Standard Model and Beyond Including Gravity in NCG

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References

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Need for new geometry

- The large scale global picture of space-time is well described in terms of Riemannian geometry, but this picture breaks down in the high energy scales where the quantum picture takes over.
- It is thus natural to look for a paradigm of geometry which starts from the quantum framework, where the role of real variables is played by self- adjoint operators in Hilbert space.
- Such a framework for geometry has been slowly emerging under the name of noncommutative geometry. One of its key features, besides the ability to handle spaces for which coordinates no longer commute with each other, is that this new geometry is spectral.

Need for new geometry

- This is in agreement with physics in which most of the data we have, either about the far distant parts of the universe or about high energy physics, are also of spectral nature. The red shifted spectra of distant galaxies or the momentum eigenstates of outgoing particles in high energy experiments both point towards a prevalence of spectral nature.

Need for new geometry

- From the mathematical standpoint it takes some doing to obtain a purely spectral (Hilbert space theoretical) counterpart of Riemannian geometry. One reason for the difficulty of this task is that, as is well known since the examples of J. Milnor, non-isometric Riemannian spaces exist which have the same spectra (for the Dirac or Laplacian operators). Another reason is that the conditions for a (compact) space to be a smooth manifold are given in terms of the local charts, whose existence and compatibility is assumed, but whose intrinsic meaning is more elusive.

Need for new geometry

- The laws of physics at low energies of order of 100 Gev are well encoded by the Standard Model action (with massive neutrinos) and the Einstein-Hilbert action. The fields in the standard model are the quarks, leptons, gauge fields and a Higgs field. These fields have different status than the gravitational field which depends only on the geometry of a Riemannian manifold M. The natural group of invariance of this theory is the semidirect product of the gauge transformations of U (l) x SU(2) x SU (3) and Diff (M)

Need for new geometry

- Many questions in the Standard Model are begging for answers, such as: Why the above gauge group? Why 16 fermions per generation? Why the particular representations of fermions? Why three generations? Why the particular Yukawa couplings and the huge hierarchy in the masses ranging from the neutrino masses to the top quark mass? Why the Higgs mechanism and spontaneous symmetry breaking? Is there gauge couplings unification? What determines the Higgs couplings? What is protecting the Higgs mass from the quadratic divergencies in what is known as the hierarchy problem.
- At present there are no compelling answers to most of these questions

Need for new geometry

- The group *G* of symmetries of the Lagrangian of gravity coupled with matter is handed to us by physics. It is the semi-direct product of the group Map(*M*, *G*) of gauge transformations of second kind by the symmetry group of gravity, namely the group Diff(*M*) of diffeomorphisms of ordinary space-time *M*: *G* = Map(*M*, *G*) t> Diff(*M*). Now for gravity coupled with matter to be pure gravity on a new space *N* the most obvious requirement is to find the manifold *N* in such a way that Diff (*N*) = *G*

Need for new geometry

There is a general mathematical result which asserts that the connected component of identity in Diff (*N*) is a *simple* group for any manifold *N*. Thus, since *G* has the non-trivial normal subgroup Map(*M*, *G*) there is no way one can solve the above equation using ordinary manifolds *N*. One can show that there is a solution, provided one searches for noncommutative solutions. i.e. that the group *G* is indeed the group of diffeomorphisms of a new space *N*.

Noncommutative Geometry

- The basic data is that of a spectral triple (A, H, D) which gives a representation in Hilbert space H of both the algebra A of coordinates and of the inverse line element D.
- Given a von Neumann algebra A of operators in Hilbert space H one can always find an anti-unitary isometry J such that the following commutators vanish :

$$[x, Jy^* J^{-1}] = 0 \qquad \forall x, y \in A$$

The basic rules are

$$J^2 = \varepsilon , \quad DJ = \varepsilon\, JD, \quad J\gamma = \varepsilon' \gamma J, \quad D\gamma = -\gamma D$$

Noncommutative Geometry

where γ is the Z/2 grading operator. The *KO*-theory comes in 8 different versions which just depend upon the dimension of the geometry modulo 8. They are distinguished by the three possible signs ε ∈ ±1 which govern the algebraic rules and whose values are according to the dimension modulo 8.

In physics terms these data have the following names and meaning:

- *H*: one particle Euclidean Fermions.
- *D*: inverse propagator.

Noncommutative Geometry

- J: charge conjugation.
- γ: chirality.

Thus the new formalism for geometry keeps a very close contact with physics. Exactly as the inner automorphisms form an "internal" part of the group of geometric symmetries, the metric admits "inner fluctuations".

Geometry of Space-Time

- Space-time could be approximated by a noncommutative space which is a product of a continuous four-dimensional Riemannian manifold times a finite dimensional space . This space is almost commutative with the noncommutativity arising from the matrix structure of the discrete space F.

- The main intrinsic reason for crossing by a finite geometry *F* has to do with the value of the dimension of space-time modulo 8. We needed this *KO*-dimension to be 2 modulo 8 (or equivalently 10) to define the Fermionic action, since this eliminates the doubling of fermions in the Euclidean framework. In other words the need for crossing by *F* is to shift the *KO*-dimension from 4 to 2 (modulo 8) . This finite geometry were derived from first principles through the following steps :

Geometry of Space-Time

- Classified the irreducible triplets (A, H, J).
- Studied the Z/2-gradings γ on H.
- Classified the subalgebras $A_F \subset A$ which allow for an operator D that does not commute with the center of A but fulfills the "order one" condition $[[D, a], b^0] = 0$

which guarantees the linearity of the connection.

Classification of finite spaces

- The classification in the first step shows that the solutions fall in two classes the first of which is inconsistent with *KO*-dimension 6. For the second class, we have shown that among the very few choices of lowest dimension we obtain the case

$$A = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

where H is the skew field of quaternions. Note that this determines the number of fermions to be $4^2 = 16$. Our main result then is that there

Classification of finite spaces

- This result is remarkable because the input that was used is minimal and the first possibility obtained consistent with the axioms of noncommutative geometry, after im- posing the symplectic-unitary symmetry condition on the algebra, is the algebra of the standard model with fermions in the correct representation. All the arbitrariness that is usually encountered in the construction of the standard model whether in the choice of the $SU(3) \times SU(2) \times U(1)$ gauge group, the fermionic representations, or the Higgs struc- ture and the electroweak spontaneous breaking mechanism disappear.

Classification of finite spaces

- The standard model becomes completely determined. In this respect we see that there is a geometrical structure responsible for all the details of the standard model. Geometrically we see that the underlying algebra is a direct sum of two algebras. The first algebra is quaternionic M_2 (H), broken to $(\mathrm{C} \oplus \mathrm{C})_R \oplus \mathrm{H}_L$, decomposes by the chirality operator into a

  left-handed and right-handed sectors. The

second algebra M_4 (C) is broken by the

Classification of finite spaces

Majorana mass for right-handed neutrino into $\mathbb{C} \oplus M_3(\mathbb{C})$ and corresponds to the splitting of the leptons and quarks. The fermions follow the product representation of the two algebras.

The Spectrum

- The spectrum of the fermionic particles, which is the number of states in the Hilbert space per family is predicted to be $4^2 = 16$.
- The 16 spinors get the correct quantum number with respect to the standard model gauge group which follow the decomposition:

$(4, 4) \rightarrow (1_R + 1_R + 2_L , 1 + 3) =$

$(1_R , 1) + (1_R , 1) + (2_L , 1) + (1_R , 3) + (1_R , 3) + (2_L , 3)$

The Spectrum

- These spinors correspond to v_R , e_R , l_L , u_R , d_R , q_L respectively, where l_L is the left-handed neutrino-electron doublet and q_L is the left-handed up-down quark doublet.
- In addition to the gauge bosons of $SU(3) \times SU(2) \times U(1)$ which are the inner fluctuations of the metric along continuous directions, we also have a Higgs doublet which corresponds to the inner fluctuations of the metric along the discrete directions.
- A singlet Scalar field whose vev gives Majorana mass to the right-handed neutrinos.
- What is peculiar about the Higgs doublet, is that its mass term as determined from the spectral action comes with a negative sign and a quartic term with a plus sign, thus predicting the phenomena of spontaneous breakdown of the electroweak symmetry.

Spectral Action

- The fermionic action takes the simple form

$(\Psi, D_A\Psi)$ where

$$\Psi = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}$$

with being the 16 dim spinor and ψ the ψ^c conjugate spinor so that Ψ satisfies both Majorana and Weyl condition $J\Psi=\Psi$ and $\gamma\Psi=\Psi$.

$$D_A=D+A+JA^*J^{-1}$$

A containes all the gauge fields, Higgs doublet and the neutrino singlet.

Spectral Action

- The missing ingredient, in the above description of the Standard Model coupled to gravity, is provided by a simple action principle—the spectral action principle, that has the geometric meaning of "pure gravity" and delivers the action functional of the Standard Model coupled to gravity when evaluated on $M \times F$. The spectral action principle is the simple statement that the physical action is determined by the spectrum of the Dirac operator D. The spectral data are available in localized form anywhere, and are (asymptotically) of the additive form

Spectral Action

Trace (f (D/Λ)). The detailed form of the function f is largely irrelevant since the spectral action can be expanded in decreasing powers of the scale Λ and the function f only appears through the scalars

$$f_k = \int_0^\infty f(y) y^{k-1} dy$$

Spectral Action

$$
\begin{aligned}
S_{\mathrm{b}} = {} & \frac{24}{\pi^2} F_4 \Lambda^4 \int d^4x \sqrt{g} \qquad (5.49) \\
& - \frac{2}{\pi^2} F_2 \Lambda^2 \int d^4x \sqrt{g} \left(R + \frac{1}{2} a \overline{H} H + \frac{1}{4} c \sigma^2 \right) \\
& + \frac{1}{2\pi^2} F_0 \int d^4x \sqrt{g} \left[\frac{1}{30} \left(-18 C^2_{\mu\nu\rho\sigma} + 11 R^* R^* \right) + \frac{5}{3} g_1^2 B_{\mu\nu}^2 + g_2^2 \left(W_{\mu\nu}^{\alpha} \right)^2 + g_3^2 \left(V_{\mu\nu}^{m} \right)^2 \right. \\
& \left. + \frac{1}{6} a R \overline{H} H + b \left(\overline{H} H \right)^2 + a \left| \nabla_\mu H_a \right|^2 + 2e \overline{H} H \, \sigma^2 + \frac{1}{2} d \, \sigma^4 + \frac{1}{12} c R \sigma^2 + \frac{1}{2} c \left(\partial_\mu \sigma \right)^2 \right] \\
& + F_{-2} \Lambda^{-2} a_6 + \cdots
\end{aligned}
$$

Spectral Action

$$
\begin{aligned}
S_{\mathrm{f}} = & \nu_R^{*}\gamma^{\mu} D_{\mu}\nu_R \\
& + e_R^{*}\gamma^{\mu}\left(D_{\mu} + i g_1 B_{\mu}\right) e_R \\
& + l_L^{a*}\gamma^{\mu}\left(\left(D_{\mu} + \frac{i}{2} g_1 B_{\mu}\right)\delta_a^b - \frac{i}{2} g_2 W_{\mu}^{\alpha}\left(\sigma^{\alpha}\right)_a^b\right) l_{bL} \\
& + u_R^{i*}\gamma^{\mu}\left(\left(D_{\mu} - \frac{2i}{3} g_1 B_{\mu}\right)\delta_i^j - \frac{i}{2} g_3 V_{\mu}^{m}\left(\lambda^{m}\right)_i^j\right) u_{jR} \\
& + d_R^{i*}\gamma^{\mu}\left(\left(D_{\mu} + \frac{i}{3} g_1 B_{\mu}\right)\delta_i^j - \frac{i}{2} g_3 V_{\mu}^{m}\left(\lambda^{m}\right)_i^j\right) d_{jR} \\
& + q_L^{ia*}\gamma^{\mu}\left(\left(D_{\mu} - \frac{i}{6} g_1 B_{\mu}\right)\delta_a^b\delta_i^j - \frac{i}{2} g_2 W_{\mu}^{\alpha}\left(\sigma^{\alpha}\right)_a^b\delta_i^j - \frac{i}{2} g_3 V_{\mu}^{m}\left(\lambda^{m}\right)_i^j\delta_a^b\right) q_{jbL} \\
& + \nu_R^{*}\gamma_5 k^{*\nu}\epsilon^{ab} H_b l_{aL} + e_R^{*}\gamma_5 k^{*e}\overline{H}^{a} l_{aL} \\
& + u_R^{i*}\gamma_5 k^{*u}\epsilon^{ab} H_b \delta_i^j q_{jaL} + d_R^{i*}\gamma_5 k^{*d}\overline{H}^{a}\delta_i^j q_{jaL} + \nu_R^{*}\gamma_5 k^{*\nu_R}\sigma\left(\nu_R^{*}\right)^{c} + \mathrm{h.c}
\end{aligned}
$$

Singlet Role

- The presence of the singlet insures that the Higgs coupling does not turn negative at high energies. The potential takes the form, after rescaling

$$V = \frac{1}{4}\left(\lambda_h \overline{h}^4 + 2\lambda_{h\sigma}\overline{h}^2\overline{\sigma}^2 + \lambda_\sigma \overline{\sigma}^4\right) - \frac{2g^2}{\pi^2} f_2 \Lambda^2 \left(\overline{h}^2 + \overline{\sigma}^2\right)$$

where

$$\lambda_h = \frac{n^2+3}{(n+3)^2}\left(4g^2\right)$$

$$\lambda_{h\sigma} = \frac{2n}{n+3}\left(4g^2\right)$$

$$\lambda_\sigma = 2\left(4g^2\right)$$

Prediction of the Spectral SM

- We have classified all noncommutative spaces formed as a product of a continuous four dimensional space times a discrete space consistent with the axioms of noncommutative geometry.
- Under the very weak physical assumptions that there are no mirror fermions in nature and the existence of a fermionic Majorana mass, we have determined uniquely that the resulting noncommutative space corresponds to the spectrum of the Standard Model of particle interactions in addition to right-handed neutrinos and an extra singlet scalar field.
- We have predicted that the number of fermions per family is 16 with the correct representations under the symmetry U(1)xSU(2)xSU(3).

Prediction of the Spectral SM

- Explained the extremely small mass of the neutrino through the see-saw mechanism.
- Determined the existence of a Higgs doublet as the fluctuation of the metric (Dirac operator) along discrete directions, and the spontaneous symmetry breaking mechanism.
- The existence of a new scalar field, whose vev gives a Majorana mass to the right-handed neutrinos, and which is essential in protecting the Higgs coupling from turning to negative at energies of order 1011Gev. This paves the way for the Standard Model to hold all the way up to very high energies.

Prediction of the Spectral SM

- Predicted the top quark mass to be of the order of 170 Gev.
- Provided a geometrical framework for the unification of all particle interactions including gravity valid up to very high energies.
- Replaced diffeomorphism invariance in general relativity and gauge symmetry of vector bosons with outer and inner automorphisms of the algebra defining the noncommutative space.

Prediction of the Spectral SM

- Euclidean formulation of quantum gravity requires the addition of the Hawking-Gibbons boundary term to teh Einstein action. The spectral action contains the Hawking-Gibbons term automatically, reflecting the fact that noncommutative geometry being dependent on the Dirac operator, which is the inverse of the fermion propagator, contains information about quantum gravity.

Beyond SM

- Removing one of the axioms of noncommutative geometry which guarantees the linearity of the connection allows for inner fluctuations which contains a quadratic dependence. These fluctuations have the very interesting property of forming a semi-group (i.e. containing non-invertible elements).

  $$D_A = D + A + JA^* J^{-1} + A_{(2)}$$

- The underlying symmetry in this case is then extended to the Pati-Salam unification group $SU(2)_R x SU(2)_L x SU(4)$ where the lepton number is the fourth color.
- The $A_{(2)}$ part of the connection depend on scalar fields, the Higgs fields that break the high energy sector that breaks both $SU(2)_R x SU(4)$ to $U(1) x SU(3)$.

Beyond SM

- Depending on the initial structure of D without fluctuations the Higgs fields could contain parts depending quadratically on the fundamental Higgs fields or contain fields in the product representations.
- The simplest Higgs fields structure is (2,2,1)+(2,1,4)+(1,1,1+15).
- The Higgs potential and interactions are fixed unambiguously.
- The Standard Model continues to hold experimentally to a very high degree of precision.
- In the noncommutative setting, any deviation from the Standard Model can only be consistent with the Pati-Salam model with a connection containing quadratic dependence.

Conclusions

- Noncommutative geometry as developed by Alain Connes provides an attractive geometric setting for the unification of all fundamental interactions including gravity.
- For finite spaces consistent with the linearity of the connection, the SM is singled out in a unique way, extended by right-handed neutrinos and a singlet.

Conclusions

- It provides explanations to many of the questions that have no answers in the SM.
- Any future deviations from the SM can only be due to a Pati-Salam unification with a very well defined structure.
- Many interesting developments are still needed such as a quantizing scheme that takes the noncommutative geometric constraints into account.