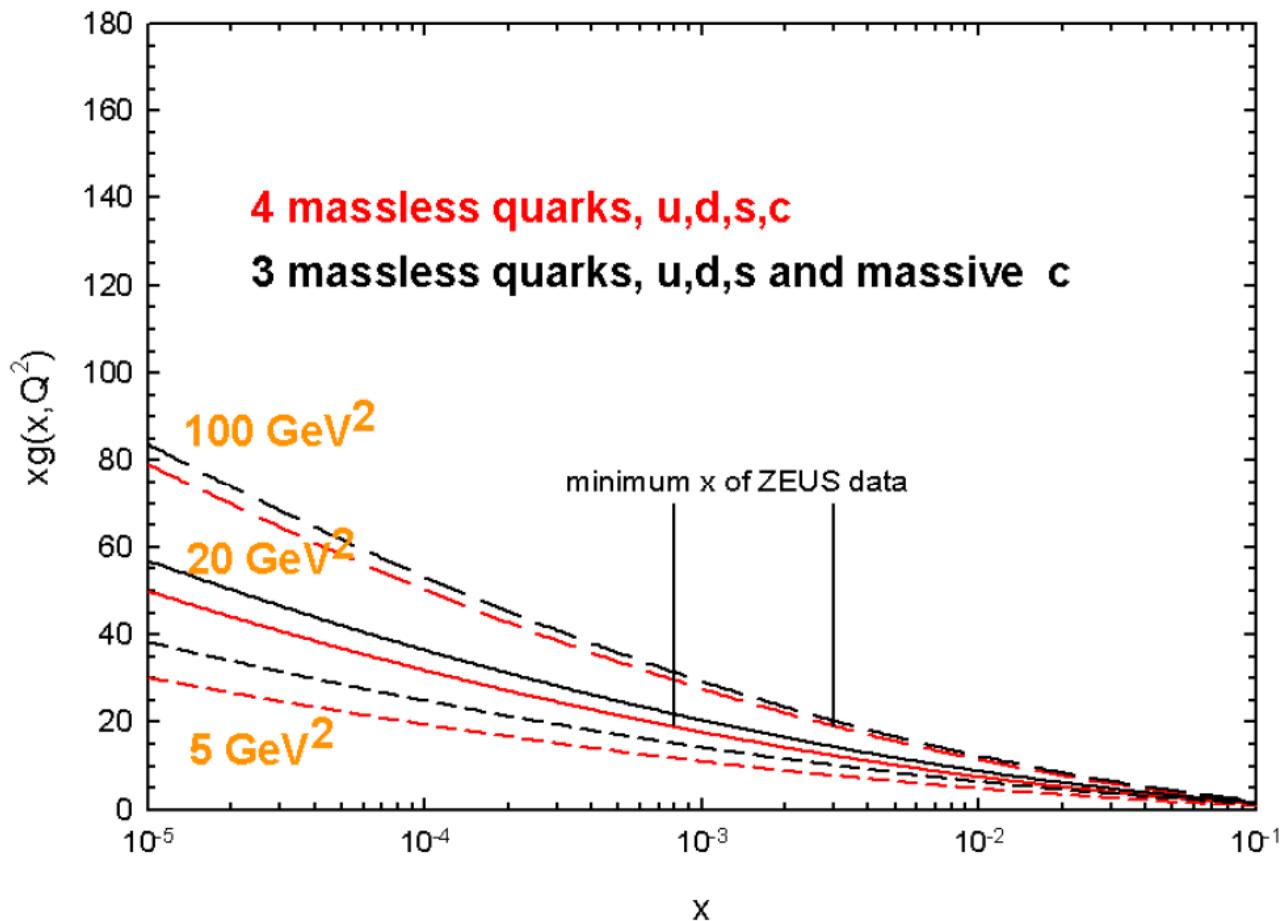
First, solve 4 quark differential equation,  
3 massless u, d, s  
and  
a massive c quark

$$\begin{aligned}\hat{\mathcal{G}}_3(v, Q^2) = & \left( \frac{\partial^2}{\partial v^2} + 3 \frac{\partial}{\partial v} + 4 \right) \hat{\mathcal{G}}_{3+c}(v, Q^2) \\ & + \frac{2}{3} \theta(v_c) \left[ A(\eta_c) \frac{\partial^2}{\partial v^2} + B(\eta_c) \frac{\partial}{\partial v} \right. \\ & \left. + C(\eta_c) \right] \hat{\mathcal{G}}_{3+c}(v_c, Q^2) \Big|_{v_c=v-\ln \eta_c}\end{aligned}$$

Note evaluation at **shifted** argument  $v_c$

The solution is given by

$$\begin{aligned}G_{3+c}(x, Q^2) &= \alpha_{3+c}(1-x)^2 + \beta_{3+c}(1-x)\ln(1/x) \\&\quad + \gamma_{3+c}\ln^2(1/x), \quad 0 < x \leq 1, \text{ with} \\ \alpha_{3+c} &= -0.68 - 0.15\ln Q^2 - 0.021\ln^2 Q^2 \\ \beta_{3+c} &= 0.48 - 0.040\ln Q^2 - 0.013\ln^2 Q^2 \\ \gamma_{3+c} &= 0.14 + 0.069\ln Q^2 + 0.011\ln^2 Q^2.\end{aligned}$$



Differential equation for five quarks:  
3 massless and massive c and massive b  
quarks

$$\begin{aligned}\hat{\mathcal{G}}_3(v, Q^2) = & \left( \frac{\partial^2}{\partial v^2} + 3 \frac{\partial}{\partial v} + 4 \right) \hat{\mathcal{G}}_{4+b}(v) \\ & + \theta(v_c) \frac{2}{3} \left[ A_c \frac{\partial^2}{\partial v^2} + B_c \frac{\partial}{\partial v} + C_c \right] \hat{\mathcal{G}}_{4+b}(v_c, Q^2) |_{v_c=v-\ln \eta_c} \\ & + \theta(v_b) \frac{1}{6} \left[ A_b \frac{\partial^2}{\partial v^2} + B_b \frac{\partial}{\partial v_b} + C_b \right] \hat{\mathcal{G}}_{4+b}(v_b, Q^2) |_{v_b=v-\ln \eta_b}\end{aligned}$$

Note the displaced arguments  $v_c$  and  $v_b$

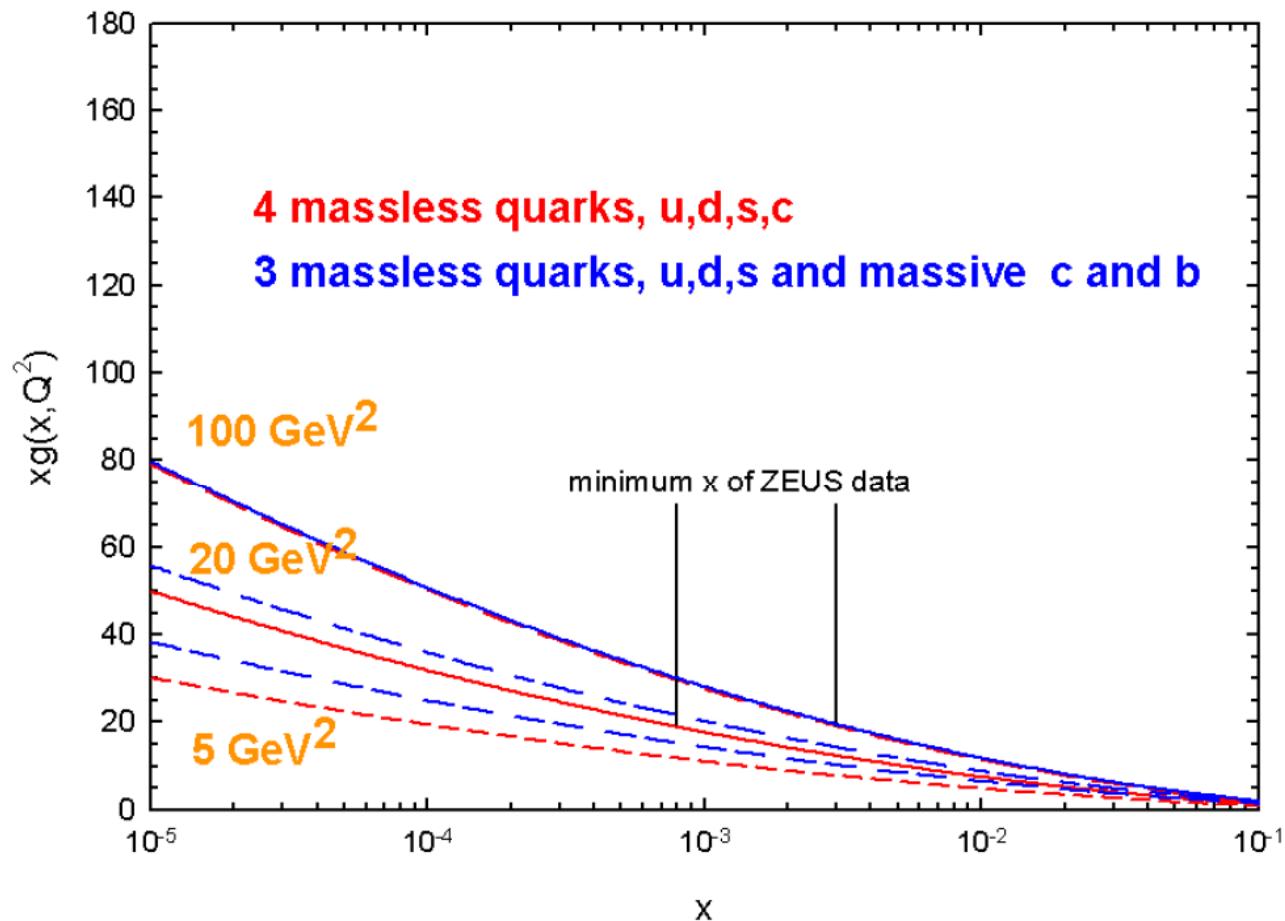
## Solution for $G_{4+b}(x, Q^2)$

5 quarks: 3 massless (u,d,s) quarks  
+ massive c quark ( $M_c = 1.3$  GeV)  
+ massive b quark ( $M_b = 4.2$  GeV)

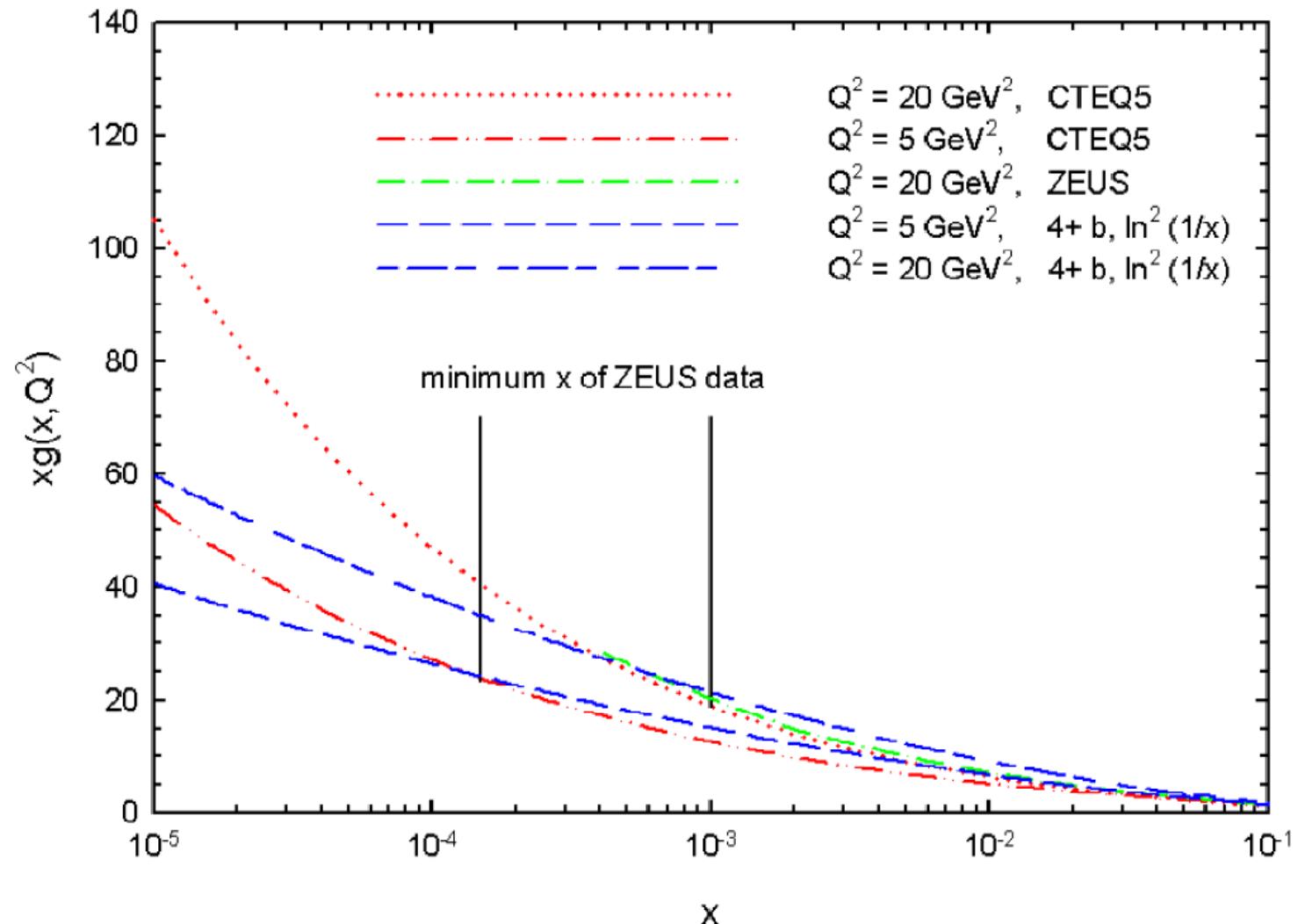
$$G_{4+b}(x, Q^2) = \alpha_{4+b}(1-x)^2 + \beta_{4+b}(1-x) \ln \frac{1}{x} + \gamma_{4+b} \ln^2 \frac{1}{x}, \quad 0 < x \leq x_P, \text{ with}$$
$$\alpha_{4+b} = -0.82 - 0.036 \ln Q^2 - 0.029 \ln^2 Q^2$$
$$\beta_{4+b} = 0.50 - 0.041 \ln Q^2 - 0.013 \ln^2 Q^2$$
$$\gamma_{4+b} = 0.14 + 0.072 \ln Q^2 + 0.0092 \ln^2 Q^2$$

$\ln^2(1/x)$ , for small  $x$

Quadratic in  $\ln Q^2$



## Comparisons with LO CTEQ5 and ZEUS



## CONCLUSIONS

- Using *only* the 1<sup>st</sup> Alterelli-Parisi equation and an *experimental global fit to the proton structure function*  $F_2(x, Q^2)$ , we showed that one can obtain an *analytic LO gluon solution for 5 quarks*: 3 massless (u,d,s) and 2 massive c + b quarks,  $G_{4+b}(x, Q^2) = xg(x, Q^2)$ , with rather small errors.
- We illustrated this procedure using a Froissart-bound fit to  $F_2(x, Q^2)$ , in  $\ln^2(1/x)$  and  $\ln^2(Q^2)$ .

To be done:

- NLO Gluon distributions:
- Quark distributions: needs much more detail.



“Global QCD analysis of parton structure of the nucleon: CTEQ5 parton distributions”,  
H.L. Lai *et al*, Eur. Phys. J. **C12**, 375 (2000).

p. 389

“Last, but by no means last, there are hidden uncertainties associated with the choice of functional forms for the non-perturbative initial parton distribution. ....”